Практика по рядам Фурье. 02.09.21.

$$f(x)=\sum_{k=0}^{\infty}\frac{f'(x_0)}{k!}(x-x_0)^k$$

Тригонометрическая вариация: $f(x),\ x\in(a,b)\to(-l,l)$

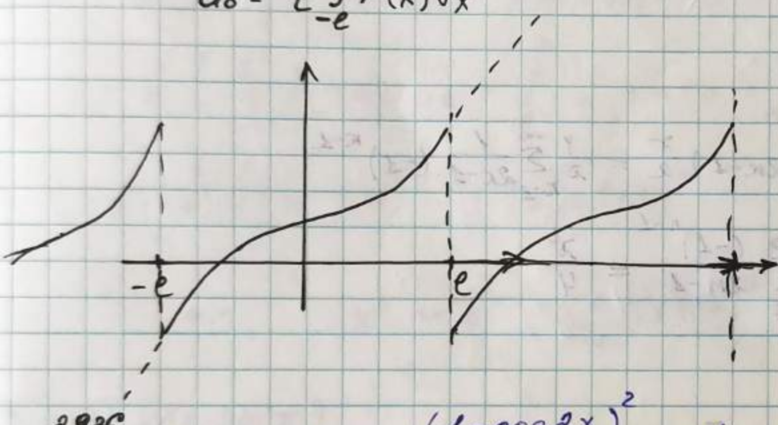
$$f(x)\sim\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos\frac{2\pi n x}{b-a}+b_n\sin\frac{2\pi n x}{b-a}$$

$t=x-\frac{a+b}{2}$ ~ смещение к симметр. интервалу.

$$\Rightarrow f(x)\sim\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cdot\cos\frac{n\pi x}{l}+b_n\sin\frac{\pi n x}{l}\text{, где}$$

$$a_n=\frac{1}{l}\int_{-l}^{l}f(x)\cos\frac{n\pi x}{l}\,dx\quad;\quad b_n=\frac{1}{l}\int_{-l}^{l}f(x)\sin\frac{n\pi x}{l}\,dx$$

$$a_0=\frac{1}{l}\int_{-l}^{l}f(x)\,dx$$



2936.

$$f(x)=\sin^4x=\left(\frac{1-\cos2x}{2}\right)^2=\left(\frac{1}{2}-\frac{\cos2x}{2}\right)^2=\frac{1}{4}(1-\cos2x)^2=$$

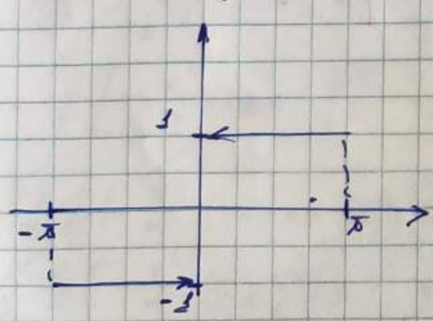
$$=\frac{1}{4}(1-2\cos2x+\cos^22x)=\frac{1}{4}\left(1-2\cos2x+\frac{1+\cos4x}{2}\right)=$$

$$=\frac{1}{4}-\frac{1}{2}\cos2x+\frac{1}{8}+\frac{1}{8}\cdot\cos4x=\frac{3}{8}-\frac{1}{2}\cos2x+\frac{1}{8}\cos4x$$

~ тоже ряд Фурье.

2938.

$f(x)=\operatorname{sgn}x\quad(-\pi<x<\pi)$. ~ ф-я нечётная, a_n можно отбросить.



$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}\underbrace{\operatorname{sgn}x\cdot\sin nx}_{\text{неч.}\cdot\text{неч}=\text{чёт}}\,dx=\frac{2}{\pi}\int_0^{\pi}\sin nx\,dx=$$

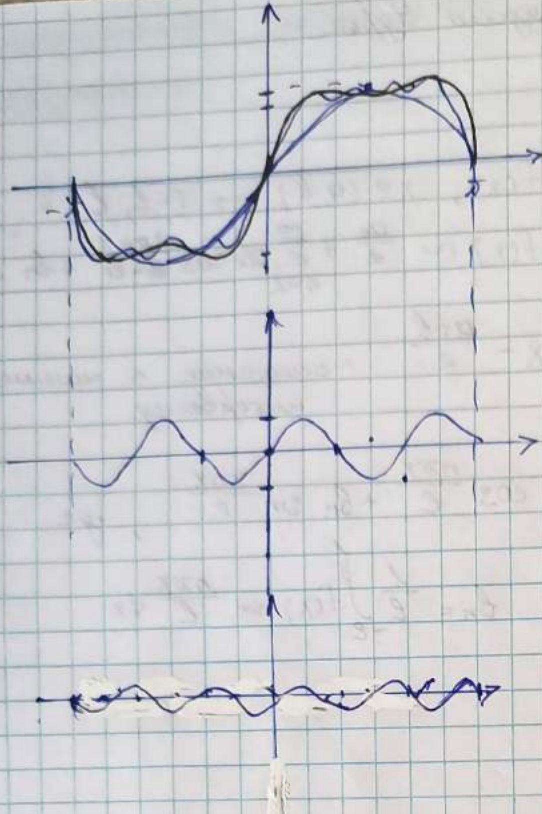
$$=-\frac{2}{\pi n}\cos nx\Big|_0^{\pi}=\frac{2}{\pi n}(1-\cos\pi n)=$$

$$=\frac{2}{\pi n}(1-(-1)^n)=\begin{cases}0, & n=2k,\ k\in\mathbb{Z}\\ \frac{4}{\pi n}, & n=2k-1\end{cases}$$

$$\operatorname{sgn}x\sim\frac{4}{\pi}\sum_{k=1}^{\infty}\frac{1}{2k-1}\cdot\sin(2k-1)x=\frac{4}{\pi}\sin x+\frac{4}{3\pi}\sin3x+\frac{4}{5\pi}\sin5x\pm\ldots$$

Найти сумму ряда:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1} = \frac{\pi}{4}$$



$$\boxed{\operatorname{sgn}\frac{\pi}{2} \sim \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{1}{2k-1}\cdot\sin(2k-1)\frac{\pi}{2}}$$

$$\sin(2k-1)\frac{\pi}{2} = \sin\left(\pi k - \frac{\pi}{2}\right) = (-1)^k \sin\left(-\frac{\pi}{2}\right)$$

$$\sin(\alpha + \pi n) = (-1)^n \sin\alpha$$

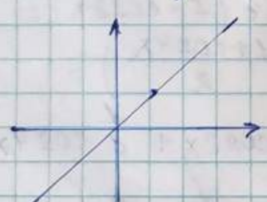
$$= (-1)^{k+1} = (-1)^{k-1}$$

$$\Rightarrow \operatorname{sgn}\frac{\pi}{2} \sim \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{1}{2k-1}\cdot\sin(2k-1)\frac{\pi}{2} = \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{1}{2k-1}(-1)^{k-1}$$

$$\Rightarrow \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{2n-1} = \frac{\pi}{4}$$

2940.

$f(x) = x,\ x\in(-\pi, \pi)$



$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x\cdot\sin nx\, dx = \left\{ \begin{array}{l} u = x \to du = dx \\ dv = \sin nx\, dx \to v = -\frac{1}{n}\cos nx \end{array} \right.$$

$$= -\frac{x}{n}\cos nx \Big|_{-\pi}^{\pi} + \frac{1}{n}\int_{-\pi}^{\pi}\cos nx\, dx = \Big| \cdot \frac{1}{\pi}$$

$$= -\frac{x}{n}\cos nx \Big|_{-\pi}^{\pi} + \frac{2}{n}\int_{0}^{\pi}\cos nx\, dx = \Big| \cdot \frac{1}{\pi}$$

$$\frac{1}{\pi}\left[-\frac{x}{n}\left(\cos\pi n - \cos(-\pi n)\right) + \frac{2}{n}\cdot\frac{1}{n}\sin nx \Big|_0^{\pi}\right] =$$

$$= -\frac{2}{n}\cos\pi n = \frac{2}{n}(-1)^{n+1}$$

$$x \sim \sum_{n=1}^{\infty}\frac{2}{n}(-1)^{n+1} \sin nx$$

2941.

$f(x) = \frac{\pi - x}{2}$ в интервале $(0; 2\pi)$

$t = x - \pi$

$f(t) = \frac{\pi - (t+\pi)}{2} = -\frac{t}{2},\quad t\in(-\pi, \pi)$

$F(t) = \left(-\frac{1}{2}\right) \cdot t$ в прошлой задаче.

$$f(t) = -\frac{1}{2} \cdot 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \cdot \sin(nt) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sin(nt)$$

$$f(x) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sin(nx - \pi n) = \sum_{n=1}^{\infty} \frac{\sin nx}{n}$$

2961. $f(x) = x^2$

а) $x \in (-\pi, \pi)$

$$x^2 \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

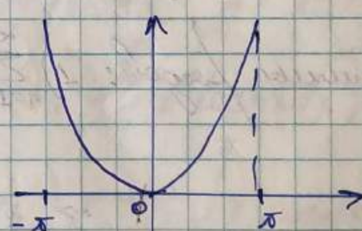
$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 dx = \frac{1}{\pi} \cdot \frac{x^3}{3}\Big|_{-\pi}^{\pi} = \frac{1}{\pi}\left(\frac{\pi^3}{3} + \frac{\pi^3}{3}\right) = \frac{2}{3}\pi^2$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \cos nx\, dx = \left\{ \begin{matrix} u = x^2 \to du = 2x\,dx \\ dV = \cos nx\, dx \quad V = \frac{1}{n}\sin nx \end{matrix} \right\} =$$

$$= \frac{1}{\pi}\left[\frac{x^2}{n} \sin nx \Big|_{-\pi}^{\pi} - \int_{-\pi}^{\pi} \frac{2x}{n} \sin nx\, dx \right] = -\frac{2}{n} \cdot \underbrace{\frac{1}{\pi}\int_{-\pi}^{\pi} x \sin nx\, dx}_{} =$$

$$= -\frac{2}{n}\left(+\frac{2}{n} \cdot (-1)^{n+1}\right) = \frac{4}{n^2}(-1)^n$$

$$\Rightarrow \quad x^2 \sim \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2}(-1)^n \cos nx$$



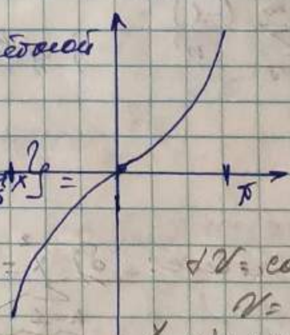
б) $x \in (0, \pi)$

$$x^2 \sim \sum_{n=1}^{\infty} b_n \sin nx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx\, dx \;\circleddash$$

Искусственно сделаем нечётной

$$f(x) = \begin{cases} x^2, & x \in (0, \pi) \\ -x^2, & x \in (-\pi, 0) \end{cases}$$



$$\circleddash\; \frac{2}{\pi}\int_0^{\pi} x^2 \sin nx\, dx = \left\{ \begin{matrix} u = x^2 \to du = 2x\,dx \\ dV = \sin nx\, dx \to V = -\frac{1}{n}\cos nx \end{matrix} \right\} =$$

$$= \frac{2}{\pi}\left[-\frac{x^2}{n}\cos nx \Big|_0^{\pi} + \frac{2}{n}\int_0^{\pi} \cos nx \cdot x\, dx \right] =$$

$u = x \Rightarrow du = dx$

$dV = \cos nx\, dx$

$V = \frac{1}{n}\sin nx$

$\frac{x}{n}\sin nx \Big|_0^{\pi} - \frac{1}{n}\int_0^{\pi} \sin nx\, dx = -\frac{\cos nx}{n^2}\Big|_0^{\pi}$

$$= \frac{2}{\pi}\left[\left(-\frac{x^2}{n} \cdot (\cos \pi n - 1)\right) + \frac{2}{n}\left(-\frac{\cos \pi n}{n^2} + \frac{1}{n^2}\right)\right] =$$

$$= \frac{2}{\pi}\left[-\frac{x^2}{n}\left((-1)^n - 1\right) + \frac{2}{n^3}\left(1 - (-1)^n\right) \right] =$$

$$= \left(\frac{2}{n}\pi + \frac{4}{\pi n^2}\right)$$

$$x^2 \sim \sum_{n=1}^{\infty}\left(\frac{2}{n}\pi + \frac{4}{\pi n^2}\right)\sin nx$$

в) $f(x)=x^2$ в интервале $(0, 2\pi)$

$t=x-\pi \to x=t+\pi \quad ; \; dx=dt$

$f(t)=(t+\pi)^2 \quad , \; t\in[-\pi;\pi]$

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}(t+\pi)^2\,dt=\frac{1}{\pi}\int_{-\pi}^{\pi}(t^2+2\pi t+\pi^2)\,dt=\frac{1}{\pi}\left(\frac{t^3}{3}\Big|_{-\pi}^{\pi}+\pi t^2\Big|_{-\pi}^{\pi}+\pi^2 t\Big|_{-\pi}^{\pi}\right)=$$

$$=\frac{1}{\pi}\left(\frac{\pi^3}{3}+\frac{\pi^3}{3}+\pi^3+\pi^3\right)=\frac{1}{\pi}\left(\frac{2\pi^3}{3}+2\pi^3\right)=\frac{8\pi^3}{3\pi}=\frac{8\pi^2}{3}$$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}(t+\pi)^2\cdot\cos nt\,dt=\frac{1}{\pi}\cdot\frac{4\pi}{n^2}(-1)^{n+1}=\frac{4}{n^2}(-1)^{n+1}$$

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}(t+\pi)^2\sin(nt)\,dt=-(t+\pi)^2\frac{\cos(nt)}{n}\Big|_{-\pi}^{\pi}+\frac{2}{n}\int_{-\pi}^{\pi}(t+\pi)\cos nt\,dt=$$

$$=-4\pi^2\frac{\cos(\pi n)}{n}+\frac{2}{n}\left[(t+\pi)\frac{\sin(nt)}{n}\Big|_{-\pi}^{\pi}-\frac{1}{n}\int_{-\pi}^{\pi}\sin(nt)\,dt\right]=$$

$$=\frac{4\pi^2(-1)^{n+1}}{n}\cdot\frac{1}{\pi}=\frac{4\pi(-1)^{n+1}}{n}$$

$$f(t)\sim\frac{4\pi^2}{3}+\sum_{n=1}^{\infty}\frac{4}{n^2}(-1)^{n+1}\cos(nt)+\frac{4\pi(-1)^{n+1}}{n}\sin(nt)$$

$$f(x)\sim\frac{4\pi^2}{3}+\sum_{n=1}^{\infty}\frac{4}{n^2}(-1)^{n+1}\cos(nx-\pi n)+\frac{4\pi(-1)^{n+1}}{n}\sin(nx-n\pi)$$

$$f(x)\sim\frac{4\pi^2}{3}+4\sum_{n=1}^{\infty}\frac{\cos(nx)}{n^2}-4\pi\sum_{n=1}^{\infty}\frac{\sin(nx)}{n}$$

Суммы рядов: 1) $\sum_{n=1}^{\infty}\frac{1}{n^2}$ из а) $x^2-\frac{\pi^2}{3}=\sum_{n=1}^{\infty}\frac{4}{n^2}(-1)^n\cos(\pi x)$

$x=\pi$

$$\Rightarrow \pi^2-\frac{\pi^2}{3}=4\sum_{n=1}^{\infty}\frac{1}{n^2} \quad \Rightarrow \underline{\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}}$$

2) $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}$; из в) : $x=\pi \to \frac{4\pi^2}{3}+4\sum_{n=1}^{\infty}\frac{\cos(\pi n)}{n^2}-0=\pi^2$

$$\frac{4\pi^2}{3}+4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}=\pi^2 \quad /\cdot(-1)$$

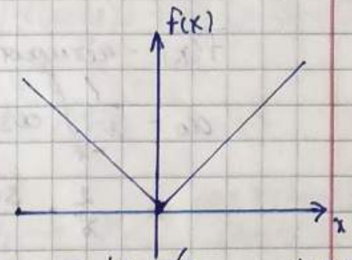
$$4\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}=\frac{\pi^2}{3}$$

$$\Rightarrow \underline{\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}=\frac{\pi^2}{12}}$$

3) $\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}=$

Домашняя работа

2942. $f(x)=|x|$ в интервале $(-\pi,\pi)$



$f(x)$ – чётная

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}|x|\sin\frac{n\pi x}{\pi}dx=\frac{1}{\pi}\int_{-\pi}^{\pi}\underbrace{|x|\sin(nx)dx}_{\text{нечётн.}}=0$$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}\underbrace{|x|\cos nx\,dx}_{\text{чёт}}=\frac{2}{\pi}\int_0^{\pi}x\cdot\cos nx\,dx=\left\{\begin{matrix}u=x\to du=dx\\ dv=\cos nx\,dx\to v=\frac{\sin nx}{n}\end{matrix}\right\}=$$

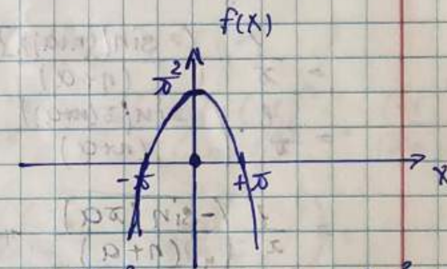
$$=\frac{2}{\pi}\left(\underbrace{\frac{x\sin(nx)}{n}\Big|_0^{\pi}}_{=0}-\frac{1}{n}\int_0^{\pi}\sin(nx)dx\right)=\frac{2}{\pi n}\left(\frac{\cos(nx)}{n}\Big|_0^{\pi}\right)=$$

$$=\frac{2}{\pi n}\left(\frac{\cos(\pi n)}{n}-\frac{1}{n}\right)=\frac{2}{\pi n^2}(\cos\pi n-1)=\frac{2}{\pi n^2}((-1)^n-1)$$

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}|x|dx=\frac{2}{\pi}\int_0^{\pi}x\,dx=\frac{2}{\pi}\cdot\left(\frac{x^2}{2}\Big|_0^{\pi}\right)=\frac{2\pi^2}{\pi 2}=\pi$$

$$|x|\sim\frac{\pi}{2}+\sum_{n=1}^{\infty}\frac{2}{\pi n^2}(\cos\pi n-1)\cdot\cos nx$$

2944. $f(x)=\pi^2-x^2$ в интервале $(-\pi,\pi)$



$f(x)$ – чётная $\Rightarrow b_n=0$

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}(\pi^2-x^2)dx=\frac{2}{\pi}\int_0^{\pi}(\pi^2-x^2)dx=$$

$$=\frac{2}{\pi}\left(\int_0^{\pi}\pi^2dx-\int_0^{\pi}x^2dx\right)=\frac{2}{\pi}\left(\pi^3-\frac{\pi^3}{3}\right)=\frac{4\pi^2}{3}\Rightarrow\frac{a_0}{2}=\frac{2\pi^2}{3}$$

$$a_n=\frac{2}{\pi}\int_0^{\pi}(\pi^2-x^2)\cos nx\cdot dx=\frac{2}{\pi}\left(\int_0^{\pi}\pi^2\cdot\cos nx\,dx-\int_0^{\pi}x^2\cos nx\,dx\right)=$$

$$=\frac{2}{\pi}\left(\underbrace{\pi^2\cdot\frac{\sin(nx)}{n}\Big|_0^{\pi}}_{=0}-\int_0^{\pi}x^2\cos nx\,dx\right)=-\frac{2}{\pi}\int_0^{\pi}x^2\cos nx\,dx \;=$$

Отдельно: $\int_0^{\pi}x^2\cos nx\,dx=\left\{\begin{matrix}u=x^2\to du=2x\,dx\\ dv=\cos nx\,dx\to v=\frac{\sin nx}{n}\end{matrix}\right\}=\underbrace{\frac{x^2}{n}\cdot\sin nx\Big|_0^{\pi}}_{=0}-2\int_0^{\pi}\frac{\sin(nx)}{n}x\,dx=$

$$=-\frac{2}{n}\int_0^{\pi}x\cdot\sin nx\,dx=\left\{\begin{matrix}u=x\to du=dx\\ dv=\sin(nx)dx\to v=-\frac{\cos(nx)}{n}\end{matrix}\right\}=$$

$$=-\frac{2}{n}\left(-\frac{x\cos(nx)}{n}\Big|_0^{\pi}+\frac{1}{n}\int_0^{\pi}\cos nx\,dx\right)=-\frac{2}{n}\left(-\frac{\pi\cos(\pi n)}{n}+\underbrace{\frac{1}{n}\cdot\frac{\sin nx}{n}\Big|_0^{\pi}}_{=0}\right)=$$

$$=+\frac{2\pi\cos(\pi n)}{n^2}$$

$$=\;-\frac{2}{\pi}\cdot\left(\frac{2\pi\cos(\pi n)}{n^2}\right)=-\frac{4\cos(\pi n)}{n^2}=-\frac{4\cdot(-1)^n}{n^2}=\frac{4(-1)^{n+1}}{n^2}$$

$$\Rightarrow f(x)=\pi^2-x^2\sim\frac{2\pi^2}{3}+4\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}\cos nx$$

2945. $f(x) = \cos ax$ в интервале $(-\pi, \pi)$ (a - не целое)

$f(x)$ - чётная $\Rightarrow b_n = 0$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos ax \, dx = \frac{2}{\pi}\int_0^{\pi} \cos ax \, dx = \frac{2}{\pi} \cdot \left(\frac{\sin ax}{a} \Big|_0^{\pi} \right) =$$

(т.к. a - не целое)

$$= \frac{2}{\pi} \cdot \frac{\sin a\pi}{a} = \frac{2}{a\pi} \sin(a\pi)$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{\cos ax \cdot \cos(nx)}_{\text{чётн.}} dx = \frac{2}{\pi}\int_0^{\pi} \cos(ax) \cdot \cos(nx) \, dx \;(=)$$

$$\cos\alpha \cdot \cos\beta = \frac{1}{2}\left(\cos(\alpha+\beta) + \cos(\alpha-\beta)\right)$$

~ или наоборот одно и то же (пох в ответе)

$$(=)\; \frac{1}{\pi}\int_0^{\pi} \left(\cos(a+n)x + \cos(a-n)x \right) dx =$$

$$= \frac{1}{\pi}\left(\int_0^{\pi} \cos[(a+n)x] \, dx + \int_0^{\pi} \cos[(a-n)x] \, dx \right) =$$

$$= \frac{1}{\pi}\left(\int_0^{\pi} \cos[(n+a)x] \, dx + \int_0^{\pi} \cos[(n-a)x] \, dx \right) =$$

$$= \frac{1}{\pi} \cdot \left(\frac{\sin[(n+a)x]}{(n+a)} \Big|_0^{\pi} + \frac{\sin[(n-a)x]}{(n-a)} \Big|_0^{\pi} \right) =$$

$$= \frac{1}{\pi} \cdot \left(\frac{\sin(\pi(n+a))}{(n+a)} + \frac{\sin(\pi n - \pi a)}{(n-a)} \right) = \frac{1}{\pi} \cdot \left(\frac{\sin(\pi n + \pi a)}{(n+a)} + \frac{\sin(\pi n - \pi a)}{(n-a)} \right) =$$

$$= \frac{1}{\pi}\left(\frac{(-1)^n \cdot \sin(\pi a)}{(n+a)} + \frac{(-1)^n \cdot \sin(-\pi a)}{(n-a)} \right) =$$

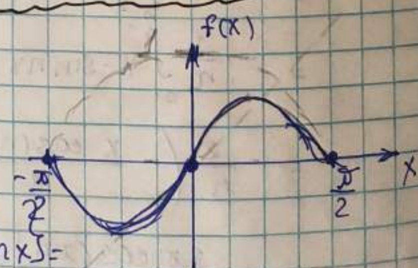
$$= \frac{(-1)^n}{\pi} \cdot \left(\frac{\sin(\pi a)}{(n+a)} - \frac{\sin(\pi a)}{(n-a)} \right) = \frac{(-1)^n}{\pi}\left(\frac{\cancel{n \cdot \sin(\pi a)} - a\sin(\pi a) - \cancel{n\sin(\pi a)} - a\sin}{n^2 - a^2} \right.$$

$$= \frac{(-1)^{n+1} \cdot a\sin(\pi a)}{\pi(n^2 - a^2)}$$

$$\Rightarrow \cos ax \sim \frac{\sin(a\pi)}{a\pi} + \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \cdot a\sin(\pi a)}{\pi(n^2 - a^2)} \cdot \cos nx =$$

$$= \frac{2\sin(\pi a)}{\pi}\left(\frac{1}{2a} + \sum_{n=1}^{\infty} (-1)^{n+1} \cdot \frac{a\cos(nx)}{n^2 - a^2} \right)$$

2951. $f(x) = x\cos x$ в интервале $\left(-\frac{\pi}{2}; \frac{\pi}{2}\right)$



$f(x)$ - нечётная $\Rightarrow a_n = 0$

$$a_0 = \frac{2}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \underbrace{x\cos x}_{\text{нечёт}} \, dx = \left\{ \begin{array}{l} u = x \to du = dx \\ dv = \cos x \, dx \to v = \sin x \end{array} \right\} =$$

$$= \frac{2}{\pi}\left(x \cdot \sin x \Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} - \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin x \, dx \right) = 0$$

$$b_n = \frac{2}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \underbrace{x\cos x \cdot \sin(2nx)}_{\text{чётная}} dx = \frac{4}{\pi}\int_0^{\frac{\pi}{2}} x\cos x \cdot \sin(2nx) \, dx \;(=)$$

$$\sin\alpha\cos\beta = \frac{1}{2}\left(\sin(\alpha+\beta) + \sin(\alpha-\beta) \right)$$

⑤ $\frac{2}{\pi}\int_0^{\frac{\pi}{2}} x\left(\sin(2nx+x)+\sin(2nx-x)\right)dx = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} x\left(\sin[x(2n+1)]+\sin[x(2n-1)]\right)dx =$

$$=\frac{2}{\pi}\cdot\left(\int_0^{\frac{\pi}{2}} x\cdot\sin(x(2n+1))\,dx + \int_0^{\frac{\pi}{2}} x\sin(x(2n-1))\,dx\right) =$$

Отдельно: $\int_0^{\frac{\pi}{2}} x\cdot\sin(x(2n+1))\,dx = \left\{ \begin{matrix} u=x & du=dx \\ dv=\sin x(2n+1)\,dx \to & v=-\frac{\cos x(2n+1)}{(2n+1)}\end{matrix}\right\} =$

$$= \underbrace{-\frac{x\cos(x(2n+1))}{(2n+1)}\Big|_0^{\frac{\pi}{2}}}_{=0} + \int_0^{\frac{\pi}{2}}\frac{\cos(x(2n+1))}{(2n+1)}\,dx =$$

$$=\frac{1}{(2n+1)}\cdot\frac{\sin(x(2n+1))}{(2n+1)}\Big|_0^{\frac{\pi}{2}} = \frac{\sin\left(\frac{\pi}{2}(2n+1)\right)}{(2n+1)^2}=$$

$$=\frac{\sin\left(\pi n+\frac{\pi}{2}\right)}{(2n+1)^2}=\frac{(-1)^n}{(2n+1)^2}$$

$$\int_0^{\frac{\pi}{2}} x\cdot\sin(x(2n-1))\,dx = \left\{ \begin{matrix} u=x & du=dx \\ dv=\sin(x(2n-1))\,dx & \to v=-\frac{\cos(x(2n-1))}{(2n-1)}\end{matrix}\right\} =$$

$$= \underbrace{-\frac{x\cos(x(2n-1))}{(2n-1)}\Big|_0^{\frac{\pi}{2}}}_{=0} + \frac{1}{(2n-1)}\int_0^{\frac{\pi}{2}}\cos(x(2n-1))\,dx =$$

$$=\frac{1}{(2n-1)^2}\sin(x(2n-1))\Big|_0^{\frac{\pi}{2}} = \frac{\sin\left(\pi n-\frac{\pi}{2}\right)}{(2n-1)^2} = \frac{(-1)\cdot(-1)^n}{(2n-1)^2}=\frac{(-1)^{n+1}}{(2n-1)^2}$$

$$= \frac{2}{\pi}\left(\frac{(-1)^n}{(2n+1)^2}+\frac{(-1)^{n+1}}{(2n-1)^2}\right)=\frac{2\cdot(-1)^n}{\pi}\left(\frac{1}{(2n+1)^2}-\frac{1}{(2n-1)^2}\right)=$$

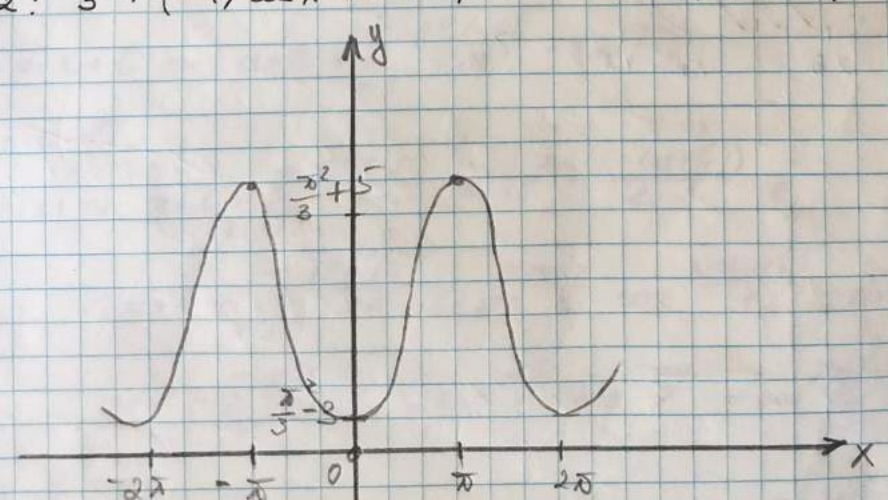
$$=\frac{2\cdot(-1)^n}{\pi}\left(\frac{\cancel{4n^2}-4n+\cancel{1}-\cancel{4n^2}-4n-\cancel{1}}{(4n^2-1)^2}\right)=\frac{16\cdot(-1)^{n+1}}{\pi(4n^2-1)^2}$$

$$\Rightarrow\quad x\cos x \sim \frac{16}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{\pi(4n^2-1)^2}\sin 2nx$$

2961. (графики)

а) $x^2\sim\frac{\pi^2}{3}+\sum_{n=1}^{\infty}\frac{4}{n^2}(-1)^n\cos nx$

$n=2:\ \frac{\pi^2}{3}+(-4)\cos x+\cos 2x = \frac{\pi^2}{3}-4\cos x+\cos 2x$



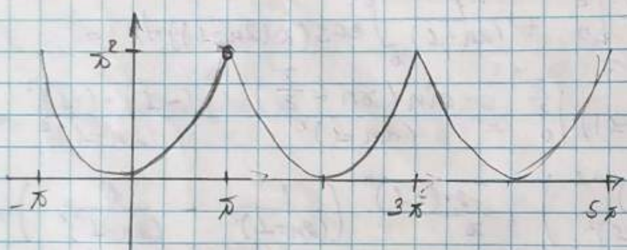
в) $x^2 \sim \frac{4\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{\cos(nx)}{n^2} - 4\pi\sum_{n=1}^{\infty}\frac{\sin(nx)}{n}$?

$n=2: \quad \frac{4\pi^2}{3} + 4\cos x + \cos(2x) - 4\pi\sin x - 2\pi\sin(2x)$

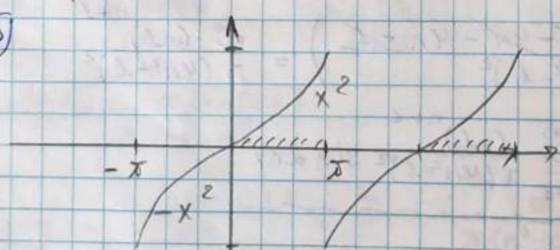
$n=1: \quad \frac{4\pi^2}{3} + 4\cos x - 4\pi\sin x$

2968. $f(x) = x^2$

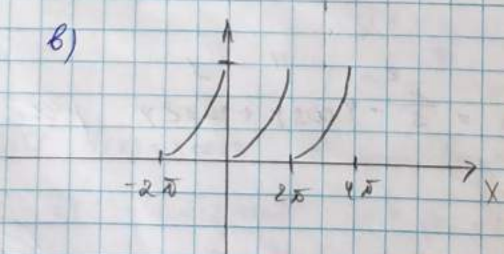
а) $x \in (-\pi, \pi)$



б)

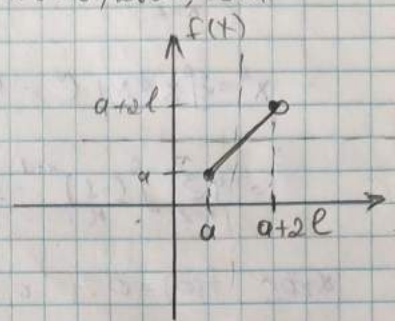


в)



Ряды Фурье (продолжение). 2949, 2962, 2963 | 2952? 08.09.21.

Д/З: 2939, 2960, 2964



2949. $f(x) = x$ в инт. $(a; a+2l)$

$\cancel{t = x-(a+b) = x-a-l}$

$x = a;\ t = -l$

$x = a+2l;\ t = l;\quad t = kx + b$

$\begin{cases} -l = ak + b \\ l = (a+2l)k + b \end{cases} \Rightarrow b = -l - a$

$-2l = ak - k(a+2l)$

$-2l = ak - ak - 2kl \rightarrow k = 1$ $\Rightarrow$ Замена: $t = x - l - a$

$x = t + l + a \Rightarrow f(t) = a + l + t,\ t \in (-l; l)$ ~ нечёт

$f_1 = a + l$

$a_0 = \frac{1}{l}\int_{-l}^{l}(a+l)\,dt = \frac{1}{l}\left(at\Big|_{-l}^{l} + lt\Big|_{-l}^{l}\right) = \frac{1}{l}(2al + 2l^2) =$

$= 2(a+l) \Rightarrow \frac{a_0}{2} = a + l$

$a_0 = \frac{1}{l}\int_{-l}^{l} t\,dt = \frac{1}{l}\left(\frac{t^2}{2}\Big|_{-l}^{l}\right) = 0$

$a_n = 0$

$b_n = \frac{1}{l}\int_{-l}^{l} t \cdot \sin\left(\frac{\pi n t}{l}\right)dt = \left\{ \begin{matrix} u = t \rightarrow du = dt \\ dv = \sin\left(\frac{\pi n t}{l}\right)dt \rightarrow v = -\frac{l}{\pi n}\cos\frac{\pi n t}{l} \end{matrix} \right\} =$

$= \frac{1}{l}\left(-\frac{l}{\pi n}\cos\frac{\pi n t}{l}\cdot t\Big|_{-l}^{l} + \frac{l}{\pi n}\int_{-l}^{l}\cos\frac{\pi n t}{l}\,dt\right) =$

$= \frac{1}{l}\left(-\frac{l^2}{\pi n}\cos(\pi n) - \frac{l^2}{\pi n}\cos(\pi n) + \underbrace{\frac{l^2}{\pi^2 n^2}\left(\sin\frac{\pi n t}{l}\Big|_{-l}^{l}\right)}_{=0}\right) =$

$= \cancel{\frac{1}{l}\left(\frac{l^2}{\pi^2 n^2}\right)} = \frac{2l\cdot(-1)^{n+1}}{\pi n}$

$f(t) = a + l + t \sim a + l + \sum_{n=1}^{\infty}\frac{2l(-1)^{n+1}}{\pi n}\cdot\sin\frac{\pi n t}{l}$

$t = x - a - l$

$f(x) \sim a + l + \sum_{n=1}^{\infty}\frac{2l(-1)^{n+1}}{\pi n}\sin\frac{\pi n(x-(a+l))}{l} =$

$= a + l + \sum_{n=1}^{\infty}\frac{2l(-1)^{n+1}}{\pi n}\cdot\left(\sin\frac{\pi n x}{l}\cos\frac{\pi n(a+l)}{l} - \sin\frac{\pi n(a+l)}{l}\cos\frac{\pi n x}{l}\right) =$

$= a + l + \sum_{n=1}^{\infty}\frac{2l(-1)^{2n+1}}{\pi n}\left(\sin\frac{\pi n x}{l}\cos\frac{\pi n a}{l} - \cos\frac{\pi n x}{l}\cdot\sin\frac{\pi n a}{l}\right)$

2962. $x = 2\sum_{n=1}^{\infty}(-1)^{n+1}\cdot\frac{\sin(nx)}{n} \qquad (-\pi<x<\pi)$

$$x^2 = 2\int x\,dx + C = 4\int\sum_{n=1}^{\infty}(-1)^{n+1}\cdot\frac{\sin(nx)}{n}\,dx =$$

$$= 4\sum_{n=1}^{\infty}\int\frac{(-1)^{n+1}}{n}\cdot\sin(nx)\,dx = 4\sum_{n=1}^{\infty}\frac{(-1)^{n}}{n^2}\cos nx + C =$$

$x=0$: $f(0)=0$: $\quad 0 = 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2} + C \to C = -4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}$

$$= 4\sum_{n=1}^{\infty}\left(\frac{(-1)^n}{n^2}\cos nx - 4\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}\right)$$

$$x^3 = 12\sum_{n=1}^{\infty}\int\left(\frac{(-1)^n}{n^2}\cos nx\,dx - 12\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}\right)dx =$$

$$= \cancel{\frac{12(-1)^n}{n^3}}\quad 12\left(\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3}\sin nx + 2\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\cdot\sin nx}{n}\right) =$$

$$= 12\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\cdot\sin nx\left(\frac{1}{n^2} - 2\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^2}\right)$$

$$x^4 = 4\int x^3\,dx = 48\int\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin nx\left(\frac{1}{n^2} - 2\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^2}\right)dx =$$

$$= 48\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\left(\frac{1}{n^2} - 2\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^2}\right)\cdot\int\sin nx\,dx =$$

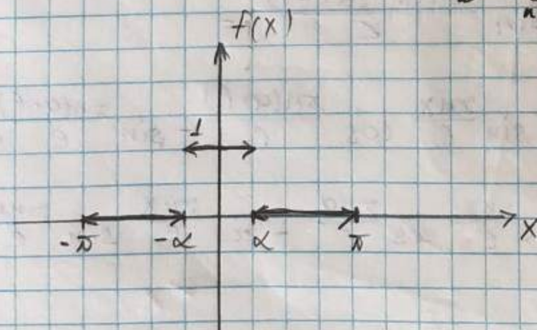
$$= 48\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}\left(\frac{1}{n^2} - 2\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^2}\right)\cos nx + C =$$

$$= 48\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}\left(\frac{1}{n^2} - 2\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^2}\right)\cos nx + C$$

$$x^4 = 0 = 48\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}\left(\frac{1}{n^2} - \frac{\pi^2}{3}\right) + C \Rightarrow C = -48\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}\left(\frac{1}{n^2} - \frac{\pi^2}{3}\right)$$

2983. $f(x) = \begin{cases} 1, & |x|<\alpha; \\ 0, \text{ при} & \alpha<|x|<\pi \end{cases}$

Равенство Ляпунова: $\frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2) = \frac{1}{l}\int_{-l}^{l} f^2(x)\,dx$



функц. – чётн.

$x \in (-\pi, \pi)$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\alpha} 1\,dx + 0 = \frac{2\alpha}{\pi}$$

$$\frac{a_0^2}{2} = \frac{4\alpha^2}{\pi^2\cdot 2} = \frac{2\alpha^2}{\pi^2}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\alpha} \cos nx\,dx = \frac{2}{\pi n}\sin n\alpha$$

$$b_n = 0$$

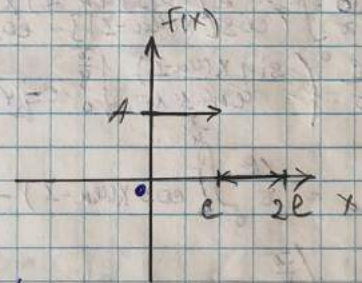
$$\frac{2\alpha^2}{\pi^2} + \sum_{n=1}^{\infty}\frac{4}{\pi^2 n^2}\sin^2 n\alpha = \frac{2}{\pi}\int_0^{\alpha} dx$$

$$\frac{2\alpha^2}{\pi^2} + \sum_{n=1}^{\infty}\frac{4}{\pi^2 n^2}\sin^2 n\alpha = \frac{2\alpha}{\pi} \quad \Big/ :$$

Домашняя работа.

2939.

$$f(x) = \begin{cases} A, & \text{если } x \in (0, l) \\ 0, & \text{если } x \in (l; 2l) \end{cases}$$



$$a_0 = \frac{1}{l}\int_0^{2l} f(x)\,dx = \frac{1}{l}\left(\int_0^{l} A\,dx + \int_l^{2l} 0\,dx\right) = A$$

$$a_n = \frac{1}{l}\int_0^{2l} f(x)\cos\frac{n\pi x}{l}\,dx = \frac{1}{l}\int_0^{l} A\cdot\cos\frac{n\pi x}{l}\,dx =$$

$$= \frac{A}{l}\cdot\frac{l}{\pi n}\left(\sin\frac{\pi n x}{l}\Big|_0^{l}\right) = 0$$

$$b_n = \frac{1}{l}\int_0^{l} A\cdot\sin\frac{\pi n x}{l}\,dx = \frac{A}{l}\cdot\left(-\frac{l}{\pi n}\cos\frac{\pi n x}{l}\Big|_0^{l}\right) =$$

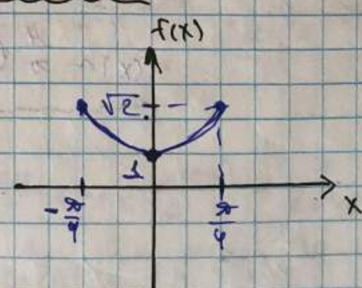
$$= -\frac{A}{\pi n}\cdot(\cos\pi n - 1) = \frac{A}{\pi n}(1 - (-1)^n)$$

$$\Rightarrow f(x) \sim \frac{A}{2} + \sum_{n=1}^{\infty}\frac{A}{\pi n}(1-(-1)^n)\sin\frac{\pi n x}{l} = \sim \text{ если } n = 2k\text{, то } \Sigma = 0$$

$$= \frac{A}{2} + \frac{2A}{\pi}\sum_{k=0}^{\infty}\frac{1}{2k+1}\cdot\sin\left[\frac{(2k+1)\pi x}{l}\right]$$

2960.

$$f(x) = \sec x \qquad \left(-\frac{\pi}{4} < x < \frac{\pi}{4}\right)$$



$$\sec x = \frac{1}{\cos x} =$$

$$\sec\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{1} = \sqrt{2}$$

$f(x)$ − чётная $\Rightarrow b_n = 0$

$$a_0 = \frac{4}{\pi}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sec(x)\,dx = \frac{8}{\pi}\int_0^{\frac{\pi}{4}} \sec(x)\,dx =$$

$$\int \sec(x)\,dx = \ln\left|tg\left(\frac{\pi}{4}+\frac{x}{2}\right)\right| + C$$

$$= \frac{8}{\pi}\left[\ln\left|tg\left(\frac{x}{2}+\frac{\pi}{4}\right)\right|\Big|_0^{\frac{\pi}{4}}\right] = \frac{8}{\pi}\left(\ln\left|tg\left(\frac{\pi}{8}+\frac{\pi}{4}\right)\right| - \ln\left|tg\frac{\pi}{4}\right|\right) =$$

$$= \frac{8}{\pi}\cdot\ln\left|\frac{\sin\left(\frac{\pi}{8}+\frac{\pi}{4}\right)}{\cos\left(\frac{\pi}{8}+\frac{\pi}{4}\right)}\right| = \frac{8}{\pi}\cdot\ln\left|\frac{\sqrt{2+\sqrt{2}}\cdot 2}{2\sqrt{2-\sqrt{2}}}\right| = \frac{8}{\pi}\ln\left(\sqrt{\frac{2+\sqrt{2}}{2-\sqrt{2}}}\right) =$$

$$= \frac{8}{\pi}\ln\left(\sqrt{\frac{(2+\sqrt{2})^2}{2}}\right) = \frac{8}{\pi}\ln\left(\sqrt{\frac{4+2\sqrt{2}+2}{2}}\right) =$$

$$= \frac{8}{\pi}\ln\left((2+\sqrt{2})\cdot\frac{1}{\sqrt{2}}\right) = \frac{8}{\pi}\ln\left(1+\sqrt{2}\right)$$

$$a_n = \frac{4}{\pi}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\frac{\cos 4nx}{\cos x}\,dx = \frac{8}{\pi}\int_0^{\frac{\pi}{4}}\frac{\cos 4nx}{\cos x}\,dx =$$

?

$$\cos 4nx - \cos(4nx-4x) = -2\sin(4nx-2x)\sin 2x =$$

$$= -4\sin(4nx-2x)\sin x\cos x =$$

$$\Rightarrow \cos(4nx) =$$

$$\Rightarrow -4\cdot\frac{1}{2}\left(\cos(4nx-3x) - \cos(4nx-x)\right)\cos x = 2\left(\cos(4nx-x) - \cos(4nx-3x)\right)\cdot$$

$$\cdot\cos x = 2\left(\cos(4n-1)x - \cos(4n-3)x\right)\cos x$$

$$\Rightarrow \cos 4nx = 2\left(\cos[(4n-1)x] - \cos[(4n-3)x]\right)\cdot\cos x + \cos[4x(n-1)]$$

$$\Rightarrow a_n = \frac{16}{\pi}\int_0^{\frac{\pi}{4}}\left[\cos x(4n-1) - \cos x(4n-3)\right]dx + \underbrace{\frac{8}{\pi}\int_0^{\frac{\pi}{4}}\frac{\cos 4x(n-1)}{\cos x}\,dx}_{=a_{n-1}} =$$

$$= \frac{16}{\pi}\left(\int_0^{\frac{\pi}{4}}\cos x(4n-1)\,dx - \int_0^{\frac{\pi}{4}}\cos x(4n-3)\,dx\right) + a_{n-1} =$$

$$= \frac{16}{\pi}\left(\frac{\sin x(4n-1)}{4n-1}\Big|_0^{\frac{\pi}{4}} - \frac{\sin x(4n-3)}{(4n-3)}\Big|_0^{\frac{\pi}{4}}\right) + a_{n-1} =$$

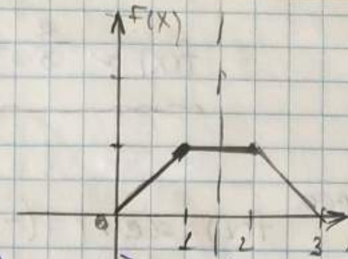
$$= \frac{16}{\pi}\left(\frac{\sin\frac{\pi}{4}(4n-1)}{(4n-1)} - \frac{\sin\frac{\pi}{4}(4n-3)}{(4n-3)}\right) + a_{n-1} =$$

$$= \frac{16}{\pi}\cdot\left(\frac{(-1)^n\frac{\sqrt{2}}{2}}{4n-3} - \frac{(-1)^n\frac{\sqrt{2}}{2}}{4n-1}\right) + a_{n-1} = \frac{8\sqrt{2}\cdot(-1)^n}{\pi}\left(\frac{1}{4n-3} - \frac{1}{4n-1}\right) + a_{n-1} =$$

$$= \frac{16\sqrt{2}(-1)^n}{\pi(4n-3)(4n-1)} + a_{n-1} \to \dots + a_{n-k} = \sum_{k=1}^{n}\frac{16\sqrt{2}(-1)^k}{\pi(4k-3)(4k-1)} + a_0$$

$$f(x) \sim \frac{4}{\pi}\ln(1+\sqrt{2}) + \sum_{n=1}^{\infty}\left(\sum_{k=1}^{n}\frac{16\sqrt{2}(-1)^k}{\pi(4k-3)(4k-1)} + \frac{8}{\pi}\ln(1+\sqrt{2})\right)\cdot\cos 4nx$$

2964. $f(x) = \begin{cases} x, & \text{если } 0 \le x \le 1; \\ 1, & \text{если } 1 < x < 2; \\ 3 - x, & \text{если } 2 \le x \le 3. \end{cases}$



$$a_0 = \frac{2}{3}\int_0^1 x\,dx + \frac{2}{3}\int_1^2 dx + \frac{2}{3}\int_2^3 (3-x)\,dx =$$

$$= \frac{1}{3} + \frac{2}{3} + \frac{2}{3}\cdot\left(3x - \frac{x^2}{2}\Big|_2^3\right) = 1 + \frac{2}{3}\left(9 - \frac{9}{2} - 6 + 2\right) =$$

$$\ell = \frac{3}{2}$$

$$= 1 + \frac{2}{3}\left(\frac{1}{2}\right) = \frac{4}{3}$$

$$a_n = \frac{2}{3}\int_0^1 x\cos\frac{2\pi n x}{3}\,dx + \frac{2}{3}\int_1^2 \cos\frac{2\pi n x}{3}\,dx + \frac{2}{3}\int_2^3 (3-x)\cos\frac{2\pi n x}{3}\,dx =$$

Каждый интеграл отдельно:

$$\frac{2}{3}\int_0^1 x\cos\frac{2\pi n x}{3}\,dx = \left\{\begin{matrix} u = x \to du = dx \\ dv = \cos\frac{2\pi n x}{3}dx \to v = \sin\frac{2\pi n x}{3}\cdot\frac{3}{2\pi n}\end{matrix}\right\} =$$

$$= \frac{2}{3}\cdot\left(\frac{3}{2\pi n}x\cdot\sin\frac{2\pi n x}{3}\Big|_0^1 - \frac{3}{2\pi n}\int_0^1 \sin\frac{2\pi n x}{3}\,dx\right) =$$

$$= \frac{1}{\pi n}\cdot\sin\frac{2\pi n}{3} + \frac{1}{\pi n}\cdot\cos\frac{2\pi n x}{3}\cdot\frac{3}{2\pi n}\Big|_0^1 =$$

$$= \frac{1}{\pi n}\cdot\sin\frac{2\pi n}{3} + \frac{3}{2\pi^2 n^2}\left(\cos\frac{2\pi n}{3} - 1\right)$$

$$\frac{2}{3}\int_1^2 \cos\frac{2\pi n x}{3}\,dx = \frac{2}{3}\cdot\frac{3}{2\pi n}\cdot\sin\left(\frac{2\pi n x}{3}\right)\Big|_1^2 = \frac{1}{\pi n}\cdot\left(\sin\frac{4\pi n}{3} - \sin\frac{2\pi n}{3}\right) =$$

$$= \frac{2}{\pi n}\sin\frac{2\pi n}{3}\cos(2\pi n) = \frac{2\cdot(-1)^n}{\pi n}\sin\frac{2\pi n}{3}$$

$$\frac{2}{3}\left(\int_2^3 3\cos\frac{2\pi n x}{3}\,dx - \int_2^3 x\cdot\cos\frac{2\pi n x}{3}\,dx\right) =$$

$$= \frac{2}{3}\left(-\frac{9}{2\pi n}\sin\frac{4\pi n}{3} + \frac{6}{2\pi n}\sin\frac{4\pi n}{3} - \frac{9\cdot(-1)^n}{4\pi^2 n^2} + \frac{9}{4\pi^2 n^2}\cos\frac{4\pi n}{3}\right) =$$

$$= \frac{2}{3}\left(\frac{9}{4\pi^2 n^2}\cos\frac{4\pi n}{3} + \frac{9(-1)^{n+1}}{4\pi^2 n^2} - \frac{3}{2\pi n}\sin\frac{4\pi n}{3}\right) =$$

$$= \frac{3}{2\pi^2 n^2}\cos\frac{4\pi n}{3} + \frac{3(-1)^{n+1}}{2\pi^2 n^2} - \frac{1}{\pi n}\sin\frac{4\pi n}{3}$$

$$= \frac{1}{\pi n}\sin\frac{2\pi n}{3} + \boxed{\frac{3}{2\pi^2 n^2}\cos\frac{2\pi n}{3}} - \frac{3}{2\pi^2 n^2} + \frac{2(-1)^n}{\pi n}\sin\frac{2\pi n}{3} + \boxed{\frac{3}{2\pi^2 n^2}\cdot\cos\frac{4\pi n}{3}} +$$
$$+ \frac{3}{2}\frac{(-1)^{n+1}}{\pi^2 n^2} - \frac{1}{\pi n}\sin\frac{4\pi n}{3} =$$

$$= \frac{1}{\pi n}\left(\sin\frac{2\pi n}{3} - \sin\frac{4\pi n}{3}\right) + \frac{3}{2\pi^2 n^2}\left(\cos\frac{2\pi n}{3} + \cos\frac{4\pi n}{3}\right) + \frac{3}{2\pi^2 n^2}\left((-1)^{n+1} - 1\right) +$$
$$+ \frac{2(-1)^n}{\pi n}\sin\frac{2\pi n}{3} = 2\cdot(-1)^n\cos\frac{2\pi n}{3} + \frac{3}{2\pi^2 n^2}\left((-1)^{n+1} - 1\right)$$

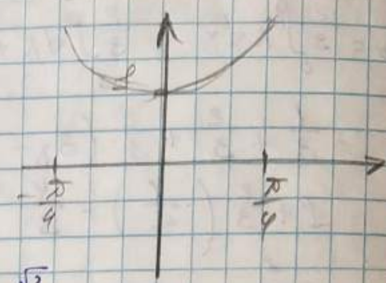
$$= \cancel{\frac{2(-1)^{n+2}}{\pi n}\sin\frac{2\pi n}{3}}$$

$b_n = 0$, т.к. отн. $x = \frac{3}{2}$ ф-я получается чётной

$$f(x) \sim \frac{2}{3} + \sum_{n=1}^{\infty}\left(2\cdot(-1)^n \cos\frac{2\pi n}{3} + \frac{3}{2\pi^2 n^2}\left((-1)^{n+1}-1\right)\right)\cos\frac{2\pi n x}{3}$$

9.09.21.

№2960. $f(x) = \sec(x) \quad \left(-\frac{\pi}{4} < x < \frac{\pi}{4}\right)$



$\sec x = \frac{1}{\cos x}$

$$a_0 = \frac{4}{\pi}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{dx}{\cos x} = \frac{8}{\pi}\int_0^{\frac{\pi}{4}} \frac{\cos x\, dx}{\cos^2 x} =$$

$$= \frac{8}{\pi}\int_0^{\frac{\pi}{4}} \frac{d(\sin x)}{1-\sin^2 x} = \{\sin x = t\} = \frac{8}{\pi}\int_0^{\frac{\sqrt{2}}{2}} \frac{dt}{1-t^2} =$$

$$= \frac{8}{\pi}\int_0^{\frac{\sqrt{2}}{2}} \frac{dt}{(1-t)(1+t)} = \frac{8}{\pi 2}\int_0^{\frac{\sqrt{2}}{2}} \left(\frac{1}{1-t} + \frac{1}{1+t}\right) dt =$$

$$= \frac{4}{\pi}\left(-\ln|1-t|\Big|_0^{\frac{\sqrt{2}}{2}} + \ln|1+t|\Big|_0^{\frac{\sqrt{2}}{2}}\right) =$$

$$= \frac{4}{\pi}\left(\ln\left|\frac{1+t}{1-t}\right|\Big|_0^{\frac{\sqrt{2}}{2}}\right) = \frac{4}{\pi}\left(\ln\left|\frac{1+\frac{\sqrt{2}}{2}}{1-\frac{\sqrt{2}}{2}}\right|\right) =$$

$$= \frac{4}{\pi}\ln\frac{2+\sqrt{2}}{2-\sqrt{2}} = \frac{4}{\pi}\ln\frac{(2+\sqrt{2})^2}{4-2} = \frac{4}{\pi}\ln\frac{4+4\sqrt{2}+2}{2} =$$

$$\cancel{= \frac{4}{\pi}\ln(}$$

$$\Rightarrow a_0 = \frac{4}{\pi}\ln\left(\frac{2+\sqrt{2}}{2-\sqrt{2}}\right)$$

$$a_n = \frac{8}{\pi}\int_0^{\frac{\pi}{4}} \frac{\cos 4nx}{\cos x}\, dx \; (=)$$

$$\cos 4nx - \cos(4nx-4x) = -2\sin 2x \cdot \sin(4nx-2x) =$$

$$= -4\sin x\cos x \cdot \sin(4nx-2x) = -2\left(\cos(4nx-3x) - \cos(4nx-x)\right)\cos x =$$

$$= 2\left(\cos x(4n-1) - \cos x(4n-3)\right)\cos x$$

$$\Rightarrow \cos 4nx = 2\left(\cos x(4n-1) - \cos x(4n-3)\right)\cos x + \cos 4x(n-1)$$

$$(=) -\frac{8}{\pi}\int_0^{\frac{\pi}{4}} (-2)\left(\cos x(4n-3) - \cos x(4n-1)\right) dx + \frac{8}{\pi}\int_0^{\frac{\pi}{4}} \frac{\cos 4x(n-1)}{\cos x}\, dx =$$

$$= \frac{16}{\pi}\left(\frac{\sin\frac{\pi}{4}(4n-3)}{4n-3} - \frac{\sin\frac{\pi}{4}(4n-1)}{4n-1}\right) + a_{n-1} =$$

$$= \frac{16}{\pi}\left(\frac{\sin\left(\pi n - \frac{3}{4}\pi\right)}{4n-3} - \frac{\sin\left(\pi n - \frac{\pi}{4}\right)}{4n-1}\right) + a_{n-1} =$$

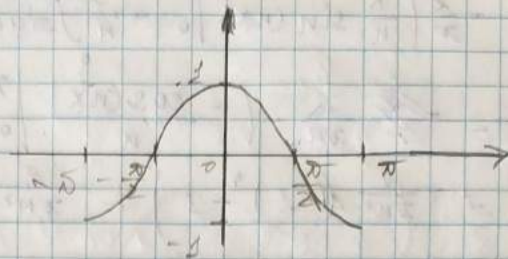
$$= \frac{\cancel{16}\sqrt{2}\; 8}{\pi \cancel{2}}(-1)^n \cdot \left(\frac{1}{4n-1} - \frac{1}{4n-3}\right) + a_{n-1} = \frac{16\sqrt{2}\cdot(-1)^{n+1}}{\pi(4n-1)(4n-3)} + a_{n-1} =$$

$$= \sum_{k=1}^{n} \frac{16\sqrt{2}(-1)^{k+1}}{\pi(4k-1)(4k-3)} + a_0 = \sum_{k=1}^{n} \frac{16\sqrt{2}(-1)^{k+1}}{\pi(4k-1)(4k-3)} + \frac{4}{\pi}\ln\left(\frac{2+\sqrt{2}}{2-\sqrt{2}}\right)$$

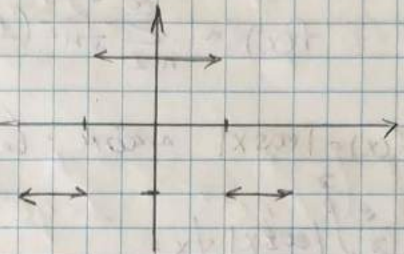
~~2952~~, 2954, 2958, 65, 71

д/з: 2953, 55, 57, 67, 69, 70.

<u>2952.</u> $f(x) = \operatorname{sgn}(\cos x)$ ~ чётн. $b_n = 0$



$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} \operatorname{sgn}(\cos x)\,dx = \frac{2}{\pi}\int_0^{\pi} \operatorname{sgn}(\cos x)\,dx = \frac{2}{\pi}\int_0^{\pi}\frac{|\cos x|}{\cos x}\,dx =$$



$$= \frac{2}{\pi}\left(\int_0^{\frac{\pi}{2}} dx - \int_{\frac{\pi}{2}}^{\pi} dx\right) = \frac{2}{\pi}\left(\frac{\pi}{2} - \left(\pi - \frac{\pi}{2}\right)\right) = 0$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \operatorname{sgn}(\cos x)\cdot\cos nx\,dx = \cancel{\frac{1}{\pi}\int_{-\pi}^{\pi}|\cos x|\,dx} =$$

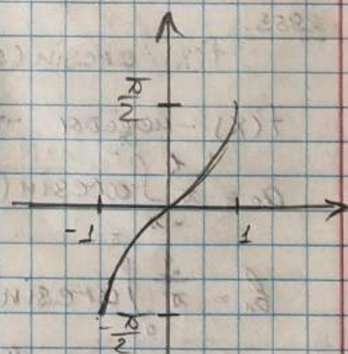
$$\cancel{= \frac{2}{\pi}\int_0^{\pi}|\cos x|\,dx = \frac{2}{\pi}\cdot\left(\int_0^{\frac{\pi}{2}}\cos x\,dx - \int_{\frac{\pi}{2}}^{\pi}\cos x\,dx\right)} =$$

$$\cancel{= \frac{2}{\pi}\left(\sin x\Big|_0^{\frac{\pi}{2}} - \sin\right)}$$

$$= \frac{2}{\pi}\left(\int_0^{\frac{\pi}{2}}\cos nx\,dx - \int_{\frac{\pi}{2}}^{\pi}\cos nx\,dx\right) = \frac{4}{\pi n}\cdot\sin\frac{\pi n}{2} = \begin{cases} 0, & \text{при } n = 2k \\ \frac{4}{\pi(2k-1)}\cdot(-1)^{k+1}, & \text{при } n = 2k-1 \end{cases}$$

$$f(x) \sim \sum_{k=1}^{\infty}\frac{4}{\pi(2k-1)}\cdot(-1)^{k+1}\cos(2k-1)x$$

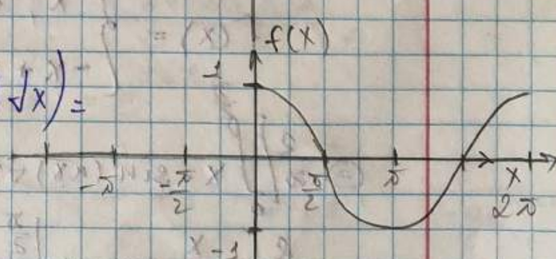
<u>2954.</u> $f(x) = \arcsin(\cos x) = \arcsin(\sqrt{1-\sin^2 x}) =$



$$= \arcsin\left(\cos\left(\left(x-\frac{\pi}{2}\right)+\frac{\pi}{2}\right)\right) =$$

$$= \arcsin\left(\sin\left(x+\frac{\pi}{2}\right)\right) =$$

$$f(x) = \begin{cases} \frac{\pi}{2} - x, & x \in (0;\pi) \\ \frac{\pi}{2} + x, & x \in (-\pi;0) \end{cases}$$

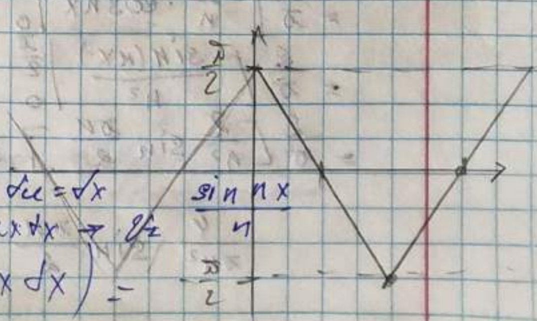


$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\left(\int_{-\pi}^{0}\left(x+\frac{\pi}{2}\right)dx + \int_0^{\pi}\left(\frac{\pi}{2}-x\right)dx\right) =$$

$$= \frac{1}{\pi}\left(\left[\frac{x^2}{2}+\frac{\pi x}{2}\right]\Big|_{-\pi}^{0} + \left[\frac{\pi x}{2}-\frac{x^2}{2}\right]\Big|_0^{\pi}\right) =$$

$$= \frac{1}{\pi}\left(-\frac{\pi^2}{2}+\frac{\pi^2}{2}+\frac{\pi^2}{2}-\frac{\pi^2}{2}\right) = 0$$

$$a_n = \frac{2}{\pi}\int_0^{\pi}\left(\frac{\pi}{2}-x\right)\cos nx\,dx =$$



$$= \int_0^{\pi}\cos nx\,dx - \frac{2}{\pi}\int_0^{\pi}x\cos nx\,dx = \left\{\begin{array}{l} u = x \to du = dx \\ dv = \cos nx\,dx \to v = \frac{\sin nx}{n}\end{array}\right\} =$$

$$= \frac{\sin nx}{n}\Big|_0^{\pi} - \frac{2}{\pi}\left(\frac{x}{n}\sin(nx)\Big|_0^{\pi} - \frac{1}{n}\int_0^{\pi}\sin nx\,dx\right) =$$

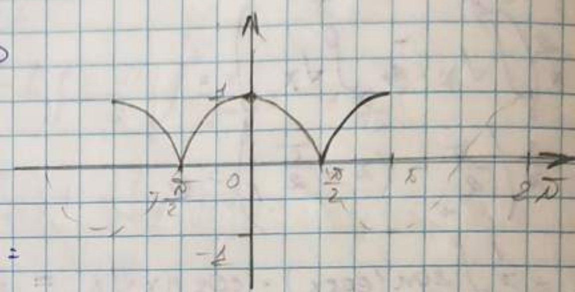
$$= -\frac{2}{\pi}\left(\frac{x}{n}\sin(nx)\Big|_0^{\pi} - \frac{1}{n}\int_0^{\pi}\sin(nx)\,dx\right) =$$

$$= -\frac{2}{\pi n}\cdot\left(-\frac{\cos(nx)}{n}\Big|_0^{\pi}\right) = -\frac{2}{\pi n^2}\cdot\left(\cos(\pi n) - 1\right) =$$

$$= -\frac{2}{\pi n^2}\left((-1)^n - 1\right) = \frac{2}{\pi n^2}\left(1 + (-1)^{n+1}\right)$$

$$f(x) \sim \sum_{n=1}^{\infty}\frac{2}{\pi n^2}\left(1+(-1)^{n+1}\right)\cdot\cos nx$$

<u>2958.</u> $f(x) = |\cos x|$ — чётн; $b_n = 0$



$$a_0 = \frac{2}{\pi}\int_0^{\pi}|\cos x|\,dx =$$

$$= \frac{2}{\pi}\left(\int_0^{\frac{\pi}{2}}\cos x\,dx - \int_{\frac{\pi}{2}}^{\pi}\cos x\,dx\right) =$$

$$= \frac{2}{\pi}\left(\sin x\Big|_0^{\frac{\pi}{2}} - \sin x\Big|_{\frac{\pi}{2}}^{\pi}\right) = \frac{2}{\pi}\left(1 - (0-1)\right) = \frac{4}{\pi}$$

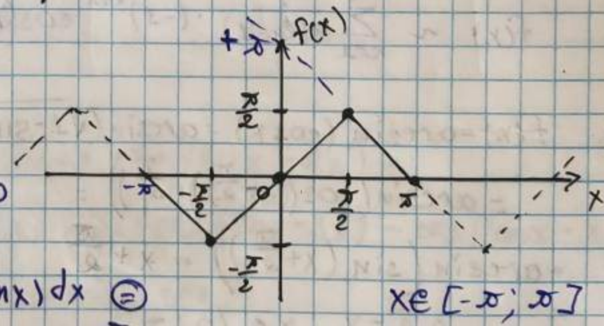
$$a_n = \frac{2}{\pi}\int_0^{\pi}|\cos x|\cdot\cos nx\,dx$$

Домашняя работа.

<u>2953.</u> $f(x) = \arcsin(\sin x)$;

$f(x)$ — нечётн $\to a_n = 0$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi}\arcsin(\sin x)\,dx = 0$$



$x \in [-\pi; \pi]$

$$b_n = \frac{2}{\pi}\int_0^{\pi}\arcsin(\sin x)\cdot\sin(nx)\,dx =$$

$$f(x) = \begin{cases} x, & \text{при } x \in [0; \frac{\pi}{2}] \\ -x+\pi, & \text{при } x \in [\frac{\pi}{2}; \pi] \end{cases}$$

$u = x \to du = dx$

$v = -\frac{1}{n}\cos x$

$$= \frac{2}{\pi}\left[\int_0^{\frac{\pi}{2}} x\cdot\sin(nx)\,dx + \int_{\frac{\pi}{2}}^{\pi}(\pi - x)\sin nx\,dx\right] =$$

$$= \frac{2}{\pi}\left[-\frac{x}{n}\cdot\cos nx\Big|_0^{\frac{\pi}{2}} + \frac{1}{n}\int_0^{\frac{\pi}{2}}\cos nx\,dx - \frac{\pi}{n}\cos nx\Big|_{\frac{\pi}{2}}^{\pi} + \left(\frac{x}{n}\cos nx - \frac{1}{n^2}\sin nx\right)\Big|_{\frac{\pi}{2}}^{\pi}\right.$$

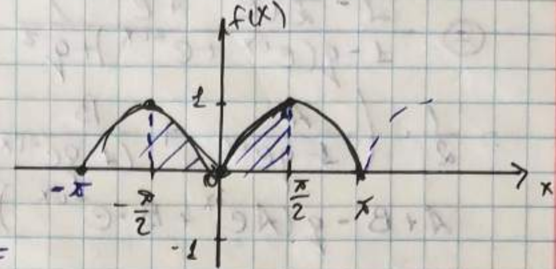
$$= \frac{2}{\pi}\left[\frac{\sin(nx)}{n^2}\Big|_0^{\frac{\pi}{2}} - \cancel{\frac{\pi}{n}\left(\cos\pi n\right.} - \underbrace{\cos\frac{\pi n}{2}}_{=0}\Big) + \cancel{\frac{\pi}{n}\cos\pi n} - 0 - 0 + \frac{1}{n^2}\sin\frac{\pi n}{2}\right.$$

$$= \frac{2}{\pi}\left[\frac{2}{n^2}\sin\frac{\pi n}{2}\right] = \frac{4}{\pi n^2}\sin\frac{\pi n}{2}$$

$$\frac{4}{\pi n^2}\cdot\sin\frac{\pi n}{2} = \begin{cases} 0, & n = 2k. \\ (-1)^k\cdot\frac{4}{\pi(2k+1)^2}, & \text{при } n = 2k+1 \end{cases}$$

$$f(x) \sim \frac{4}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^2}\sin(2k+1)x$$

2957. $f(x) = |\sin x|$



$f(x)$ - чётная $\to b_n = 0$

$x \in [-\frac{\pi}{2}; \frac{\pi}{2}]$

$$a_0 = \frac{2}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}|\sin x|\,dx = \frac{4}{\pi}\int_0^{\frac{\pi}{2}}\sin x\,dx =$$

$$= \frac{4}{\pi}\cdot\left(-\cos x\Big|_0^{\frac{\pi}{2}}\right) = \frac{4}{\pi}$$

$$a_n = \frac{4}{\pi}\int_0^{\frac{\pi}{2}}\sin x\cdot\cos 2nx\,dx = \frac{4}{\pi}\int_0^{\frac{\pi}{2}}(\sin(2n+1)x - \sin(2n-1)x)\,dx =$$

$$= \frac{4}{\pi}\cdot\left[\int_0^{\frac{\pi}{2}}\sin(2n+1)x\,dx - \int_0^{\frac{\pi}{2}}\sin(2n-1)x\,dx\right] =$$

$$= \frac{4}{\pi}\cdot\left[-\frac{\cos(2n+1)x}{(2n+1)}\Big|_0^{\frac{\pi}{2}} + \frac{\cos(2n-1)x}{(2n-1)}\Big|_0^{\frac{\pi}{2}}\right] =$$

$$= \frac{4}{\pi}\left[\frac{1}{2n+1} - \frac{1}{2n-1}\right] = \frac{4}{\pi}\left[\frac{\cancel{2n}-1-\cancel{2n}-1}{4n^2-1}\right] = -\frac{4}{\pi(4n^2-1)}$$

$$f(x) \sim \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\cos 2nx}{(4n^2-1)}$$

2967. $\frac{1-q^2}{1-2q\cos x+q^2}$ $(|q|<1)$

$$\boxed{\cos x = \frac{1}{2}(e^{ix}+e^{-ix})}$$

$1 - qe^{ix} - q^{-ix} + q^2 = 0$

$$\frac{1-q^2}{1-2q\cos x+q^2} = \frac{1-q^2}{1-q(e^{ix}+e^{-ix})+q^2} = \frac{1-q^2}{\dots} \ominus$$

$$q^2 - q(e^{ix}-e^{-ix}) + 1 = 0$$

$$D = (e^{ix}+e^{-ix})^2 - 4$$

$$q_{1,2} = \frac{(e^{ix}-e^{-ix}) \pm\sqrt{(e^{ix}+e^{-ix})^2-4}}{2} = \frac{2\cos x \pm 2\sqrt{\cos^2 x - 1}}{2} =$$

$$= \cos x \pm \sqrt{i^2\sin^2 x} = \cos x \pm i\sin x = e^{\pm ix}$$

$$\ominus \frac{(1-q^2)}{(q-e^{ix})(q-e^{-ix})} = \frac{(1-q^2)}{(1-qe^{ix})(1-qe^{-ix})} \stackrel{?}{=} -1 + \frac{1}{1-qe^{ix}} + \frac{1}{1-qe^{-ix}} =$$

$$= -1 + (1+qe^{ix}+q^2e^{2ix}+\dots) + (1+qe^{-ix}+q^2e^{-2ix}+\dots) =$$ деление столбиком

$$= 1 + 2\sum_{n=1}^{\infty}q^n\cos nx.$$

См. нерудовек с переходами

2968. $\dfrac{1-q\cos x}{1-2q\cos x+q^2} \circeq (|q|<1)$ (см. 2967.)

$$\circeq \frac{1-\frac{q}{2}(e^{ix}+e^{-ix})}{1-q(e^{ix}+e^{-ix})+q^2} = \frac{1}{2}\cdot\frac{(2-q(e^{ix}+e^{-ix}))}{(1-qe^{ix})(1-qe^{-ix})} =$$

$$= \frac{1}{2}\cdot\left(\frac{A}{1-qe^{ix}} + \frac{B}{(1-qe^{-ix})}\right) \circeq$$

$$A+B-q(Ae^{ix}+B\cdot e^{-ix}) = 2-q(e^{ix}+e^{-ix})$$

$$\Rightarrow A=B=1$$

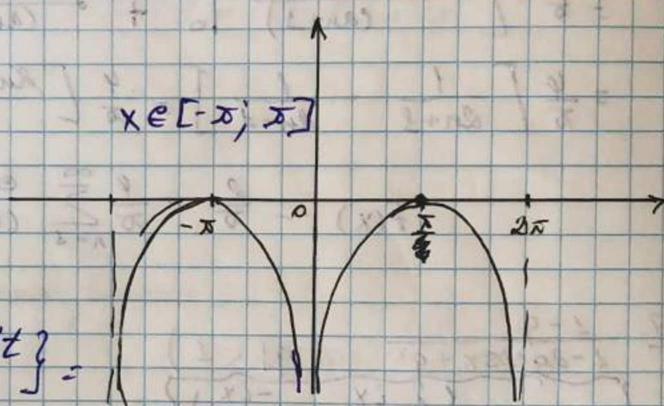
$$\circeq \frac{1}{2}\left(\frac{1}{1-qe^{ix}} + \frac{1}{1-qe^{-ix}}\right) = \frac{1}{2}\Big[(1+qe^{ix}+q^2e^{2ix}+\dots)+$$

$$+(1+qe^{-ix}+q^2e^{-2ix}+\dots)\Big] = 1+q\cos x+q^2\cos 2x+\dots =$$

$$= \sum_{n=0}^{\infty} q^n \cos nx$$

2970. $f(x) = \ln|\sin\frac{x}{2}|$

$f(x) \underset{x}{=}$ чётн $\to b_n = 0$ $\qquad x\in[-\pi;\pi]$



$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} \ln|\sin\frac{x}{2}|\,dx =$$

$$= \frac{2}{\pi}\int_0^{\pi} \ln(\sin\frac{x}{2})\,dx =$$

$$= \left\{\begin{array}{ll}\frac{x}{2}=t & dx = 2dt \\ t=\frac{\pi}{2} & \end{array}\right\} =$$

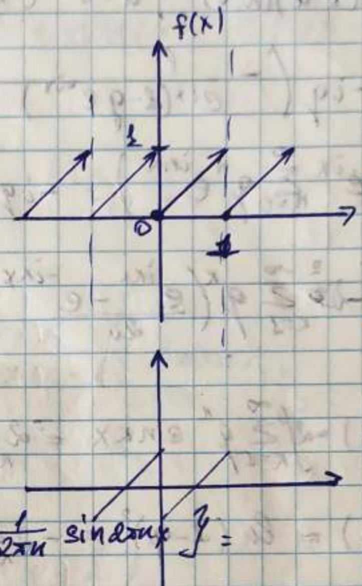
$$= \frac{4}{\pi}\int_0^{\frac{\pi}{2}} \ln\sin t\,dt \qquad ?$$

$\cos x = \frac{1}{2}(e^{ix}+e^{-ix})$

2965. $\cos^{2m} x = \frac{1}{2^{2m}}(e^{ix}+e^{-ix})^{2m} = \frac{1}{2^{2m}}\sum_{n=0}^{2m} C_{2m}^{n} e^{(2m-n)ix} e^{-nix} =$

$= \frac{1}{2^{2m}}\sum_{n=0}^{2m} C_{2m}^{n} e^{2(m-n)ix}$

2955. $f(x) = x - [x]$



$t = x - 1$.

$y(x+1) = x+1-[x+1] = x-[x] = f(x)$

$a_0 = \frac{1}{\frac{1}{2}}\int_0^1 (x-[x])\,dx = 2\int_0^1 (x-[x])\,dx =$

$= 2\int_0^1 x\,dx = 1.$

$a_n = 2\int_0^1 (x-[x])\cos 2n\pi x\,dx =$

$= 2\int_0^1 x\cos 2\pi n x\,dx = \left\{ \begin{matrix} u = x \Rightarrow du = dx \\ dv = \cos 2\pi n x\,dx \Rightarrow v = \frac{1}{2\pi n}\sin 2\pi n x \end{matrix} \right\} =$

$= 2\cdot\left(\underbrace{\frac{x}{2\pi n}\sin 2\pi n x\Big|_0^1}_{=0} - \frac{1}{2\pi n}\int_0^1 \sin 2\pi n x\,dx\right) = +\frac{1}{2\pi^2 n^2}\frac{\cos 2\pi n x}{1}\Big|_0^1 = 0$

$b_n = 2\int_0^1 (x-[x])\sin 2\pi n x\,dx = 2\int_0^1 x\sin 2\pi n x\,dx = \left\{ \begin{matrix} u = x & du = dx \\ & v = -\frac{\cos 2\pi n x}{2\pi n} \end{matrix} \right.$

$= 2\left(-\frac{x}{2\pi n}\cos 2\pi n x\Big|_0^1 + \frac{1}{2\pi n}\int_0^1 \cos 2\pi n x\,dx\right) = -\frac{\cos 2\pi n}{\pi n} + \frac{1}{2\pi^2 n^2}\sin 2\pi n x\Big|_0^1 =$

$= -\frac{\cos 2\pi n}{\pi n} = \frac{-1}{\pi n}$

$f(x) \sim \frac{1}{2} - \frac{2}{\pi n}\sum_{n=0}^{\infty} \sin 2\pi n x$

2969.
$f(x) = \ln(1 - 2q\cos x + q^2) \quad (|q|<1)$

$f' = \frac{2q\sin x}{1-2q\cos x+q^2} = \frac{2q\cdot\frac{e^{ix}-e^{-ix}}{2i}}{1-2q\frac{e^{ix}+e^{-ix}}{2}+q^2} = \frac{-iq(e^{ix}-e^{-ix})}{1-q(e^{ix}+e^{-ix})+q^2} =$

$e^{i\alpha} = \cos\alpha + i\sin\alpha$

$\cos\alpha = \frac{e^{i\alpha}+e^{-i\alpha}}{2}$; $\sin\alpha = \frac{e^{i\alpha}-e^{-i\alpha}}{2i}$

$\sum_{k=0}^{\infty} d^k = \frac{1}{1-d} \qquad \frac{1}{1-d} \qquad \underline{|d|<1}$

$q^2 - q(e^{ix}+e^{-ix}) + 1 = 0$

$$D=(e^{ix}+e^{-ix})^2-4=(e^{ix})^2+2+(e^{-ix})^2-4=(e^{ix}-e^{-ix})^2$$

$$q_1=\frac{(e^{ix}+e^{-ix})+e^{ix}-e^{-ix}}{2}=e^{ix}\quad ;\ q_2=e^{-ix}$$

$$= \frac{-iq(e^{ix}-e^{-ix})}{(q-e^{ix})(q-e^{-ix})}=-iq(e^{ix}-e^{-ix})\cdot\cancel{\frac{1}{(e^{ix}-e^{-ix})}}\left(\frac{1}{q-e^{ix}}-\frac{1}{q-e^{-ix}}\right) =$$

$$\frac{1}{(x-a)(x-b)}=\frac{1}{a-b}\left(\frac{1}{x-a}-\frac{1}{x-b}\right)$$

$$= -iq\left(-\frac{1}{e^{ix}(1-qe^{-ix})}+\frac{1}{e^{-ix}(1-qe^{ix})}\right)=-iq\left(-e^{-ix}\sum_{n=0}^{\infty}q^n e^{-inx}+\right.$$

$$\left.+e^{ix}\sum_{n=0}^{\infty}q^n e^{inx}\right)=-iq\sum_{n=0}^{\infty}q^n\left(e^{i(n+1)x}-e^{-i(n+1)x}\right)=\{k=n+1\}=$$

$$=-2i^2\sum_{k=1}^{\infty}q^k\left(\frac{e^{ikx}-e^{-ikx}}{2i}\right)=2\sum_{k=1}^{\infty}q^k\sin kx$$

$$f(x)=2\int\sum_{k=1}^{\infty}q^k\sin kx=-2\sum_{k=1}^{\infty}\frac{q^k}{k}\cos kx+C$$

$$f(0)=\ln(1-q)^2=-2\sum_{k=1}^{\infty}q^k:k+C=-2(-\ln|1-q|)+C\Rightarrow C=0$$

$$S(q)=\sum_{k=1}^{\infty}\frac{q^k}{k}=\int\frac{1}{1-q}dq=-\ln|1-q|$$

$$S'=\sum_{k=1}^{\infty}q^{k-1}=\{n=k-1\}=\sum_{0}^{\infty}q^n=\frac{1}{1-q}$$

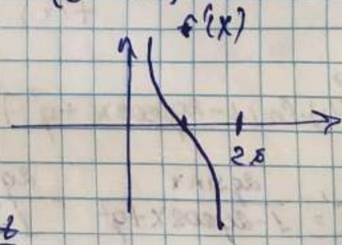
Если $q=0\to S(q)=$

№2970.

$$f(x)=\ln\left|\sin\frac{x}{2}\right|$$

$$f'(x)=\frac{1}{2}\frac{\cos\frac{x}{2}}{\sin\frac{x}{2}}=\frac{i}{2}\cdot\frac{e^{i\frac{x}{2}}+e^{-i\frac{x}{2}}}{e^{i\frac{x}{2}}-e^{-i\frac{x}{2}}}=\frac{i}{2}\frac{e^{-i\frac{x}{2}}(e^{ix}+1)}{e^{-i\frac{x}{2}}(e^{ix}-1)}=$$

$$=\frac{i}{2}\cdot\frac{e^{ix}+1}{e^{ix}-1}=\frac{i}{2}\left(1+\frac{2}{e^{ix}-1}\right)$$



$$\cancel{b_n=\frac{1}{\pi}\cdot\frac{1}{2}\int_0^{\pi}\cos\frac{x}{2}}$$

$$f'(x)=\frac{1}{2}\frac{\cos\frac{x}{2}}{\sin\frac{x}{2}}=\frac{1}{2}\frac{\cos\frac{t+\pi}{2}}{\sin\frac{t+\pi}{2}}=-\frac{1}{2}\frac{\sin\frac{t}{2}}{\cos\frac{t}{2}}$$

$\pm\cos$

$$b_n=\frac{2}{\pi}\int_0^{\pi}\left(-\frac{1}{2}\right)\frac{\sin\frac{t}{2}}{\cos\frac{t}{2}}\sin nt\,dt=\frac{1}{2\pi}\int_0^{\pi}\frac{\cos(n+\frac{1}{2})t-\cos(n-\frac{1}{2})t}{\cos\frac{t}{2}}dt=$$

$$\cos\alpha-\cos\beta=-2\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}$$

$\parallel$
$\sin t$

$\frac{\alpha-\beta}{2} = t$

$\alpha = (n+\frac{1}{2})t \quad \to (n+\frac{1}{2})t - \beta = 2t \quad \to \beta = n - \frac{3}{2}t$

$$= \frac{1}{2\pi}\int_0^\pi \frac{\cos(n+\frac{1}{2})t - \cos(n-\frac{3}{2})t}{\cos\frac{t}{2}\; 2\sin\frac{t}{2}} + \frac{\cos(n-\frac{3}{2})t - \cos(n-\frac{1}{2})t}{2\sin\frac{t}{2}}\,dt =$$

$$= \frac{1}{2\pi}\int_0^\pi \frac{-2\sin(n-\frac{1}{2})t\,\sin t}{\cos\frac{t}{2}} + \frac{2\sin(n-1)t\,\sin t}{\cos\frac{t}{2}}\,dt =$$

$$= \frac{2}{2\pi}\int_0^\pi (\underbrace{\cos nt}_{\to 0} - \cos(n-1)t - \cos(n-\frac{1}{2})t + \cos(n-1)t)\,dt =$$

$$= \frac{-1}{\pi}\,\frac{\sin(n-\frac{1}{2})t}{(n-\frac{1}{2})}\Big|_0^\pi = \frac{(-1)^n}{\pi(n-\frac{1}{2})}$$

$$f'(t) = \sum_{n=1}^{\infty} \frac{(-1)^n}{\pi(n-\frac{1}{2})}\sin(nt)$$

$$f'(x) = -\sum_{n=1} \frac{(-1)^n(-1)^n}{\pi(n-\frac{1}{2})}\sin(nx) = -\sum_{n=1}^{\infty}\frac{1}{\pi(n-\frac{1}{2})}\sin(nx)$$

$$f(x) = \sum_{n=1}^{\infty}\frac{1}{\pi n(n-\frac{1}{2})}\cos nx + C$$

$$C = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{\pi n(n-\frac{1}{2})} \qquad (\text{при } x = \frac{\pi}{2})$$

$$f(x) = \sum_{n=1}^{\infty}\frac{1}{\pi n(n-\frac{1}{2})}\cos nx + \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{\pi n(n-\frac{1}{2})}$$

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n(n-\frac{1}{2})} = 2\left(\frac{1}{1/2} - 1 - \frac{1}{3/2} + \frac{1}{2} + \frac{1}{5/2} - \frac{1}{3}\dots\right)$$

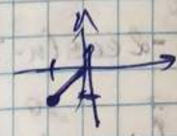
$$\frac{1}{n(n-\frac{1}{2})} = +2\left(\frac{1}{n-\frac{1}{2}} - \frac{1}{n}\right)$$

16.09.21. Операции с комплексными числами.

N: 1(2), 2(1, 2, 4), 4(2, 4, 7)

Д/З:

1.1. 2) $\frac{1-i}{1+i} = \frac{(1-i)^2}{1^2-i^2} = \frac{1-2i-1}{2} = -i$



1.2.

1) $3i = z \rightarrow |z| = 3, \arg z = \frac{\pi}{2}$

2) $z = -2 \rightarrow |z| = 2, \arg z = \pi$

4) $z = -1-i \rightarrow |z| = \sqrt{2}, \arg z = -\frac{3\pi}{4}$

1.4

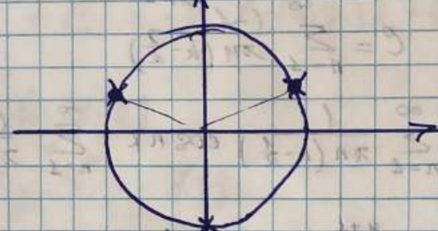
2) $\sqrt[3]{i} = e^{\frac{i\left(\frac{\pi}{2}+2\pi k\right)}{3}} = e^{i\left(\frac{\pi}{6}+\frac{2\pi k}{3}\right)}$

$|i| = 1, \arg(i) = \frac{\pi}{2}$

$k=0: z_1 = e^{i\frac{\pi}{6}} = \cos\frac{\pi}{6} + i\sin\frac{\pi}{6} = \frac{\sqrt{3}}{2} + i\frac{1}{2}$

$k=1: z_2 = e^{i\frac{5\pi}{6}} = \cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6} = -\frac{\sqrt{3}}{2} + i\cdot\frac{1}{2}$

$k=2: z_3 = e^{i\frac{3\pi}{2}} = -i$



4) $\sqrt[6]{-8} = \sqrt{2}\cdot e^{\frac{i(\pi+2\pi k)}{6}} =$

$|-8| = 8; \arg(-8) = \pi$

$k=0: z_1 = \sqrt{2}\cdot e^{i\frac{\pi}{6}} = \sqrt{2}\cdot\left(\frac{\sqrt{3}}{2} + i\frac{1}{2}\right) = \frac{\sqrt{6}}{2} + i\frac{\sqrt{2}}{2}$

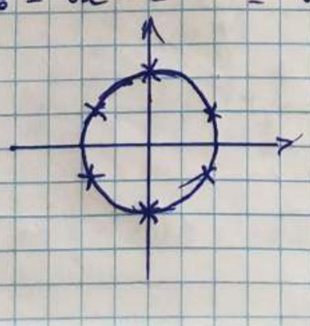
$k=1: z_2 = \sqrt{2}\cdot e^{i\frac{\pi}{2}} = \sqrt{2}(0+i) = \sqrt{2}\cdot i$

$k=2: z_3 = \sqrt{2}\cdot e^{i\frac{5\pi}{6}} = \sqrt{2}\left(-\frac{\sqrt{3}}{2} + i\frac{1}{2}\right) = -\frac{\sqrt{6}}{2} + i\frac{\sqrt{2}}{2}$

$k=3: z_4 = \sqrt{2}\cdot e^{i\frac{7\pi}{6}} = \sqrt{2}\left(-\frac{\sqrt{3}}{2} - i\frac{1}{2}\right) = -\frac{\sqrt{6}}{2} - i\frac{\sqrt{2}}{2}$

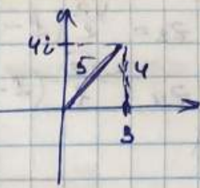
$k=4: z_5 = \sqrt{2}\cdot e^{i\frac{3}{2}\pi} = \sqrt{2}\cdot(-i) = -\sqrt{2}i$

$k=5: z_6 = \sqrt{2}\cdot e^{i\frac{11}{6}\pi} = \sqrt{2}\cdot\left(\frac{\sqrt{3}}{2} - i\frac{1}{2}\right) = \frac{\sqrt{6}}{2} - i\cdot\frac{\sqrt{2}}{2}$



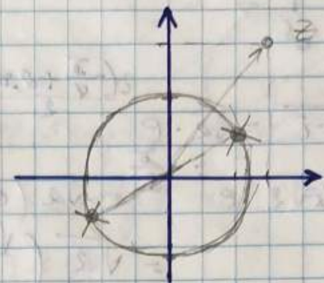
7) $z=\sqrt{3+4i} = \sqrt{5}\cdot e^{i\frac{\mathrm{arctg}\frac{4}{3}+2\pi k}{2}}$

$|z| = \sqrt{9+16} = 5$



$k=0: \quad z_1 = \sqrt{5}\cdot e^{i\frac{\mathrm{arctg}\frac{4}{3}}{2}} =$

$k=1: \quad z_2 = \sqrt{5}\cdot e^{i\left(\frac{\mathrm{arctg}\frac{4}{3}}{2}+\pi\right)}$



Домашняя работа

1.1.

1) $\frac{1}{i} = \frac{-i}{-i^2} = -i$

3) $\frac{2}{1-3i} = \frac{2(1+3i)}{1-9i^2} = \frac{2+6i}{10} = \frac{1}{5}+\frac{3}{5}i = \frac{1}{5}(1+3i)$

4) $(1+i\sqrt{3})^3 = (1+i\sqrt{3})(1+2i\sqrt{3}-3) = (1+i\sqrt{3})(2i\sqrt{3}-2) =$

$= 2i\sqrt{3}-2+2\cdot 3\cdot i^2-2i\sqrt{3} = -2-6 = -8$

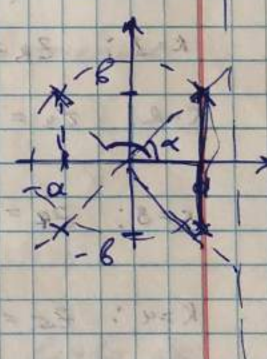
1.2.

3) $z=1+i \to |z|=\sqrt{2}$; $\arg z = \frac{\pi}{4}$

5) $z=2+5i \to |z|=\sqrt{29}$; $\arg z = \mathrm{arctg}\,\frac{5}{2}$

10) $z=a+bi\ (a\neq 0) \to |z| = \sqrt{a^2+b^2}$;

$$\arg z = \begin{cases} \mathrm{arctg}\,\frac{b}{a}, & b\geq 0,\ a\geq 0 \\ \mathrm{arctg}\,\frac{b}{a}+\pi, & \text{при } a<0,\ b\geq 0 \\ \mathrm{arctg}\,\frac{b}{a}-\pi, & a<0,\ b<0 \end{cases}$$



1.4

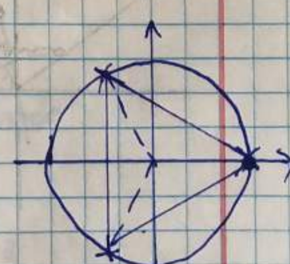
1) $\sqrt[3]{1} = 1\cdot e^{\frac{i2\pi k}{3}}$

$\arg(1) = 0$

$k=0: \quad z_1 = 1\cdot e^0 = 1.$

$k=1: \quad z_2 = 1\cdot e^{\frac{2\pi}{3}i} = \cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3} = -\frac{1}{2}+\frac{\sqrt{3}}{2}i$

$k=2: \quad z_3 = 1\cdot e^{\frac{4\pi}{3}i} = \cos\frac{4\pi}{3}+i\sin\frac{4\pi}{3} = -\frac{1}{2}-\frac{\sqrt{3}}{2}i$



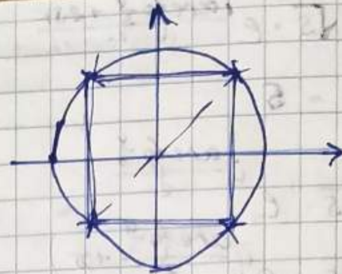
3) $\sqrt[4]{-1} = 1\cdot e^{\frac{i(\pi+2\pi k)}{4}}$

$k=0: \quad z_1 = e^{i\frac{\pi}{4}} = \cos\frac{\pi}{4}+i\sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{2}(1+i)$

$k=1: \quad z_2 = e^{i\frac{3\pi}{4}} = -\frac{\sqrt{2}}{2}(1+i)$

$z_3 = \frac{\sqrt{2}}{2}(1 - i)$

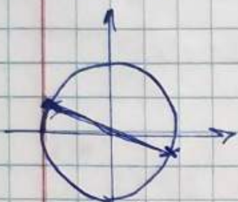
$z_4 = -\frac{\sqrt{2}}{2}(1 - i)$



в) $\sqrt{1 - i} = \sqrt{2} \cdot e^{i\frac{\left(-\frac{\pi}{4} + 2\pi k\right)}{2}}$

$k = 0: z_1 = \sqrt{2} \cdot e^{i \cdot -\frac{\pi}{8}} = \sqrt{2}\left(\cos\left(-\frac{\pi}{8}\right) + i\sin\left(-\frac{\pi}{8}\right)\right) =$

$= \sqrt{2}\left(\frac{\sqrt{2+\sqrt{2}}}{2} - i\frac{\sqrt{2-\sqrt{2}}}{2}\right) =$

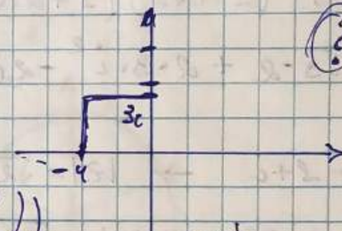


$= \frac{\sqrt{2}}{2}\sqrt{2+\sqrt{2}}\,(1 - i) = \sqrt{\frac{2+\sqrt{2}}{2}}\,(1 - i)$

$k = 1: z_2 = \sqrt{2} \cdot e^{i\left(-\frac{9\pi}{8}\right)} = \sqrt{2} \cdot \left(-\frac{\sqrt{2+\sqrt{2}}}{2} + i\frac{\sqrt{2-\sqrt{2}}}{2}\right) =$

$= -\sqrt{\frac{2+\sqrt{2}}{2}}\,(1 - i)$

г) $\sqrt[5]{-4 + 3i} = \sqrt{25} \cdot e^{i\frac{\left(-\operatorname{arctg}\left(\frac{3}{4}\right) + 2\pi k\right)}{5}}$?



$k = 0: z_1 = 5 \cdot e^{i\frac{-\operatorname{arctg}\left(\frac{3}{4}\right)}{5}} =$

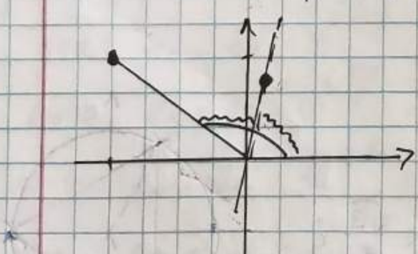
$= 5\left(\cos\left(\frac{\operatorname{arctg}\frac{3}{4}}{5}\right) - i\sin\left(\frac{\operatorname{arctg}\frac{3}{4}}{5}\right)\right)$

$k = 1: z_2 = 5\left(\cos\left(\frac{\operatorname{arctg}\left(\frac{3}{4}\right) - 2\pi}{5}\right) - i\sin\left(\frac{\operatorname{arctg}\left(\frac{3}{4}\right) - 2\pi}{5}\right)\right)$

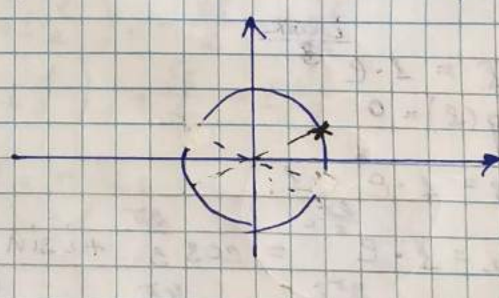
$k = 2: z_3 = 5\left(\cos\left(\frac{\operatorname{arctg}\left(\frac{3}{4}\right) - 4\pi}{5}\right) - i\sin\left(\frac{\operatorname{arctg}\left(\frac{3}{4}\right) - 4\pi}{5}\right)\right)$

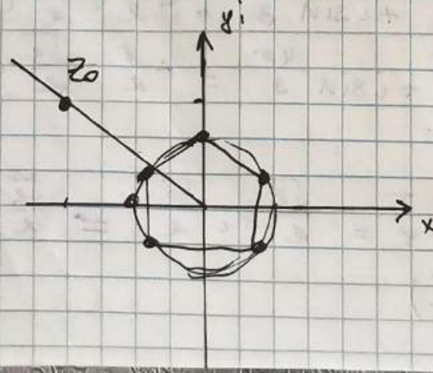
$k = 3: z_4 = 5\left(\cos\left(\frac{\operatorname{arctg}\left(\frac{3}{4}\right) - 6\pi}{5}\right) - i\sin\left(\frac{\operatorname{arctg}\left(\frac{3}{4}\right) - 6\pi}{5}\right)\right)$

$k = 4: z_5 = 5\left(\cos\left(\frac{\operatorname{arctg}\frac{3}{4} - 8\pi}{5}\right) - i\sin\left(\frac{\operatorname{arctg}\frac{3}{4} - 8\pi}{5}\right)\right)$



$\sqrt{-4 + 3i}$





1.8. 1)

1.23.

1.24. |

1.27. 1)

$z = x$

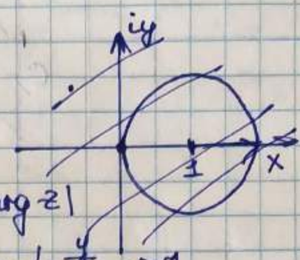
$\mathrm{Re}\, z =$

1)

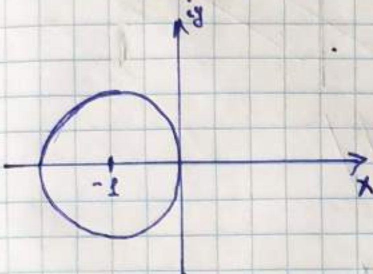
Геометрическая интерпретация неравенств и уравнений для комплексного переменного.

20.09.21.

1.8. 1) $\left|\frac{z}{|z|}-1\right| \le |\arg z|$



$z=|z|\cdot e^{i\varphi} \to |e^{i\varphi}-1|$



$|(-1+\cos\varphi)+i\sin\varphi| \le |\arg z|$

$|\underbrace{(-1+\cos\varphi)}_{x}+i\underbrace{\sin\varphi}_{y}| \le \operatorname{arctg}\frac{y}{x} = |\varphi|$

$|(-1+\cos\varphi)+i\sin\varphi| \le |\varphi|$

$\sqrt{(-1+\cos\varphi)^2+\sin^2\varphi} \le |\varphi|$

$\sqrt{(1-2\cos\varphi+\cos^2\varphi+\sin^2\varphi)} \le |\varphi|$

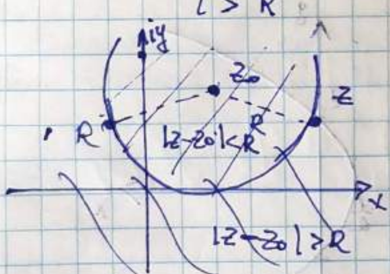
$\sqrt{2(1-\cos\varphi)} \le |\varphi|$

$\sqrt{4\cdot\sin^2\frac{\varphi}{2}} \le |\varphi|$

$2\left|\sin\frac{\varphi}{2}\right| \le |\varphi|$

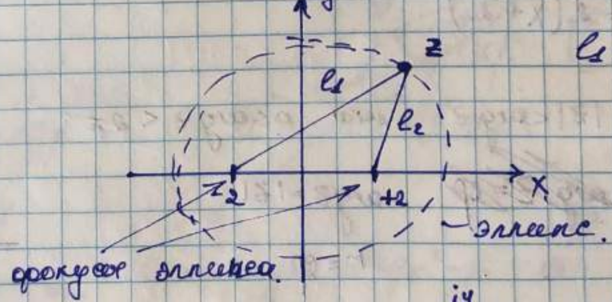
$\left|\sin\frac{\varphi}{2}\right| \le \left|\frac{\varphi}{2}\right|$ — выполняется всегда

1.23. $|z-z_0| \begin{cases} < R \\ = R \\ > R \end{cases}$



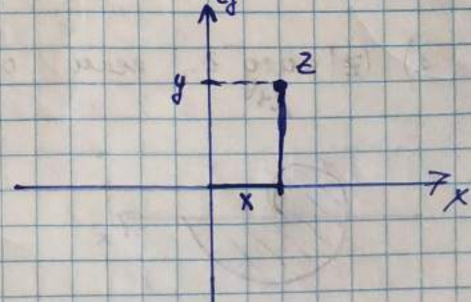
$|z-z_0|$ — расстояние между точками.

1.24. $|z-2|+|z+2|=5$



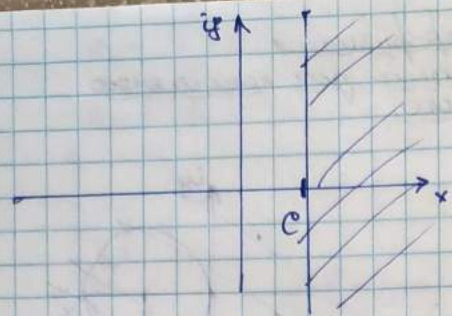
$l_1+l_2=5$

1.27. 1) $\operatorname{Re} z \ge C$; 2) $\operatorname{Im} z < C$



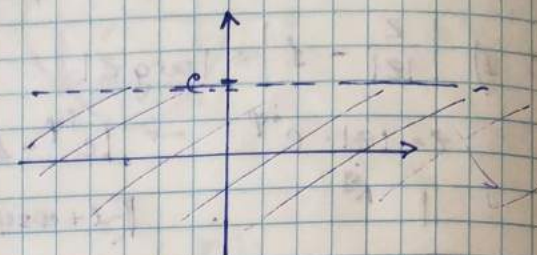
$z=x+iy$
$\operatorname{Re} z = x$

1) $x \ge C$



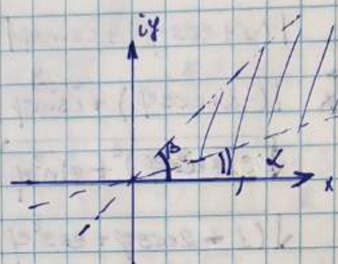
$\operatorname{Im} z < c$

$y < c.$



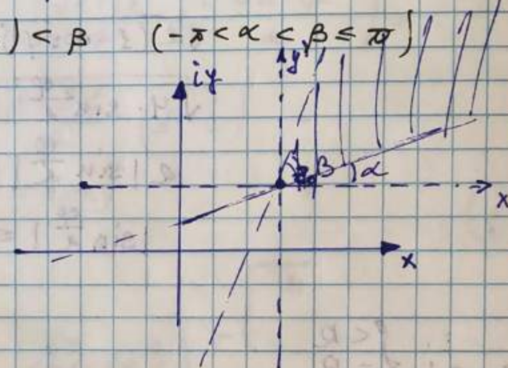
1.29. $\alpha < \arg z < \beta$

$\alpha < \varphi < \beta$



$\alpha < \arg(z - z_0) < \beta \quad (-\pi < \alpha < \beta \le \pi)$

$w = z - z_0$



1.30. $|z| = \operatorname{Re} z + 1$

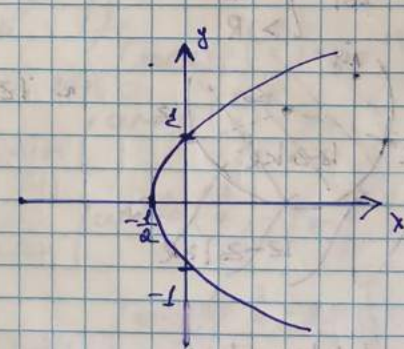
$|z| = \sqrt{x^2 + y^2}$

$\sqrt{x^2+y^2} = x + 1 \geq 0 \quad \Rightarrow x \geq -1.$

$x^2 + y^2 = x^2 + 2x + 1.$

$y^2 = 2x + 1$

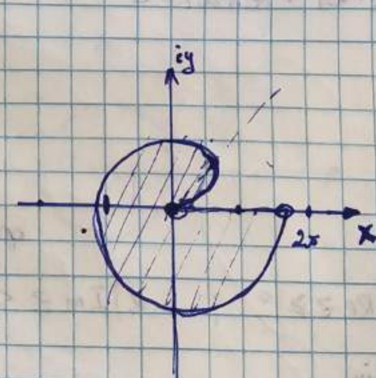
$y^2 = 2\left(x + \frac{1}{2}\right)$



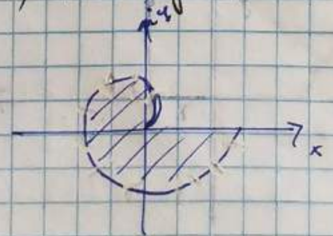
1.34. 1) $|z| < \arg z$, если $0 \le \arg z < 2\pi$

~~$|z|$ $\arg z = \varphi$~~ $\quad \arg z = |z|$

$r = \varphi$

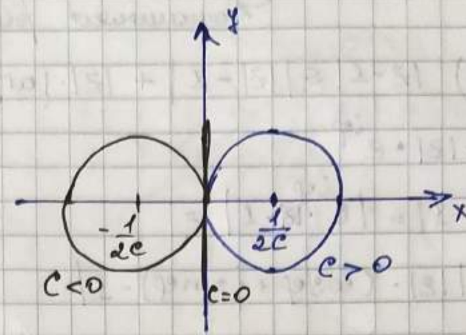


2) $|z| < \arg z$, если $0 < \arg z \le 2\pi$



– тоже самое, только выколото в нуле.

1.35 1) $\operatorname{Re}\frac{1}{z} = C$



$\frac{\bar z}{z\cdot\bar z} = \frac{\bar z}{|z|^2}$

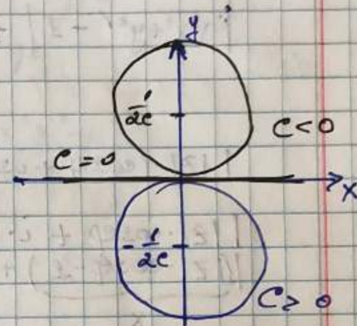
$\operatorname{Re}\frac{\bar z}{|z|^2} = C$

$\frac{x}{x^2+y^2} = C$

$Cx^2 + Cy^2 = x \quad \to \quad x^2 - 2\cdot\frac{1}{2C}x + \frac{1}{4C^2} + y^2 = \frac{1}{4C^2}$

$\cancel{Cx^2 -}$ $\qquad (x - \frac{1}{2C})^2 + y^2 = \frac{1}{4C^2}$

2) $\operatorname{Im}\frac{1}{z} = C \quad (-\infty < C < \infty)$

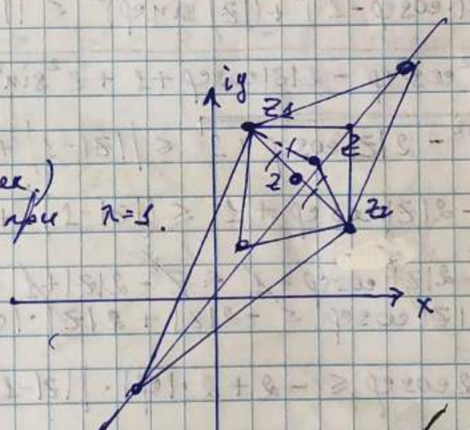


$\operatorname{Im}\frac{x-iy}{x^2+y^2} = C$

$\frac{-y}{x^2+y^2} = C \quad ; \quad x^2+y^2+2\cdot\frac{1}{2C}y + \frac{1}{4C^2} = \frac{1}{4C^2}$

$x^2 + (y+\frac{1}{2C})^2 = \frac{1}{4C^2}$

1.37. $\left|\frac{z-z_1}{z-z_2}\right| = \lambda \quad (\lambda > 0)$



$\lambda = 1$.

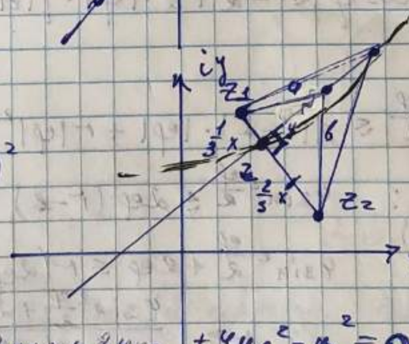
1) Прямая (лежит на сер. перпендик.)
Если $z_1 = z_2$ ~ вся плоскость, при $\lambda = 1$.

2) $0<\lambda<1$; $(\lambda = \frac{1}{2})$

$\frac{b}{a} = \sqrt{\frac{\frac{1}{9}x^2+y^2}{\frac{4}{9}x^2+y^2}}$

$a = \sqrt{\frac{1}{9}x^2+y^2}$
$b = \sqrt{\frac{4}{9}x^2+y^2}$

$\lambda = \frac{1}{2}$



? - окр.

$4(x-x_1)^2 + 4(y-y_1)^2 = (x-x_2)^2 + (y-y_2)^2$

$4x^2 - 8xx_1 + 4x_1^2 + 4y^2 - 8yy_1 + 4y_1^2 =$

$= x^2 - 2xx_2 + x_2^2 + y^2 - 2yy_2 + y_2^2$

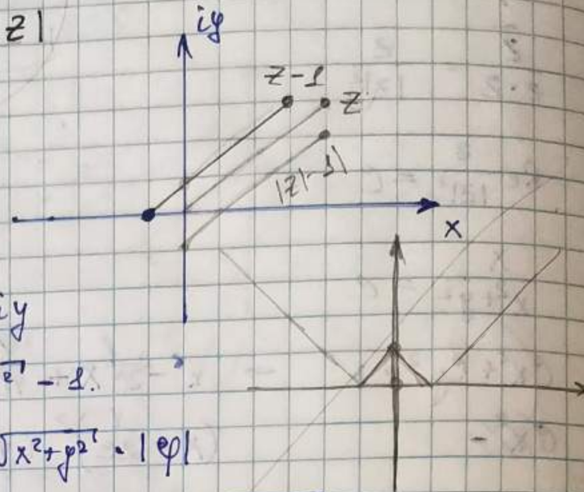
$3x^2 - 8xx_1 + 2xx_2 + 4x_1^2 - x_2^2 + 3y^2 - 8yy_1 + 2yy_2 + 4y_1^2 - y_2^2 = 0$ — изобр.

5/8 *

=> окружность.

Домашняя работа.

1.8. 2) $|z-1| \leq \big||z|-1\big| + |z|\cdot|\arg z|$



$z = |z|\cdot e^{i\varphi}$

$|z-1| = |e^{i\varphi}\cdot|z|-1| =$

$= \big||z|\cdot(\cos\varphi + i\sin\varphi) - 1\big|$

$z = x+iy \to z-1 = (x-1)+iy$

$|z| = \sqrt{x^2+y^2} \to |z|-1 = \sqrt{x^2+y^2}-1.$

$|z|\cdot|\arg z| = \sqrt{x^2+y^2}\cdot|\varphi|$

$\big|\sqrt{x^2+y^2}-1\big| + \sqrt{x^2+y^2}\cdot|\varphi| \Rightarrow \sqrt{x^2+y^2} - 1 + \sqrt{x^2+y^2}\cdot\varphi =$

$= \sqrt{x^2+y^2}(1+\varphi) - 1$

$\big||z|\cdot(\cos\varphi + i\sin\varphi) - 1\big| \leq \big||z|-1\big| + |z|\cdot|\varphi|$

$\big||z|\cdot\cos\varphi + i\cdot|z|\sin\varphi - 1\big| \leq \big||z|-1\big| + |z|\cdot|\varphi|$

$\big|\underbrace{(|z|\cdot\cos\varphi - 1)}_{x} + i\underbrace{|z|\sin\varphi}_{y}\big| \leq \big||z|-1\big| + |z|\cdot|\varphi|$

$\sqrt{(|z|\cdot\cos\varphi - 1)^2 + (|z|\sin\varphi)^2} \leq \big||z|-1\big| + |z|\cdot|\varphi|$

$\sqrt{z^2\cos^2\varphi - 2|z|\cos\varphi + 1 + z^2\sin^2\varphi} \leq \big||z|-1\big| + |z|\cdot|\varphi|$

$\sqrt{z^2 - 2|z|\cos\varphi + 1} \leq \big||z|-1\big| + |z|\cdot|\varphi|$

$z^2 - 2|z|\cos\varphi + 1 \leq (|z|-1)^2 + 2|z||\varphi|\cdot\big||z|-1\big| + z^2\varphi^2$

$\cancel{z^2} - 2|z|\cos\varphi + \cancel{1} \leq \cancel{z^2} - 2|z| + \cancel{1} + 2|z||\varphi|\cdot\big||z|-1\big| + z^2\varphi^2$

$-2|z|\cos\varphi \leq -2|z| + 2|z|\cdot|\varphi|\cdot\big||z|-1\big| + z^2\varphi^2 \quad / : |z|$

$-2\cos\varphi \leq -2 + 2\cdot|\varphi|\cdot\big||z|-1\big| + |z|\varphi^2$

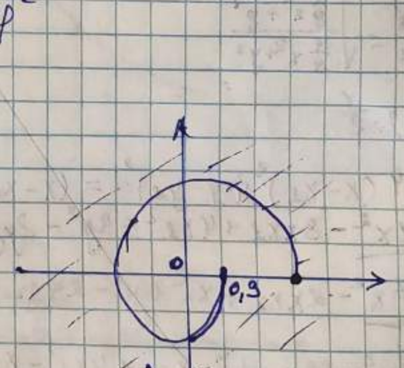
$|z| = r$

$4\sin^2\frac{\varphi}{2} \leq 2|r-1|\cdot|\varphi| + r|\varphi|^2$

при $r > 1$: $\quad 4\sin^2\frac{\varphi}{2} \leq 2\varphi(r-1) + \varphi^2 r$

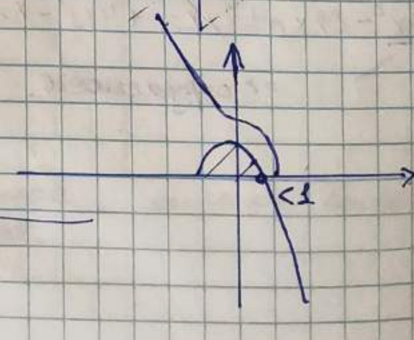
$\varphi \in (0; 2\pi)$ $\quad 4\sin^2\frac{\varphi}{2} + 2\varphi \leq r(2\varphi + \varphi^2)$

$\Rightarrow r \geq \dfrac{4\sin^2\frac{\varphi}{2} + 2\varphi}{(2\varphi + \varphi^2)}$



при $r < 1$: $\quad 4\sin^2\frac{\varphi}{2} \leq 2\varphi(1-r) + r\varphi^2$

$\varphi \in (0; 2\pi]$ $\quad r \geq \dfrac{4\sin^2\frac{\varphi}{2} - 2\varphi}{\varphi^2 - 2\varphi}$



$|z-1| \leq |z|-1|$

$\big||z|-1\big| \leq \big||z|-|1|\big| = |z|-1$

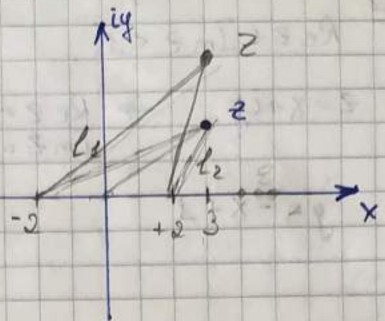
$\Rightarrow |z-1| \leq |z|-1 \leq |z|-1 + \overbrace{|z||\arg z|}^{>0}$

$\Rightarrow \underline{|z-1| \leq \big||z|-1\big| + |z|\cdot|\arg z|}$

1.25.

$|z-2| - |z+2| > 3$

$l_1 - l_2 > 3$ → Переставим местами



$\sqrt{(x+2)^2+y^2} - \sqrt{(x-2)^2+y^2} > 3$

$(x+2)^2+y^2 - 2\sqrt{((x+2)^2+y^2)((x-2)^2+y^2)} +$

$+ (x-2)^2 + y^2 > 9$

$x^2 + \cancel{4x} + 4 + y^2 - 2\sqrt{x^4+16+y^4-8x^2+2x^2y^2+8y^2} + x^2 - \cancel{4x} + 4 + y^2 > 9$

$2x^2 + 8 + 2y^2 - 2\sqrt{x^4+16+y^4-8x^2+2x^2y^2+8y^2} > 9$

$2\sqrt{x^4+16+y^4-8x^2+2x^2y^2+8y^2} < 2x^2+8+2y^2-9 \quad |:2$

$\sqrt{x^4+16+y^4-8x^2+2x^2y^2+8y^2} < x^2+4+y^2-\frac{9}{2} \quad |^2$

$\cancel{x^4}+16+\cancel{y^4}-8x^2+2x^2y^2+8y^2 < \cancel{x^4}+\frac{1}{4}+\cancel{y^4}-x^2+2x^2y^2-y^2$

$16-8x^2+\cancel{2x^2y^2}+8y^2+x^2-\cancel{2x^2y^2}+y^2 < \frac{1}{4}$

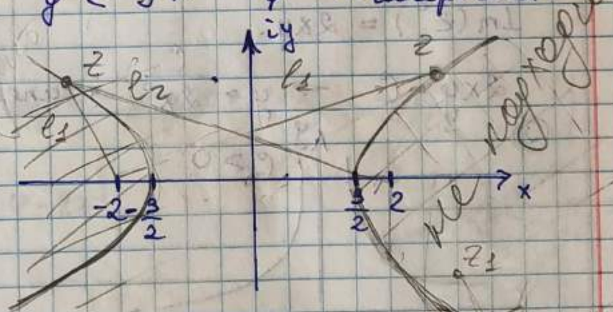
$16-7x^2+9y^2 < \frac{1}{4}$

$9y^2 < 7x^2 + \frac{1}{4} - 16$

$9y^2 < 7x^2 - \frac{63}{4} \quad |:9 \quad \rightarrow \quad y^2 < \frac{7}{9}x^2 - \frac{7}{4}$ ~ гипербола

$\frac{7}{9}x^2 - \frac{7}{4} = 0$

$x_{1,2} = \pm\frac{3}{2}$



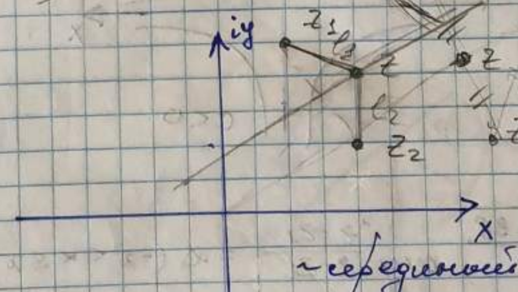
$|z| = \sqrt{x^2+y^2} \qquad z = x+iy$

$|z-2| = (x-2)+iy$

1.26.

$|z-z_1| = |z-z_2|$

z_1 и z_2 - нек. фикс. точки



$|z-z_1| = l_1$

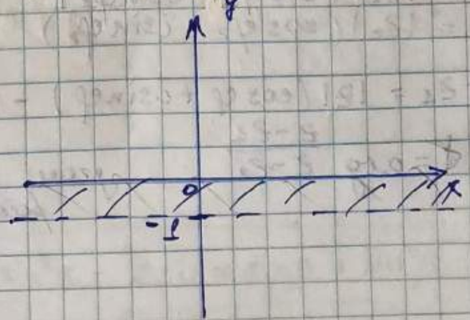
$|z-z_2| = l_2$

~ серединный перпенд.

~ Прямая, явл. серединным перпендикуляром к отрезку z_1, z_2 (и перпенд. z_1, z_2)!

1.28.

$0 < \mathrm{Re}(iz) < 1.$



$iz = i(x+iy) = -y+ix$

$\mathrm{Re}(iz) = -y$

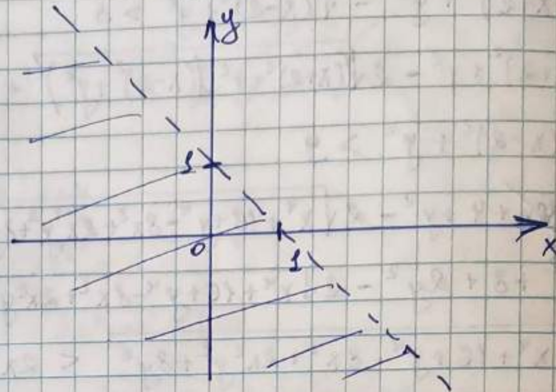
$0 < -y < 1 \rightarrow -1<y<0$

=> участок $-1<y<0$

1.31. $\operatorname{Re} z + \operatorname{Im} z < 1$

$z = x + iy \;\to\; \operatorname{Re} z = x,\ \operatorname{Im} z = y \;\Rightarrow\; x + y < 1$

$y = -x + 1$

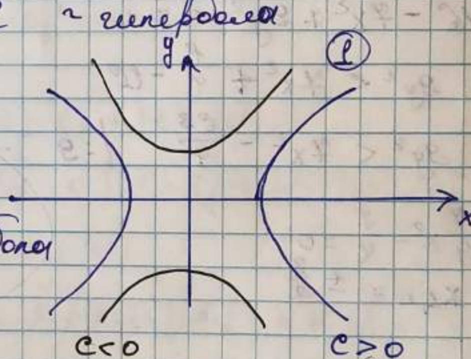


1.36. 1) $\operatorname{Re} z^2 = C$

$z = x + iy$

$z^2 = (x+iy)^2 = x^2 + 2xy \cdot i - y^2$

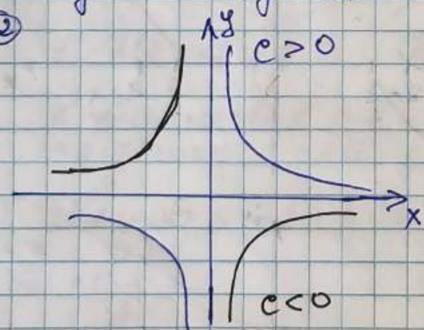
$\operatorname{Re}(z^2) = x^2 - y^2 \;\to\; x^2 - y^2 = C$ — гипербола

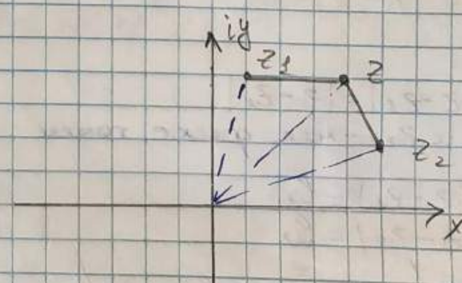


2) $\operatorname{Im} z^2 = C \quad (-\infty < C < \infty)$

$\operatorname{Im}(z^2) = 2xy.$

$2xy = C \;\to\; y = \dfrac{C}{2x}$ — гипербола





1.38. $\arg \dfrac{z - z_1}{z - z_2} = \alpha \quad (-\pi < \alpha \le \pi).$

$z = |z|(\cos\varphi + i\sin\varphi)$

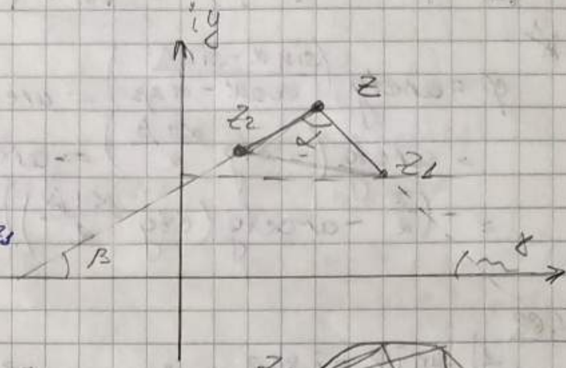
$z_1 = |z_1|(\cos\varphi_1 + i\sin\varphi_1)$

$z_2 = |z_2|(\cos\varphi_2 + i\sin\varphi_2)$

$z - z_1 = |z|(\cos\varphi + i\sin\varphi) - |z_1|(\cos\varphi_1 + i\sin\varphi_1)$

$\gamma = \arg \dfrac{z - z_1}{z - z_2}$ угол γ равен отношению расстояний $z - z_1$ и $z - z_2$

1.38. $\arg \dfrac{z-z_1}{z-z_2} = \alpha \qquad (-\pi < \alpha \le \pi)$



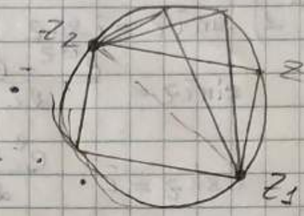
$\gamma = \beta + \angle z_2 z z_1$

$\gamma - \beta = \angle z_2 z z_1$

$\arg(z-z_1) - \arg(z-z_2) = \angle z_2 z z_1$

$\arg\left(\dfrac{z-z_1}{z-z_2}\right) = \alpha$

$\Rightarrow$ окружность



Элементарные функции комплексного переменного.

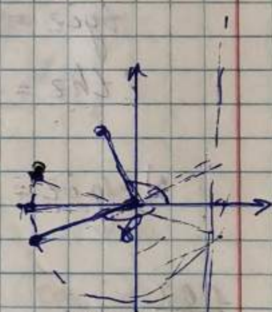
1.60. $e^{\pm i\frac{\pi}{2}}, \; e^{k\pi i} \quad (k = 0, \pm 1, \pm 2, \dots)$

$e^{ix} = \cos x + i \sin x$

$e^{\pm i\frac{\pi}{2}} = \cos\frac{\pi}{2} \pm i \sin\frac{\pi}{2} = \pm i$

1.69. 3) e^{3+4i}

$e^{3+4i} = e^3 \cdot e^{4i}$ $\qquad$ $\operatorname{Arg} z = \arg z + 2\pi k$

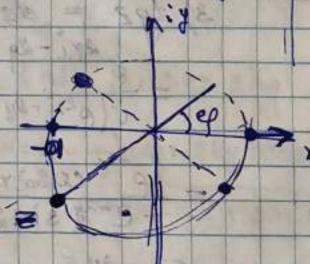


$|z| = e^3$; $\arg(z) = \qquad \varphi - 2\pi$

4) $e^{-3-4i} = e^{-3} \cdot e^{-4i}$

$|z| = e^{-3}$, $\arg(z) = -4 + 2\pi$

5) $-a e^{i\varphi} \; (a > 0, \; |\varphi| \le \pi)$



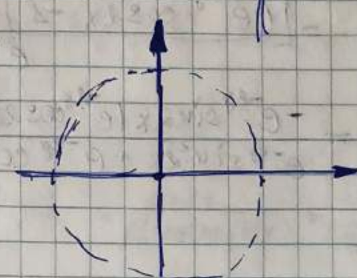
$|z| = a$; $\arg(z) = \varphi - \pi \; (\varphi > 0)$

$\pi + \varphi \; (\varphi < 0)$

6) $e^{-i\varphi} \quad (|\varphi| \le \pi)$

$|z| = 1$; $\arg(z) = -\varphi$

7) $e^{i\alpha} - e^{i\beta} \quad (0 \le \beta < \alpha \le 2\pi) =$



$= \cos\alpha + i\sin\alpha - \cos\beta - i\sin\beta =$

$= (\cos\alpha - \cos\beta) + i(\sin\alpha - \sin\beta)$

$|z| = \sqrt{(\cos\alpha - \cos\beta)^2 + (\sin\alpha - \sin\beta)^2} =$

$= \sqrt{\cos^2\alpha - 2\cos\alpha\cos\beta + \cos^2\beta + \sin^2\alpha - 2\sin\alpha\sin\beta + \sin^2\beta} =$

$= \sqrt{2 - 2(\cos\alpha\cos\beta + \sin\alpha\sin\beta)} = \sqrt{2 - 2\cos(\alpha - \beta)} =$

$$= \sqrt{2}\cdot\sqrt{1-\cos(\alpha-\beta)} = \sqrt{2}\cdot\sqrt{2\sin^2\frac{(\alpha-\beta)}{2}} = 2\sin\frac{\alpha-\beta}{2}$$

$tg\alpha \cdot ctg\alpha = 1$

$$\varphi = arctg\left(\frac{\sin\alpha-\sin\beta}{\cos\alpha-\cos\beta}\right) = arctg\left(\frac{2\sin\frac{\alpha-\beta}{2}\cos\frac{\alpha+\beta}{2}}{-2\sin\frac{\alpha-\beta}{2}\sin\frac{\alpha+\beta}{2}}\right) =$$

$$= arctg\left(-ctg\frac{\alpha+\beta}{2}\right) = -arctg\left(ctg\frac{\alpha+\beta}{2}\right) =$$

$$= -\left(\frac{\pi}{2} - arcctg\left(ctg\frac{\alpha+\beta}{2}\right)\right) = \frac{\alpha+\beta}{2} - \frac{\pi}{2} = \frac{\alpha+\beta-\pi}{2}$$

1.66

1) $\sin iz = i\,sh z$

$$\sin iz = \frac{e^{i^2 z}-e^{-i^2 z}}{2i} = \frac{e^{-z}-e^{z}}{2i} = \frac{i}{2}(e^{z}-e^{-z})$$

$$i\,sh z = i\cdot\frac{e^{z}-e^{-z}}{2}$$

$$\Rightarrow \sin(iz) = i\,sh z$$, ч.т.д.

2) $\cos iz = ch z$

$$\cos iz = \frac{e^{i^2 z}+e^{-i^2 z}}{2} = \frac{e^{-z}+e^{z}}{2} = ch z$$, ч.т.д

3) $tg\,iz = i\,th z$

$$tg\,iz = \frac{\sin(iz)}{\cos(iz)} = \frac{(e^{-z}-e^{z})\cancel{2}}{\cancel{2}i(e^{-z}+e^{z})} = i\cdot\frac{(e^{z}-e^{-z})}{(e^{z}+e^{-z})} = i\,th z$$

$$i\,th z = i\frac{sh z}{ch z} = i$$

ч.т.д.

4) $ctg\,iz = -i\left(\frac{e^{-z}-e^{z}}{e^{z}+e^{-z}}\right) = -i\,cth z$, ч.т.д.

1.67. 3)

$$tg z = \frac{\sin z}{\cos z} = \frac{e^{iz}-e^{-iz}}{i(e^{iz}+e^{-iz})} = \frac{e^{2iz}-1}{i(e^{2iz}+1)} =$$

$$= \frac{e^{2xi-2y}-1}{i(e^{2xi-2y}+1)} = \frac{e^{-2y}(\cos 2x + i\sin 2x)-1}{i(e^{-2y}(\cos 2x+i\sin 2x)+1)} =$$

$$= \frac{(e^{-2y}\cos 2x-1)+ie^{-2y}\sin 2x}{-e^{-2y}\sin 2x + i(e^{-2y}\cos 2x+1)} =$$

$$= \frac{[(e^{-2y}\cos 2x-1)+ie^{-2y}\sin 2x][-e^{-2y}\sin 2x - i(e^{-2y}\cos 2x+1)]}{e^{-4y}\sin^2 2x+(e^{-2y}\cos 2x+1)^2} =$$

$$= \frac{-e^{-2y}\sin 2x(e^{-2y}\cos 2x-1)+e^{-2y}(e^{-2y}\cos 2x+1) - ie^{-4y}\sin^2 2x - i(e^{-4y}\cos^2 2x-1)}{e^{-4y}\sin^2 2x + e^{-4y}\cos^2 2x + 2e^{-2y}\cos 2x+1} =$$

$=$

1.68. 2) $\sin 2i = \dfrac{e^{-2}-e^{2}}{2i} = -\dfrac{sh 2}{i} = \underline{i\, sh 2}$

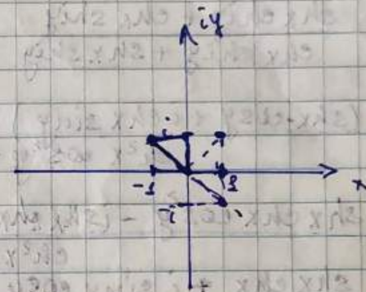
3) $tg(2-i) = \dfrac{\sin(2-i)}{\cos(2-i)} = tg\, i\left(\dfrac{2}{i}-1\right) = i\, th\left(\dfrac{2}{i}-1\right) =$

$$tg(2-i) = \frac{\sin(2-i)}{\cos(2-i)} = \frac{-i(e^{2i+1}-e^{-2i-1}+e^{2i-1}-e^{2i-1})}{(e^{2i+1}+e^{-2i-1})+e^{2i-1}-e^{2i-1}} =$$

$$= -i\,\frac{-(e^{2i+1}+e^{2i-1})-(e^{-2i-1}+e^{2i-1})}{(e^{2i+1}+e^{2i-1})+(e^{-2i+1}-e^{2i-1})} = -i\,\frac{e^{2i}(2cth\,1)-e^{-1}(e^{-2i}+e^{2i})}{e^{2i}(2cth\,1)+e^{-1}(e^{-2i}-e^{2i})} =$$

$$= -i\; e^{2i}(2cth\,1) - e^{-1}2$$

Домашняя работа.



1.59
1) $1 = e^{i\cdot 0} = 1$
2) $-1 = 1\cdot e^{i\pi}$
3) $i = e^{i\frac{\pi}{2}}$
4) $-i = e^{-\frac{\pi}{2}i}$
5) $1+i = \sqrt{2}\cdot e^{i\frac{\pi}{4}}$
6) $1-i = \sqrt{2}\cdot e^{-\frac{\pi}{4}i}$
7) $-1+i = \sqrt{2}\cdot e^{\frac{3}{4}\pi i}$
8) $-1-i = \sqrt{2}\cdot e^{-\frac{3}{4}\pi i}$

1.67 1) $\sin z = \sin(x+iy) = \dfrac{e^{i(x+iy)}-e^{-i(x+iy)}}{2i} = \dfrac{e^{ix}\cdot e^{-y}-e^{-ix}\cdot e^{y}}{2i} =$

$$= \frac{i(e^{-ix}\cdot e^{y}-e^{ix}e^{-y})}{2} =$$

$= \sin x\cos(iy) + \cos x \sin(iy) = \sin x\, ch y + i\cos x\, sh y$ (см. 1.66.)

2) $\cos z = \cos(x+iy) = \cos x\cdot\cos(iy) - \sin x\, \sin(iy) = \cos x\, ch y - i\sin x\, sh y$

6) $tg\, z = \dfrac{\sin(x+iy)}{\cos(x+iy)} = \dfrac{\sin x\, ch y + i\cos x\, sh y}{\cos x\, ch y - i\sin x\, sh y} = \dfrac{(\sin x\, ch y + i\cos x\, sh y)(\cos x\, ch y + i\sin x\, sh y)}{\cos^2 x\, ch^2 y + \sin^2 x\, sh^2 y}$

$$= \frac{\sin x\cos x\, ch^2 y + i\sin^2 x\, ch y\, sh y + i\cos^2 x\, sh y\, ch y - \sin x\cos x\, sh^2 y}{\cos^2 x\, ch^2 y + \sin^2 x\, sh^2 y} =$$

$$= \frac{\sin x\cos x(\overbrace{ch^2 y - sh^2 y}^{=1}) + i\, ch y\, sh y}{\cos^2 x\, ch^2 y + \sin^2 x\, sh^2 y} = \frac{\sin x\cos x + i\, ch y\, sh y}{(1-\sin^2 x)(1+sh^2 y)+\sin^2 x\, sh^2 y} =$$

$$= \frac{\sin x\cos x + i\, ch y\, sh y}{1+sh^2 y - \sin^2 x} = \frac{\sin x\cos x + i\, ch y\, sh y}{\cos^2 x + sh^2 y} = \frac{\sin 2x}{2(\cos^2 x + sh^2 y)} + \frac{i\, sh 2y}{2(\cos^2 x + sh^2 y)}$$

1) $|\sin z| = \sqrt{\sin^2 x\, ch^2 y + \cos^2 x\, sh^2 y}$

2) $|\cos z| = \sqrt{\cos^2 x\, ch^2 y + \sin^2 x\, sh^2 y}$

в) $th(z) = \dfrac{sh(x+iy)}{ch(x+iy)} = \dfrac{shx\,ch\,iy + chx\,sh\,iy}{chx\,ch\,iy + shx\,sh\,iy} \ominus$

$= \left\{ \begin{array}{l} ch\,iy = \dfrac{e^{iy}+e^{-iy}}{2} = \cos y \\ sh\,iy = \dfrac{e^{iy}-e^{-iy}}{2} = i\sin y \end{array} \right\} = \dfrac{shx\cdot\cos y + i\,chx\sin y}{chx\,ch y + i\,shx\sin y} =$

$= \dfrac{(shx\cdot\cos y + i\,chx\sin y)(shx\cdot\cos y - i\,chx\sin y)}{ch^2x\,ch^2y + sh^2x\sin^2y} =$

$= \dfrac{sh^2x\cos^2y - i\,shx\cdot chx\cdot\cos y\cdot\sin y + i\,shx\cdot chx\cdot\cos y\cdot\sin y + ch^2x\sin^2y}{ch^2x\,ch^2y + sh^2x\sin^2y} =$

$= \dfrac{sh^2x\cos^2y + ch^2x\sin^2y}{ch^2x\,ch^2y + sh^2x\sin^2}$

$\ominus \dfrac{shx\,ch\,iy + chx\,sh\,iy}{chx\,ch\,iy + shx\,sh\,iy} = \dfrac{shx\cdot\cos y + i\,chx\sin y}{chx\cdot\cos y + i\,shx\sin y} =$

$= \dfrac{(shx\cdot\cos y + i\,chx\sin y)(chx\cdot\cos y - i\,shx\sin y)}{ch^2x\cos^2y + sh^2x\sin^2y} =$

$= \dfrac{shx\,chx\cos^2y - i\,sh^2x\cos y\sin y + i\,ch^2x\sin y\cos y + shx\,chx\sin^2y}{ch^2x\cos^2y + sh^2x\sin^2y} =$

$= \dfrac{shx\,chx + i\sin y\cos y}{ch^2x\cos^2y + sh^2x\sin^2y} = \dfrac{1}{2}\cdot\dfrac{sh2x + i\sin 2y}{ch^2x\cos^2y + sh^2x\sin^2y}$

$\Rightarrow |th\,z| = \dfrac{1}{2}\sqrt{\left(\dfrac{sh2x}{ch^2x\cos^2y + sh^2x\sin^2y}\right)^2 + \left(\dfrac{\sin 2y}{ch^2x\cos^2y + sh^2x\sin^2y}\right)^2}$

Комплексный логарифм.

Опр. Натуральный логарифм $Ln\,z$ комплексного числа z представляет собой решение ω, $e^{\omega} = z$

Ненулевое число z можно представить: $z = r\cdot e^{i(\varphi + 2\pi k)}$

$\Rightarrow \quad Ln\,z = \ln r + i(\varphi + 2\pi k)$

$Ln(-x) = \ln x + i\pi(2k+1)$

Примеры:

$\ln(1) = 0; \quad Ln(1) = 2k\pi i$

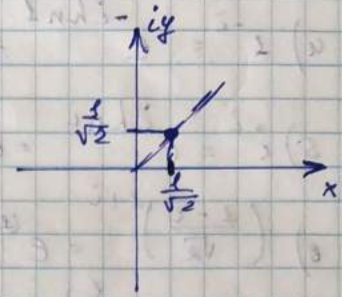
$\ln(-1) = i\pi \quad; \quad Ln(-1) = (2k+1)\,i\pi$

$\ln(i) = i\dfrac{\pi}{2} \quad; \quad Ln(i) = i\,\dfrac{4k+1}{2}\,\pi$

1.68.

1) $\cos(2+i) = \cos 2 \cos i1 - \sin 2 \sin i1 = \cos 2 \cdot ch1 - i \sin 2 \cdot sh1$

1.71.



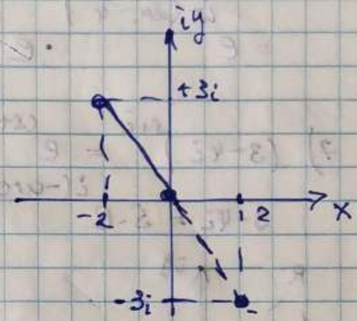
1) $Ln 4 = \ln 4 + 2\pi k i$

$Ln(-1) = \ln 1 + i(\pi + 2\pi k) = (2k+1)\pi i$

$\ln(-1) = \ln(1 \cdot e^{i\pi}) = i\pi$

2) $Ln\, i = i\left(\frac{\pi}{2} + 2\pi k\right) = \pi i\left(2k + \frac{1}{2}\right)$

$\ln i = \frac{i\pi}{2}$



3) $Ln \frac{1 \pm i}{\sqrt{2}} = \pi i\left(2k \pm \frac{1}{4}\right)$

$\frac{1}{\sqrt{2}} \pm i\frac{1}{\sqrt{2}} = 1 \cdot e^{i\left(\pm\frac{\pi}{4} + 2\pi k\right)}$

4) $Ln(2-3i) = \frac{1}{2}\ln 13 + i\left(2\pi k - arctg\frac{3}{2}\right)$

$2 - 3i = \sqrt{13} \cdot e^{i\left(-arctg\frac{3}{2} + 2\pi k\right)}$

$Ln(-2+3i) = \frac{1}{2}\ln 13 + i\left((2k+1)\pi - arctg\frac{3}{2}\right)$

$-2+3i = \sqrt{13} \cdot e^{i\left(\pi - arctg\frac{3}{2} + 2\pi k\right)}$

1.74. По определению для любых комплексных чисел $a \neq 0$ и α

$$a^{\alpha} = \exp\{\alpha\, Ln\, a\}$$

или $a^{\alpha} = e^{\alpha Ln a}$

1) $1^{\sqrt{2}} = e^{\sqrt{2}\, Ln(1)} = e^{\sqrt{2} \cdot 2k\pi i} = \cos(2k\sqrt{2}\pi) + i\sin(2k\sqrt{2}\pi)$

2) $(-2)^{\sqrt{2}} = e^{\sqrt{2}\, Ln(-2)} = e^{\sqrt{2}(\ln 2 + \pi i(2k+1))} \circleddash$

$Ln(-2) = \cancel{\ln} \ln 2 + i(\pi + 2\pi k) = \ln 2 + \pi i(2k+1)$

$-2 = 2 \cdot e^{i(\pi + 2\pi k)}$

$\circleddash\ e^{\ln 2^{\sqrt{2}}} \cdot e^{i\sqrt{2}\pi(2k+1)} = 2^{\sqrt{2}}\left(\cos(2k+1)\sqrt{2}\pi + i\sin(2k+1)\sqrt{2}\pi\right)$

3) $2^{i} = e^{i Ln 2} = e^{i(\ln 2 + 2\pi k i)} = 2^{i}(\cos(2\pi k) + i\sin(2\pi k)) \circleddash$

$Ln(2) = \ln 2 + i2\pi k$

$= e^{i\ln 2} \cdot \cancel{e^{2\pi k i}} = \cancel{(\cos(\ln 2) + i\sin\ln 2)(\cos 2\pi k + i\sin 2\pi k)} =$

$= \cos(\ln 2)\cos(2\pi k) + i\cos(\ln 2)\sin(2\pi k) + \cancel{i\sin\ln 2\cos 2\pi k - \sin 2\pi k \cdot \sin\ln 2} =$

$$\ominus\ e^{i\ln 2}\cdot e^{-2\pi k} = e^{-2\pi k}(\cos\ln 2 + i\sin\ln 2) = e^{2\pi k}(\cos\ln 2 + i\sin\ln 2)$$

4) $1^{-i} = e^{-i\,\mathrm{Ln}\,1} = e^{-i\cdot 2\pi k i} = e^{2\pi k}.$

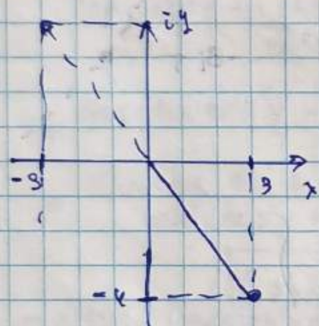
5) $i^{i} = e^{i\,\mathrm{Ln}\,i} = e^{i\cdot\pi i(2k+\frac{1}{2})} = e^{-\pi(2k+\frac{1}{2})}$

6) $\left(\frac{1-i}{\sqrt{2}}\right)^{1+i} = e^{(1+i)\,\mathrm{Ln}\left(\frac{1-i}{\sqrt{2}}\right)} = e^{(1+i)\cdot\pi i(2k-\frac{1}{4})} =$

$= e^{i\pi(2k-\frac{1}{4})}\cdot e^{-\pi(2k-\frac{1}{4})} = e^{\pi(\frac{1}{4}-2k)}\left(\cos\pi(2k-\frac{1}{4}) + i\sin\pi(2k-\frac{1}{4})\right)$

7) $(3-4i)^{1+i} = e^{(1+i)\,\mathrm{Ln}(3-4i)}$ $\ominus$

$3-4i = 5\cdot e^{i(-\mathrm{arctg}\frac{4}{3}+2\pi k)}$



$\mathrm{Ln}(3-4i) = \ln 5 + i(-\mathrm{arctg}\frac{4}{3} + 2\pi k)$

$\ominus\ e^{(1+i)(\ln 5 + i(-\mathrm{arctg}\frac{4}{3}+2\pi k))} = e^{\ln 5}\cdot e^{i(-\mathrm{arctg}\frac{4}{3}+2\pi k)}\cdot e^{i\ln 5}\cdot$

$\cdot e^{(\mathrm{arctg}\frac{4}{3}-2\pi k)} = e^{\ln 5 + (\mathrm{arctg}\frac{4}{3}-2\pi k)}$

$= 5\cdot e^{\mathrm{arctg}\frac{4}{3}+2\pi k}\left[\cos(\ln 5 - \mathrm{arctg}\frac{4}{3}) + i\sin(\ln 5 - \mathrm{arctg}\frac{4}{3})\right]$

8) $(-3+4i)^{1+i} = e^{(1+i)\,\mathrm{Ln}(-3+4i)}$ $\ominus$

$\mathrm{Ln}(-3+4i) = \ln 5 + i(\pi - \mathrm{arctg}\frac{4}{3} + 2\pi k)$

$\ominus\ e^{(1+i)(\ln 5 + i(\pi-\mathrm{arctg}\frac{4}{3}+2\pi k))} = e^{(1+i)(\ln 5 + i(-\mathrm{arctg}\frac{4}{3}+\pi(2k+1)))} =$

$= e^{\ln 5}\cdot e^{i(-\mathrm{arctg}\frac{4}{3}+\pi(2k+1))}\cdot e^{i\ln 5}\cdot e^{(\mathrm{arctg}\frac{4}{3}-\pi(2k+1))} =$

$= 5\cdot e^{(\mathrm{arctg}\frac{4}{3}-\pi(2k+1))}\cdot\left(\cos[\ln 5 - \mathrm{arctg}\frac{4}{3} + \pi(2k+1)] +\right.$

$\left.+ i\sin[\ln 5 - \mathrm{arctg}\frac{4}{3} + \pi(2k+1)]\right) =$

$= -5\cdot e^{(\mathrm{arctg}\frac{4}{3}+\pi(2k+1))}\cdot\left(\cos[\ln 5 - \mathrm{arctg}\frac{4}{3}] + i\sin[\ln 5 - \mathrm{arctg}\frac{4}{3}]\right)$

1.77.

2) $Arcsin\, z = -i\, Ln\left(i(z+\sqrt{z^2-1})\right) = u$

$z = \sin u = \dfrac{e^{iu}-e^{-u}}{2i}$

$e^{iu} - e^{-iu} = 2iz$

$e^{2iu} - 2iz\cdot e^{iu} - 1 = 0$

$e^{iu} = iz + \sqrt{-z^2+1}$

$u = \frac{1}{i} Ln\left(iz + \sqrt{-z^2+1}\right)$

$u = -i\, Ln\left(i\left(z+\sqrt{z^2-1}\right)\right)$

$Ln\, z = Ln\, r e^{i(\varphi+2\pi k)} = \ln r + i(\varphi+2\pi k)$

$z = r\cdot e^{i(\varphi+2\pi k)}$

$sh 2x = 2\, sh x\, ch x$

$ch 2x = ch^2 x + sh^2 x$

$ch^2 z = \dfrac{1+ch 2z}{2}$

$sh^2 z = \dfrac{ch 2z - 1}{2}$

30.09.21.

Условие Коши-Римана

1.135.

$z = r\cdot e^{i\varphi}$

$$\boxed{\begin{matrix} u'_x = v'_y \\ v'_x = -u'_y \end{matrix}}$$

$f(z) = u(r,\varphi) + iV(r,\varphi)$

$f'_x = u'_x + i v'_x\;, \quad f'_y = u'_y + i v'_y$

$f'_x = u'_r r'_x + u'_\varphi \varphi'_x + i(v'_r r'_x + v'_\varphi \varphi'_x)$

$f'_y = u'_r r'_y + u'_\varphi \varphi'_y + i(v'_r r'_y + v'_\varphi \varphi'_y)$,

$\Rightarrow \quad u'_r r'_x + u'_\varphi \varphi'_x = v'_r r'_y + v'_\varphi \cdot \varphi'_y$

$\begin{cases} x = r\cos\varphi \\ y = r\sin\varphi \end{cases} \Rightarrow \begin{cases} 1 = r'_x \cos\varphi - r\sin\varphi\cdot\varphi'_x & \Big| \cos\varphi \\ 0 = r'_x \sin\varphi + r\cos\varphi\cdot\varphi'_x & \Big| \sin\varphi \end{cases} + \Big| \begin{matrix}\cdot\sin\varphi \\ \cdot\cos\varphi\end{matrix} \; -$

$$\boxed{\begin{matrix} r'_x = \cos\varphi \\ \varphi'_x = -\dfrac{\sin\varphi}{r} \end{matrix}}$$

Аналогично: $\begin{cases} 0 = r'_y \cos\varphi - r\sin\varphi\cdot\varphi'_y \\ 1 = r'_y \sin\varphi + r\cos\varphi\cdot\varphi'_y \end{cases}$

$$r'_y = \sin\varphi$$

$$\varphi'_y = \frac{1}{r}\cos\varphi$$

$$u'_r\cos\varphi + u'_\varphi\left(-\frac{1}{r}\sin\varphi\right) = v'_r\sin\varphi + v'_\varphi\cdot\frac{1}{r}\cos\varphi$$

$$\underbrace{\left(u'_r - \frac{1}{r}v'_\varphi\right)}_{=0}\cos\varphi + \underbrace{\left(-v'_r - \frac{1}{r}u'_\varphi\right)}_{=0}\sin\varphi = 0$$

$$\Rightarrow \boxed{\begin{cases} u'_r = \frac{1}{r}v'_\varphi \\ v'_r = -\frac{1}{r}u'_\varphi \end{cases}}$$ — Условия Коши-Римана в полярных координатах.

1.131. z^n, e^z, $\cos z$, $\ln z$, и показать что

$$(z^n)' = nz^{n-1}, \quad (e^z)' = e^z, \quad (\cos z)' = -\sin z, \quad (\ln z)' = \frac{1}{z}$$

1) $z^n = r^n\cdot e^{i\varphi n} = r^n(\cos\varphi n + i\sin\varphi n) = \underbrace{r^n\cos\varphi n}_{u} + i\underbrace{r^n\sin\varphi n}_{v}$

$u'_r = nr^{n-1}\cos\varphi n = \frac{1}{r}v'_\varphi = \frac{1}{r}r^n n\cos\varphi n$

$v'_r = nr^{n-1}\sin\varphi n = -\frac{1}{r}r^n(-n\sin\varphi n)$

2) $e^z = e^x\cdot e^{iy} = e^x\cdot(\cos y + i\sin y) = \underbrace{e^x\cos y}_{u} + i\underbrace{e^x\sin y}_{v}$

$u'_x = e^x\cos y = v'_y = e^x\cos y$ $\Rightarrow$ выполняется.

$v'_x = e^x\sin y = -u'_y = e^x\sin y$.

3) $\cos z = \cos(x+iy) = \cos x\cdot\cos(iy) - \sin x\cdot\sin(iy) =$

$= \underbrace{\cos x\cdot\operatorname{ch}y}_{u} - i\underbrace{\sin x\,\operatorname{sh}y}_{v}$

$u'_x = -\operatorname{ch}y\sin x = v'_y = -\operatorname{ch}y\sin x$ $\Rightarrow$ выполняется.

$v'_x = -\cos x\cdot\operatorname{sh}y = -u'_y = -\cos x\cdot\operatorname{sh}y$

или: $\cos z = \frac{e^{iz} + e^{-iz}}{2} =$, а экспонента удовл. усл. Коши-Римана

$= \frac{e^{ix-y} + e^{y-ix}}{2} = \frac{e^{-y}(\cos x + i\sin x) + e^{y}(\cos x - i\sin x)}{2} =$

$= \cos x\cdot\operatorname{ch}y - i\sin x\,\operatorname{sh}y$

4) $\ln z = \ln(r\cdot e^{i\varphi}) = \ln r + i(\varphi + 2\pi k) = \underbrace{\ln r}_{u} + i\underbrace{\varphi}_{v}$

$u'_r = \frac{1}{r} = v'_\varphi\cdot\frac{1}{r} = \frac{1}{r}$

$\vartheta'_r = 0 = -\frac{1}{r}\cdot 0 = 0$ $\Rightarrow$ выполняется.

$$(z^n)' = \lim_{\Delta z\to 0}\frac{(z+\Delta z)^n - z^n}{\Delta z} = \lim_{\substack{\Delta z\to 0\\ (\Delta r\to 0)\\ (\Delta\varphi\to 0)}} \frac{n r^{n-1} e^{i\varphi n}(\Delta r + i r\Delta\varphi)}{e^{i\varphi}(\Delta r + i r \Delta\varphi)} = n z^{n-1}, \text{ ч.т.д.}$$

$$f(r,\varphi) = r^n e^{i\varphi n}$$

Вывод: $f(r+\Delta r, \varphi+\Delta\varphi) - f(r,\varphi) \simeq f'_r\Delta r + f'_\varphi\Delta\varphi = n r^{n-1} e^{i\varphi n}\Delta r + i r^n e^{i\varphi n}\cdot n\Delta\varphi$

Вход: $\Delta z = (r+\Delta r)e^{i(\varphi+\Delta\varphi)} - r e^{i\varphi} = r\cdot e^{i\varphi}(\underbrace{e^{i\Delta\varphi} - 1}_{\simeq i\Delta\varphi}) + \Delta r e^{i\varphi}\cdot e^{i\Delta\varphi} \simeq \{$при малых $\Delta\varphi\}$

$$\simeq r\cdot e^{i\varphi}\cdot i\Delta\varphi + \Delta r e^{i\varphi}$$

$$e^{\alpha x} - 1 \sim \alpha x$$

$$(e^z)' = \lim_{\Delta z\to 0}\frac{e^{z+\Delta z} - e^z}{\Delta z} = e^z \lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}} \frac{e^{\Delta x}(\cos\Delta y + i\sin\Delta y) - 1}{\Delta x + i\Delta y} =$$

$$= e^z \lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}} \frac{(1+\Delta x)(1+i\Delta y) - 1}{\Delta x + i\Delta y} = e^z \lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}} \frac{\Delta x + i\Delta y + i\Delta x\Delta y}{\Delta x + i\Delta y} = e^z$$

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}$$

$$(\cos z)' = \frac{ie^{iz} - ie^{-iz}}{2} = -\frac{e^{iz} - e^{-iz}}{2i} = -\sin z$$

$$(\ln z)' = \lim_{\Delta z\to 0}\frac{\ln(z+\Delta z) - \ln z}{\Delta z} = \lim_{\Delta z\to 0}\frac{\ln\left(1+\frac{\Delta z}{z}\right)}{\Delta z} =$$

$$= \lim_{\Delta z\to 0}\frac{\Delta z}{\Delta z\cdot z} = \frac{1}{z}$$ Но эквивалентность нужно док-ть

1.192.

1) $f(z) = \underbrace{x + ay}_{u} + i(\underbrace{bx + cy}_{\vartheta})$

$u'_x = 1 = \vartheta'_y = c \quad \to c = 1.$

$\vartheta'_x = b = -u'_y = -a \quad \to a = -b$

2) $f(z) = \cos x(\operatorname{ch} y + a\operatorname{sh} y) + i\sin x(\operatorname{ch} y + b\operatorname{sh} y)$

$u'_x = -(\operatorname{ch} y + a\operatorname{sh} y)\sin x = \vartheta'_y = \sin x(\operatorname{sh} y + b\operatorname{ch} y)$

$\vartheta'_x = (\operatorname{ch} y + b\operatorname{sh} y)\cos x = -u'_y = -\cos x(\operatorname{sh} y + a\operatorname{ch} y)$

$$\begin{cases} -\operatorname{ch} y - a\operatorname{sh} y = \operatorname{sh} y + b\operatorname{ch} y \\ \operatorname{ch} y + b\operatorname{sh} y = -\operatorname{sh} y - a\operatorname{ch} y \end{cases} \Leftrightarrow \begin{cases} b\operatorname{ch} y + a\operatorname{sh} y = \operatorname{sh} y + \operatorname{ch} y \\ b\operatorname{sh} y + a\operatorname{ch} y = -\operatorname{sh} y - \operatorname{ch} y \end{cases}$$

$\Rightarrow a = b = -1$

1.137. $f(z) = \bar{z}$ — док-ть, что нигде не дифф-ма.

$\bar{z} = x - iy$

$u'_x = 1 \neq v'_y = -1$ — не выполняется, ч.т.д.

1.165. $u = x^2 - y^2 + 5x + y - \frac{y}{x^2+y^2}$; $f(z) = u + iv$

$u'_x = 2x + 5 + \frac{y \cdot 2x}{(x^2+y^2)^2} = v'_y$

$\Rightarrow v = 2xy + 5y - \frac{x}{x^2+y^2} + \varphi(x)$

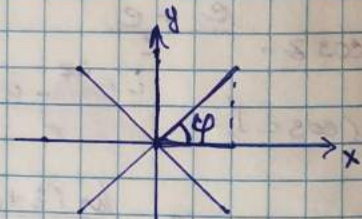
$v'_x = 2y - \frac{x^2+y^2-2x^2}{(x^2+y^2)^2} + \varphi'(x) = -u'_y =$

$= +2y - 1 + \frac{x^2+y^2-2y^2}{(x^2+y^2)} = 2y - 1 + \frac{x^2-y^2}{(x^2+y^2)}$

$\Rightarrow \varphi'(x) = -1$

$\varphi = -x + C$

Домашняя работа.



1.133. $f(z) = |x^2-y^2| + 2i|xy|$

1.138. $w = z\cdot \mathrm{Re}\,z$, диф-ма только в точке $z_0=0$

$$w = z\cdot \mathrm{Re}\,z = (x+iy)\cdot x = \underbrace{x^2}_{u} + i\underbrace{xy}_{v}.$$

$u'_x = 2x \neq v'_y = x$ ~ => ф-я не аналитическая.
$v'_x = y \neq -u'_y = 0$

$$w'(0) = \lim_{\Delta z\to 0}\frac{f(z_0+\Delta z)-f(z_0)}{\Delta z} = \lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}}\frac{f(0+\Delta z)-f(0)}{\Delta z} =$$

$$= \lim_{\Delta z\to 0}\frac{f(\Delta z)-f(0)}{\Delta z} = \lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}}\frac{f(0+\Delta x, 0+\Delta y)-f(0,0)}{\Delta x+i\Delta y}$$

$$w' = \lim_{\Delta z\to 0}\frac{(z+\Delta z)\,\mathrm{Re}(z+\Delta z) - z\,\mathrm{Re}\,z}{\Delta z} = \lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}}\frac{(x+iy+\Delta x+i\Delta y)\,\mathrm{Re}(x+iy+\Delta x+i\Delta y) - (x+iy)\cdot x}{\Delta x+i\Delta y} = \lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}}\frac{(x+\Delta x+i(y+\Delta y))(x+\Delta x) - (x+iy)x}{\Delta x+i\Delta y} =$$

$$= \lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}}\frac{\cancel{x(x+iy)} + x(\Delta x+i\Delta y) + \Delta x(x+iy) + \Delta x(\Delta x+i\Delta y) - \cancel{x(x+iy)}}{\Delta x+i\Delta y} =$$

$$= \lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}}\frac{(\Delta x+i\Delta y)(x+\Delta x) + \Delta x(x+iy)}{\Delta x+i\Delta y} = \lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}}(x+\Delta x) + \lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}}\frac{\Delta x(x+iy)}{\Delta x+i\Delta y} =$$

$$= x + \lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}}\frac{\Delta x(x+iy)(\Delta x-i\Delta y)}{\Delta x^2+\Delta y^2} = x + \lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}}\frac{(\Delta x\cdot x + \Delta x\,y\cdot i)(\Delta x - i\Delta y)}{\Delta x^2+\Delta y^2} =$$

$$= x + \lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}}\frac{x\cdot\Delta x^2 - ix\cdot\Delta x\cdot\Delta y + iy\cdot\Delta x^2 + y\cdot\Delta x\Delta y}{\Delta x^2+\Delta y^2}\ (\nexists) =$$

$$= x + \lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}}\frac{x\cdot\Delta x^2 + y\cdot\Delta x\Delta y + i(y\cdot\Delta x^2 - x\Delta x\cdot\Delta y)}{\Delta x^2+\Delta y^2} = \left\{\frac{0}{0}\right\}$$ (в окр. точки)

$z_0 = 0 \to$ ~~$(x+iy)$~~ $x+iy=0$ ~ $x_0=0$, $y_0=0$

$$\Rightarrow w'(0) = 0 + 0 = \underline{\underline{0}}$$

1.166. $u = e^x(x\cos y - y\sin y) + 2\sin x\,\mathrm{sh}\,y + x^3 - 3xy^2 + y$

$u'_x = e^x(x\cos y - y\sin y) + \cos y\cdot e^x + 2\cos x\cdot\mathrm{sh}\,y + 3x^2 - 3y^2 = v'_y$

$\Rightarrow v'_y = x\cdot e^x\cdot\cos y - e^x y\sin y + e^x\cos y + 2\cos x\cdot\mathrm{sh}\,y + 3x^2 - 3y^2$

$v = x\cdot e^x\sin y + e^x y\cos y - \cancel{e^x\sin y} + \cancel{e^x\sin y} + 2\cos x\cdot\mathrm{ch}\,y + 3x^2y - y^3 + \varphi(x) =$

$= e^x(x\sin y + y\cos y) + 2\cos x\,\mathrm{ch}\,y + 3x^2y - y^3 + \varphi(x)$

$v'_x = e^x(x\sin y + y\cos y) + e^x\sin y - 2\sin x\,\mathrm{ch}\,y + 6xy + \varphi'(x) = -u'_y =$

$-\left(e^x(-x\sin y - (\sin y + y\cos y)) + 2\sin x\,\mathrm{ch}\,y - 6xy + 1\right) =$

$$= -\left(e^x(-x\sin y - \sin y - y\cos y) + 2\sin x\, ch y - 6xy + 1\right) =$$

$$= -e^x(-x\sin y - \sin y - y\cos y) - 2\sin x\, ch y + 6xy - 1$$

$$\cancel{e^x\cdot x\sin y} + \cancel{e^x \sin y} + \cancel{e^x y\cos y} - \cancel{2\sin x\, ch y} + \cancel{6xy} - 1 =$$

$$= \cancel{e^x\cdot x\sin y} + \cancel{e^x y\cos y} + \cancel{e^x\sin y} - \cancel{2\sin x\, ch y} + \cancel{6xy} + \varphi'(x)$$

$$\Rightarrow \varphi'(x) = -1.$$

$$\varphi(x) = -x + C$$

$f(z) = u + iv$

$$\begin{cases} u = e^x(x\cos y - y\sin y) + 2\sin x\, sh y + x^3 - 3xy^2 + y \\ v = e^x(x\sin y + y\cos y) + 2\cos x\, ch y + 3x^2y - y^3 - x + C \\ f(x,y) = e^x(x\cos y - y\sin y) + 2\sin x\, sh y + x^3 - 3xy^2 + y + i\left(e^x(x\sin y + y\cos y) + 2\cos x\, ch y + 3x^2y - y^3 - x\right) + C \end{cases}$$

1.167. $v = 3 + x^2 - y^2 - \frac{y}{2(x^2+y^2)}$

$$v'_y = -2y - \frac{2(x^2+y^2) - y\cdot 4y}{4(x^2+y^2)^2} = -2y - \frac{2x^2+2y^2-4y^2}{4(x^2+y^2)^2} = -2y - \frac{2x^2-2y^2}{4(x^2+y^2)^2} =$$

$$= -2y + \frac{y^2 - x^2}{2(x^2+y^2)^2} = \frac{y^2-x^2}{2(x^2+y^2)^2} - 2y = u'_x$$

$$u'_x = \frac{y^2}{2(x^2+y^2)^2} - \frac{x^2}{2(x^2+y^2)^2} - 2y$$

$$v'_x = 2x - \frac{-4xy}{4(x^2+y^2)^2} = 2x + \frac{xy}{(x^2+y^2)^2} = -u'_y$$

$$-u'_y = 2x + \frac{xy}{(x^2+y^2)^2}$$

$$u'_y = -2x - \frac{xy}{(x^2+y^2)^2}$$

$$u = -2xy - \frac{x}{2}\int (x^2+y^2)^{-2}\, d(x^2+y^2) = -2xy + \frac{x}{2(x^2+y^2)} + \varphi(x)$$

$$u'_x = -2y + \frac{2(x^2+y^2) - 4x^2}{4(x^2+y^2)^2} + \varphi'(x) = v'_y$$

$$v'_y = \frac{y^2-x^2}{2(x^2+y^2)^2} - \cancel{2y} = -\cancel{2y} + \frac{y^2-x^2}{2(x^2+y^2)^2} + \varphi'(x)$$

$$\Rightarrow \varphi'(x) = 0 \rightarrow \varphi(x) = C$$

$f(z) = u + iv.$

$$\begin{cases} u = -2xy + \frac{x}{2(x^2+y^2)} + \varphi(x) \\ v = 3 + x^2 - y^2 - \frac{y}{2(x^2+y^2)} \\ f(x,y) = -2xy + \frac{x}{2(x^2+y^2)} + i\left(3 + x^2 - y^2 - \frac{y}{2(x^2+y^2)}\right) + C \end{cases}$$

1.168 $v = \ln(x^2+y^2) + x - 2y$

$v'_y = \frac{2y}{x^2+y^2} - 2 = u'_x$

$v'_x = \frac{2x}{x^2+y^2} + 1 = -u'_y$

$u'_x = \frac{2y}{x^2+y^2} - 2 \rightarrow u = 2y \cdot \frac{1}{y} \operatorname{arctg}\frac{x}{y} - 2x + \varphi(y) =$

$= 2\operatorname{arctg}\frac{x}{y} - 2x + \varphi(y)$

$u'_y = 2 \cdot \frac{1}{1+\frac{x^2}{y^2}} \cdot \left(-\frac{x}{y^2}\right) + \varphi'(y) = -\frac{2 \cdot y^2}{(x^2+y^2)} \cdot \frac{x}{y^2} + \cancel{\varphi'(y)} = -\frac{2x}{x^2+y^2} + \varphi'(y)$

$u'_y = -v'_x$

$\varphi'(y) - \cancel{\frac{2x}{x^2+y^2}} = -\cancel{\frac{2x}{x^2+y^2}} - 1 \rightarrow \varphi'(y) = -1$

$\varphi(y) = -y + C$

$\Rightarrow u = 2\operatorname{arctg}\frac{x}{y} - 2x - y + C$

$v = \ln(x^2+y^2) + x - 2y$

$f(x,y) = 2\operatorname{arctg}\frac{x}{y} - 2x - y + i\left(\ln(x^2+y^2) + x - 2y\right) + C$

1.169. $u = \varphi(x)$

Ур-е Лапласа $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

$u'_x = \varphi'(x)$; $u''_x = \varphi''(x)$

$\Rightarrow \varphi'' = 0$

$\varphi'(x) = C_1 \longrightarrow \varphi(x) = C_1 x + C_2.$

Ответ: $\varphi(x) = C_1 x + C_2$

4.10.21 1.133

$f(z) = |x^2 - y^2| + 2i|xy|$

1) $x > 0,\ y > 0$ или $x < 0,\ y < 0$

а) $x^2 > y^2$: $f(z) = x^2 - y^2 + 2ixy$

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = 2x$

$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = -2y$ $\Rightarrow$ аналит.

б) $x^2 < y^2$: $f(z) = y^2 - x^2 + 2xiy$

$\frac{\partial u}{\partial x} = -2x$

$\frac{\partial v}{\partial y} = 2x$ $\Rightarrow$ неаналит.

2) $x > 0,\ y < 0$ или $x < 0,\ y > 0$

а) $x^2 > y^2$; $f(z) = x^2 - y^2 - 2ixy$

$\frac{\partial u}{\partial x} = 2x$; $\frac{\partial v}{\partial y} = -2x$ $\rightarrow$ неаналит.

б) $x^2 < y^2$: $f(z) = y^2 - x^2 - 2ixy$

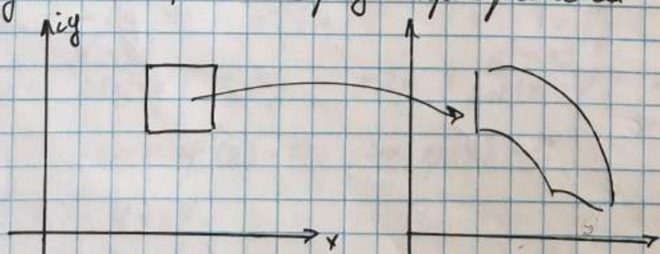
$\frac{\partial v}{\partial x} = \frac{\partial u}{\partial y} = -2x$; $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = 2y$ $\Rightarrow$ аналит.

Геометрический смысл модуля и аргумента производной

$w = f(z)$

$|w'(z_0)| = \alpha$ – растяжение пространства в т. z_0

$\arg w'(z_0) = \varphi$ – поворот пространства в т. z_0



1.187.

$w = z^2$, $w = z^3$

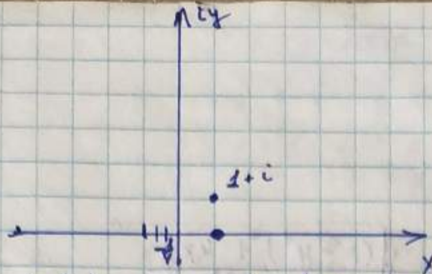
1) $z_0 = 1$ 2) $z_0 = -\frac{1}{4}$ 3) $z_0 = 1 + i$ 4) $z_0 = -3 + 4i$

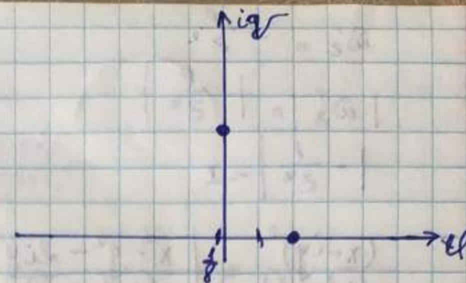
1) $w = z^2 \rightarrow w'_z = 2z_0$

$w'_z(1) = 2 \cdot 1 = 2$

$|\omega'_z(1)| = 2$

$\arg \omega'_z(1) = 0$

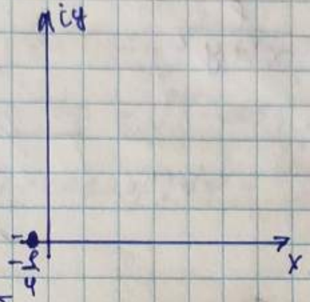


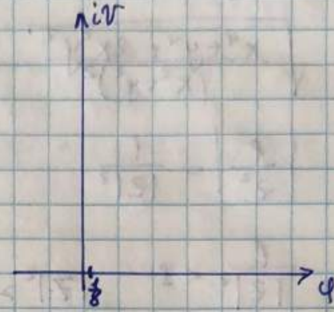


2) $\omega'_z\left(-\frac{1}{4}\right) = 2\cdot\left(-\frac{1}{4}\right) = -\frac{1}{2} = \frac{1}{2}\cdot e^{i\pi}$

$\left|\omega'_z\left(-\frac{1}{4}\right)\right| = \frac{1}{2}$

$\arg \omega_z\left(-\frac{1}{4}\right) = \pi$





3) $\omega'_z(1+i) = 2(1+i) = 2+2i$

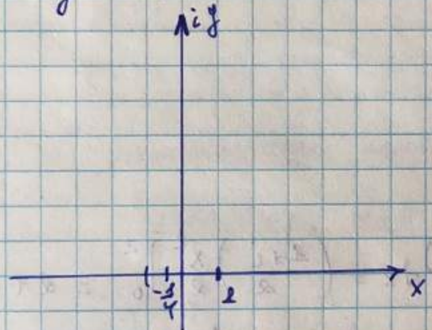
$|\omega'_z(1+i)| = \sqrt{2^2+2^2} = 2\sqrt{2}$

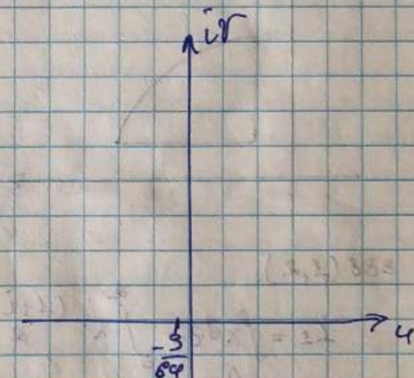
$\arg \omega'_z(1+i) = \operatorname{arctg}\frac{2}{2} = \frac{\pi}{4}$

4) $\omega'_z(-3+4i) = 2(-3+4i) = -6+8i$

$|\omega'_z(-3+4i)| = \sqrt{6^2+8^2} = 10$

$\arg \omega'_z(-3+4i) = \operatorname{arctg}\left(-\frac{4}{3}\right)$





$w = z^3$

$\omega'_z = 3z^2$

1) $\omega'_z(1) = 3\cdot 1 = 3$

$|\omega'_z(1)| = 3$

$\arg \omega'_z(1) = 0$

2) $\omega'_z\left(-\frac{1}{4}\right) = 3\cdot\left(-\frac{1}{4}\right)^2 = \frac{3}{16}$

$\left|\omega'_z\left(-\frac{1}{4}\right)\right| = \frac{3}{16}$

$\arg\left(\omega'_z\left(-\frac{1}{4}\right)\right) = 0$

Д/З: дорешать

1.188 $\alpha < 1$ сжимается, $\alpha > 1$ растяжение

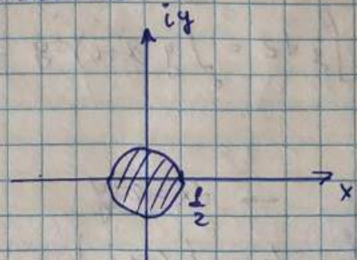
1) $\omega = z^2$

$\omega'_z = 2z$

$|\omega'_z| = |2z| < 1$

$|z| < \frac{1}{2}$ сжимается

$|z| > \frac{1}{2}$ растяжение



3) $\omega = \frac{1}{z}$

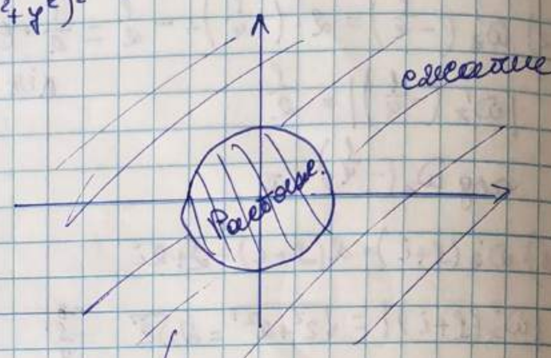
$\omega_z' = -\frac{1}{z^2}$

$|\omega_z'| = \left|-\frac{1}{z^2}\right|$

$\left|-\frac{1}{z^2}\right| < 1$

$\frac{(x-iy)^2}{(x^2+y^2)} = \left|\frac{x^2-y^2-2iy}{(x^2+y^2)}\right| = \sqrt{\frac{(x^2-y^2)^2+4xy}{(x^2+y^2)^2}}$

$\sqrt{\frac{x^4+y^4+2x^2y^2}{(x^2+y^2)^2}} = \frac{1}{x^2+y^2}$

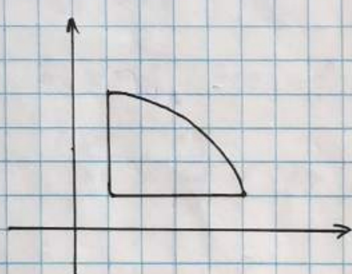


$\left|\frac{1}{z^2}\right| = \frac{1}{|z|^2}$

$\frac{1}{|z|^2} < 1 \qquad |z|^2 > 1$

$1 < x^2+y^2$ – внешняя часть круга

§3. Интегралы и степенные ряды.

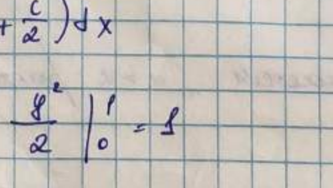


388 (1,2)

$I_1 = \int x\,dz = \int_0^2 x\cdot\frac{(1+i)}{2}\,dx = \left(1+\frac{i}{2}\right)\frac{x^2}{2}\Big|_0^2 = 2+i$

$I_2 = \int y\,dz$

1) радиус-вектор т. $z=2+i$



$y = \frac{1}{2}x$

$dy = \frac{1}{2}dx$

$dz = dx + i\,dy = dx + i\cdot\frac{1}{2}dx = \left(1+\frac{i}{2}\right)dx$

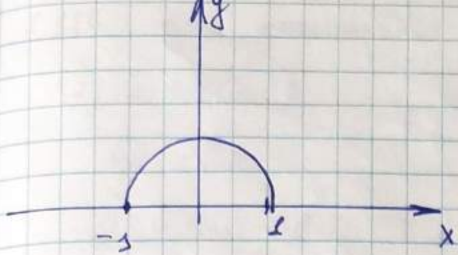
$I_2 = \int y\,dz = \int_0^1 y(2+i)\,dy = \frac{y^2}{2}\Big|_0^1 = 1$

$y = \frac{1}{2}x$

$dy = \frac{1}{2}dx \rightarrow dx = 2dy$

$dz = 2dy + i\,dy = dy(2+i)$

2) $|z| = 1 \qquad 0 \le \arg z \le \pi$



$x = r\cos\varphi$
$y = r\sin\varphi$

$r = 1 \Rightarrow x = \cos\varphi$
$y = \sin\varphi$

$dz = dx + i\,dy$
$\cancel{dx}$ $dx = -\sin\varphi\, d\varphi$
$dy = \cos\varphi\, d\varphi$ $\qquad dz = (-\sin\varphi + i\cos\varphi)\,d\varphi$

$$I_1 = \int_0^\pi \cos\varphi(-\sin\varphi + i\cos\varphi)\,d\varphi = \int_0^\pi(-\cos\varphi\sin\varphi + i\cos^2\varphi)\,d\varphi =$$
$$= \int_0^\pi \cos\varphi\, d(\cos\varphi) + \frac{i}{2}\int_0^\pi(1+\cos 2\varphi)\,d\varphi = \frac{i\pi}{2}$$

$$I_2 = \int y\,dz$$

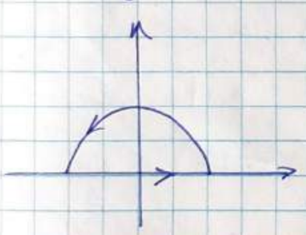
$$dz = dx + i\,dy = -\sin\varphi\,d\varphi + i\cos\varphi\,d\varphi = (-\sin\varphi + i\cos\varphi)\,d\varphi$$

$$I_2 = \int_0^\pi \sin\varphi(-\sin\varphi + i\cos\varphi)\,d\varphi = \int_0^\pi(-\sin^2\varphi + i\sin\varphi\cos\varphi)\,d\varphi =$$
$$= -\int_0^\pi \sin^2\varphi\,d\varphi + \frac{i}{2}\int_0^\pi \sin 2\varphi\,d\varphi = -\int_0^\pi \frac{1-\cos 2\varphi}{2}\,d\varphi + \frac{i}{4}(-\cos 2\varphi)\Big|_0^\pi =$$
$$= -\frac{\pi}{2}$$

390 (3, 5)

$I = \oint_C |z|\bar z\,dz$ $\qquad$ C - замкнутый контур сост из $|z| = 1$ и

$-1 \le x \le 1,\ y = 0$



$I_1 = z = r\cdot e^{i\varphi}$

$\bar z = r\cdot e^{-i\varphi}$

$dz = r\cdot e^{i\varphi}\,i\,d\varphi = e^{i\varphi}\,i\,d\varphi$; $r = 1$

$$I_1 = \int_0^\pi e^{-i\varphi}\cdot i\cdot e^{i\varphi}\,d\varphi = i\int_0^\pi d\varphi = i\pi$$

I_2: $z = x$; $\bar z = x$; $|z| = |x|$ $\qquad dz = dx$

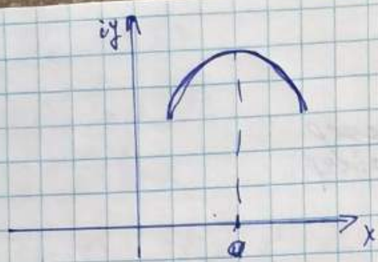
$$I_2 = \int_{-1}^1 |x|\cdot x\,dx = -\int_{-1}^0 x^2\,dx + \int_0^1 x^2\,dx = -\frac{x^3}{3}\Big|_{-1}^0 + \frac{x^3}{3}\Big|_0^1 = -\frac{1}{3} + \frac{1}{3} = 0$$

$\Rightarrow I = i\pi$

392 (3.7.)

$$\int (z-a)^n\,dz$$

1) $|z-a| = R,\quad 0 \le \arg(z-a) \le \pi$



$z = a + R$

$z - a = r \cdot e^{i\varphi}$

$|z - a| = r$

$(z-a)^n = r^n e^{i\varphi n}$

$z = a + r e^{i\varphi}$

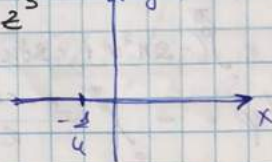
$dz = i e^{i\varphi} r \, d\varphi$

$$\int_0^{\pi} r^n e^{i\varphi n} r i e^{i\varphi} d\varphi = i r^{n+1} \int_0^{\pi} e^{i\varphi(n+1)} d\varphi = \frac{i r^{n+1} \cdot e^{i\varphi(n+1)}}{i(n+1)} \Big|_0^{\pi} =$$

$$= \frac{r^{n+1}}{n+1}\left(e^{i\pi(n+1)} - 1\right) = \frac{r^{n+1}}{n+1}\left((-1)^{n+1} - 1\right)$$

Домашняя работа.

1.187. $\omega = z^2$ и $\omega = z^3$



2) $z_0 = -\frac{1}{4}$

$\omega'_z = 2z$

$\left|\omega'_z\left(-\frac{1}{4}\right)\right| = \left|-2 \cdot \frac{1}{4}\right| = \left|-\frac{1}{2}\right| = \frac{1}{2}$ $\Rightarrow k = \frac{1}{2}$

$\arg\left(\omega'_z\left(-\frac{1}{4}\right)\right) = \pi$ $\Rightarrow \vartheta = \pi$

$\omega = z^3 \rightarrow \omega'_z = 3z^2$

$\left|\omega'_z\left(-\frac{1}{4}\right)\right| = \frac{3}{16}$ $\Rightarrow k = \frac{3}{16}$

$\arg\left(\omega'_z\left(-\frac{1}{4}\right)\right) = 0$ $\Rightarrow \vartheta = 0$

4) $z_0 = -3 + 4i$; $\omega = z^2$

$|\omega'_z(-3+4i)| = |-6+8i| = \sqrt{36+64} = 10 = k$

$\arg(\omega'_z(-3+4i)) = \operatorname{arctg}\left(-\frac{4}{3}\right) = \vartheta$

$\omega = z^3$

~~$|\omega'_z(-3+4i)| \Rightarrow$~~

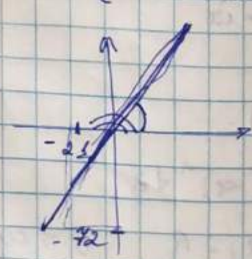
$\omega'_z(z_0) = 3 \cdot (-3+4i)^2 = 3(9 - 24i - 16) = 3(-7 - 24i) =$

$= -21 - 72i$

$|\omega'_z(z_0)| = \sqrt{21^2 + 72^2} = 75 = k$

$\arg(\omega'_z(z_0)) = \cancel{\operatorname{arctg} \frac{24}{7}}$

$= \pi + \operatorname{arctg} \frac{24}{7}$



1.188. $\alpha < 1$ сжимается; $\alpha > 1$ - растяжение.

2) $\omega = z^2 + 2z$

$\omega'_z = 2z + 2$

$|2z+2| < 1$ => Сжатие при $|z+1| < \frac{1}{2}$

растяжение при $|z+1| > \frac{1}{2}$.

4) $\omega = e^z$

$\omega'_z = e^z$

$|e^z| < 1$ →

$|e^{x+iy}| < 1$

5) $\omega = \ln(z-1)$

$\omega' = \frac{1}{z-1}$

$\left|\frac{1}{z-1}\right| < 1$ → Сжатие при $|z-1| > 1$

растяжение при $|z-1| < 1$

387. (3.2) Пусть C - замкнутый контур, огранич. площадь S, док-ть следующие равенства.

1) $\int\limits_C x\,dz = iS$

$z = x+iy$; $dz = dx + i\,dy$

$\int\limits_C x(dx + i\,dy) = \int\limits_C x\,dx + i\int\limits_C x\,dy = \int\limits_C x\,dx + ix\,dy$

$\frac{\partial Q}{\partial x} = i$; $\frac{\partial P}{\partial y} = 0$

$=> \int\limits_C P\,dx + Q\,dy = \iint\limits_S \left\{\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right\} dx\,dy$

$=> \int\limits_C x\,dz = \int\limits_C x\,dx + ix\,dy = \iint\limits_S (i-0)\,dx\,dy = i\iint\limits_S dx\,dy = iS$, ч.т.д.

2) $\int\limits_C y\,dz = -S$

$\int\limits_C y(dx + i\,dy) = \int\limits_C y\,dx + iy\,dy$.

$\frac{\partial Q}{\partial x} = 0$; $\frac{\partial P}{\partial y} = 1$.

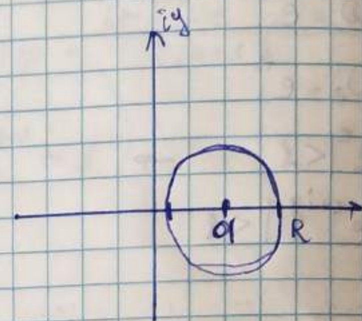
$=> \int\limits_C y\,dz = \int\limits_C y\,dx + iy\,dy = \iint\limits_S (0-1)\,dx\,dy = -S$, ч.т.д.

3) $\int\limits_c \bar{z}\,dz = 2iS$

$$\int\limits_c (x-iy)(dx+idy) = \int\limits_c x\,dx - iy\,dx + ix\,dy + y\,dy =$$

$$= \int\limits_c (x-iy)\,dx + (y+ix)\,dy = \iint\limits_S 2i\,dx\,dy = 2iS$$

$$\frac{\partial Q}{\partial x} = i \ ; \ \frac{\partial P}{\partial y} = -i$$



<u>388(3).</u> $I_1 = \int x\,dz$, $I_2 = \int y\,dz$

3) По окружности $|z-a| = R$

$z - a = (x-a) + iy$

$(x-a)^2 + y^2 = R^2$

$$\begin{cases} x - a = r\cos\varphi \\ y = r\sin\varphi \end{cases} \rightarrow \begin{matrix} dx = -R\sin\varphi\,d\varphi \\ dy = R\cos\varphi\,d\varphi \end{matrix}$$

$dz = dx + idy = -R\sin\varphi\,d\varphi + iR\cos\varphi\,d\varphi =$

$$I_1 = \int\limits_0^{2\pi} (a + R\cos\varphi)(-R\sin\varphi + iR\cos\varphi)\,d\varphi =$$

$$= \int\limits_0^{2\pi} -aR\sin\varphi\,d\varphi + iRa\int\limits_0^{2\pi}\cos\varphi\,d\varphi - R^2\int\limits_0 \sin\varphi\,d(\sin\varphi) +$$

$$+ iR^2\int\limits_0^{2\pi}\frac{1+\cos2\varphi}{2}\,d\varphi =$$

$$= -aR\cos\varphi\Big|_0^{2\pi} + 0 - 0 + \frac{iR^2}{2}\cdot 2\pi + \underbrace{\frac{iR^2}{2}\cdot\frac{\sin2\varphi}{2}\Big|_0^{2\pi}}_{=0} =$$

$$= -aR(1-1) + iR^2\pi = i\pi R^2$$

$$I_2 = \int\limits_0^{2\pi} R\sin\varphi(-R\sin\varphi + iR\cos\varphi)\,d\varphi =$$

$$= \int\limits_0^{2\pi} -R^2\sin^2\varphi\,d\varphi + i\int\limits^{2\pi} R^2\sin\varphi\,d(\sin\varphi) =$$

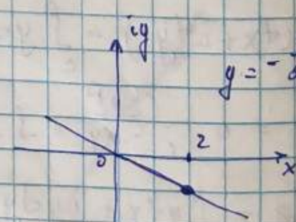
$$= -R^2\int\limits_0^{2\pi}\frac{1-\cos2\varphi}{2} + \underbrace{iR^2\cdot\frac{\sin^2\varphi}{2}\Big|_0^{2\pi}}_{=0} =$$

$$= -\frac{R^2}{2}\int\limits_0^{2\pi}d\varphi + \frac{R^2}{2}\int\limits_0^{2\pi}\cos2\varphi\,d\varphi = -R^2\pi + \underbrace{\frac{R^2}{2}\cdot\frac{\sin2\varphi}{2}\Big|_0^{2\pi}}_{=0} =$$

$$= -R^2\pi$$

<u>389 (3.4.)</u>

Вычислить $\int |z|\,dz$



1) по радиус-вектору точки $z = 2 - i$

$|z| = \sqrt{2^2+1^2} = \sqrt{5}$

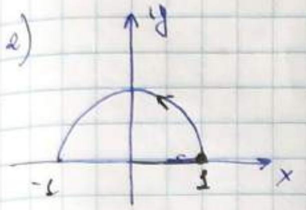
$dy = -\frac{1}{2}dx$

$$dz = dx - \frac{1}{2} i\, dx = \left(1 - \frac{1}{2} i\right) dx$$

$$|z| = \sqrt{x^2 + \frac{1}{4} x^2} = x \frac{\sqrt{5}}{2} \quad (x \geq 0)$$

$$\int_C |z|\, dz = \frac{\sqrt{5}}{2} \int_0^2 x\left(1 - \frac{1}{2} i\right) dx = \frac{\sqrt{5}(2-i)}{2}$$

2)

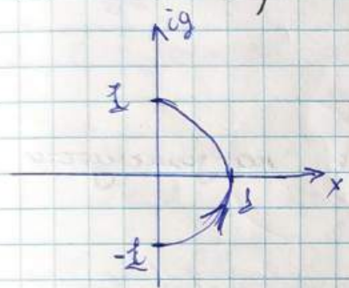


$$\begin{cases} x = r\cos\varphi \\ y = r\sin\varphi \end{cases} \qquad |z| = 1, \quad 0 \leq \arg z \leq \pi$$

$$\int_C |z|\, dz = \int_C dz = \int_C dx + i\, dy = \int_0^{\pi} -\sin\varphi\, d\varphi + i\cos\varphi\, d\varphi =$$

$$= \int_0^{\pi} (i\cos\varphi - \sin\varphi)\, d\varphi = \cos\varphi \Big|_0^{\pi} = -2$$

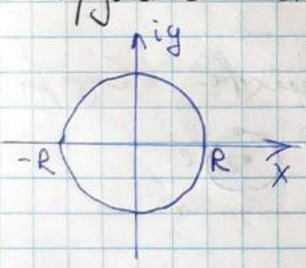
3) По полуокружности $|z| = 1$, $-\frac{\pi}{2} \leq \arg z \leq \frac{\pi}{2}$ (начало пути в точке $z = -i$)



$$\int_C |z|\, dz = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (i\cos\varphi - \sin\varphi)\, d\varphi =$$

$$= i\sin\varphi \Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = 2i$$

4) По окружности $|z| = R$



$$\int_C |z|\, dz = \int_C R\, dz = R\int_0^{2\pi} (i\cos\varphi - \sin\varphi)\, d\varphi =$$

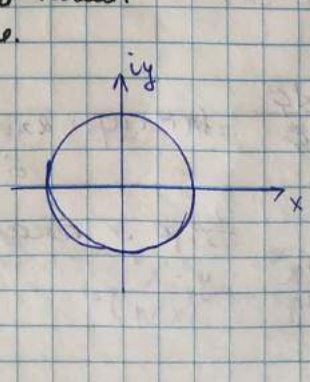
$$= \underbrace{iR\sin\varphi \Big|_0^{2\pi}}_{=0} + R\cos\varphi \Big|_0^{2\pi} = 0$$

3.2. $\int_C x\, dx + i x\, dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\, dy = i \iint_D dx\, dy = iS$

7.10.21.

Интегральная формула Коши. Теорема Коши.

394. $\int_C \operatorname{Ln} z\, dz$



1) $|z| = 1 \qquad \operatorname{Ln} 1 = 0$

$$z = r \cdot e^{i\varphi} = e^{i\varphi} \quad (r = 1)$$

$$\operatorname{Ln} z = \ln 1 + i(\varphi + 2\pi n)$$

$dz = ie^{i\varphi}d\varphi$

$$\int_C \ln z\, dz = i\int_0^{2\pi k} i(\varphi+2\pi n)e^{i\varphi}d\varphi = -\int_0^{2\pi k}(\varphi+2\pi n)e^{i\varphi}d\varphi =$$

$$= -\int_0^{2\pi k}\varphi\cdot e^{i\varphi}d\varphi \underbrace{- 2\pi n \frac{e^{i\varphi}}{i}\Big|_0^{2\pi k}}_{=0} = -\frac{\varphi e^{i\varphi}}{i}\Big|_0^{2\pi k} + \frac{1}{i}\int \frac{1}{i}e^{i\varphi}\Big|_0^{2\pi k} =$$

$$= \frac{2\pi k e^{i2\pi k}}{i} = i\cdot 2\pi k$$

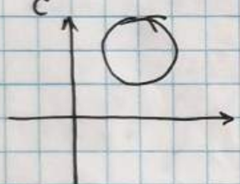
$\cancel{2) \oint \ln z = \frac{\pi i}{2}}$

4) $\ln z = \ln R + i(\varphi+2\pi k)$

$$\int_C \ln z\,dz = \int_0^{2\pi k}(\ln R + i(\varphi+2\pi n))iRe^{i\varphi}d\varphi = \underbrace{\int_0^{2\pi k}(iR\ln R - 2\pi n R)e^{i\varphi}d\varphi}_{=0}$$

$$= 2\pi k R\cdot\frac{e^{i2\pi k}}{i} = 2\pi k i R$$

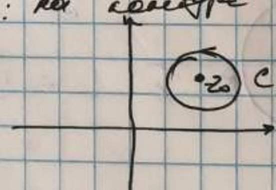
1) $\oint_C f(z)dz = 0$ Если $f(z)$ - аналитична на замкнутом контуре.



$\int_{z_1}^{z_2} f(z)dz$, когда аналит., непрерывна, то $\to$ не зависит от формы кривой

2) Если ф-я аналитична и непрерывна: на контуре и внутри него

$$\frac{1}{2\pi i}\int \frac{f(z)}{z-z_0}dz = f(z_0)$$



дифф. по z_0:

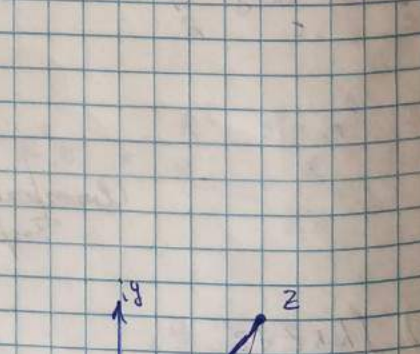
$$\frac{1}{2\pi i}\, 1\cdot\int_C \frac{f(z)}{(z-z_0)^2}dz = f'(z_0)$$

$$\frac{1}{2\pi i}\cdot 2\int_C \frac{f(z)}{(z-z_0)^3}dz = f''(z_0)$$

$$\vdots$$

$$\to \frac{n!}{2\pi i}\int_C\frac{f(z)}{(z-z_0)^{n+1}} = f^{(n)}(z_0)$$

403 (3.18) $\int_1^z \frac{d\zeta}{\zeta} = \ln r + i\varphi + 2\pi i k$ ⊖



$\frac{1}{z} = \frac{x-iy}{x^2+y^2}$ → ф-я аналит.

$u'_x = \frac{x^2+y^2-2x^2}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$

$v'_y = \frac{y^2-x^2}{(x^2+y^2)^2}$

$$\int_1^z \frac{d\xi}{\xi} = \int_{C_1} + \int_{C_2} = \int_0^{\varphi+2\pi k} \frac{i e^{i\alpha} d\alpha}{e^{i\alpha}} + \int_1^r \frac{e^{i\varphi} d\rho}{\rho \cdot e^{i\varphi}} = \ln r - \ln 1 + i(\varphi + 2\pi k)$$

C_1: $\rho = 1$

$z = \rho \cdot e^{i\alpha}$, $\xi = \rho \cdot e^{i\alpha} = e^{i\alpha}$ (на круге)

$d\xi = i e^{i\alpha} d\alpha$

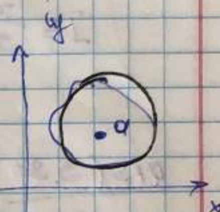
C_2:

$\xi = \rho e^{i\varphi}$

$d\xi = e^{i\varphi} d\rho$

ч.т.д.

405. (3.20)



$$\int_C (z-a)^n dz = \begin{cases} 0, & \text{если } n \neq -1 \\ 2\pi i, & \text{если } n = -1, \ a \text{ внутри } C \\ 0, & \text{если } n = -1, \ a \text{ вне } C \end{cases}$$

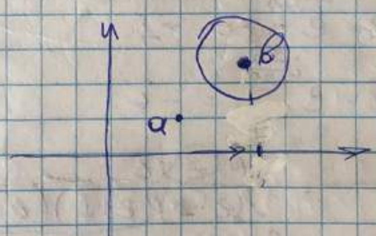
Не зависит от контура, выберем окружность

$$\int_{C,\ n\neq -1} (z-a)^n dz = \int_0^{2\pi k} R^n e^{i\varphi n} R i e^{i\varphi} d\varphi = R^{n+1} \frac{\cancel{i} e^{i\varphi(n+1)}}{\cancel{i}(n+1)} \Big|_0^{2\pi k} =$$

$$\left.\begin{aligned} z &= a + R e^{i\varphi} \\ dz &= R i e^{i\varphi} d\varphi \end{aligned}\right\} = \frac{R^{n+1}}{n+1} \left(\underline{\underline{e^{i2\pi k(n+1)}}} - 1 \right) \equiv 0$$

$n = -1$:

$$\int_C (z-a)^{-1} dz = \int_0^{2\pi k} \frac{R i e^{i\varphi}}{R \cdot e^{i\varphi}} d\varphi = 2\pi k i = \{k = 1\} = 2\pi i$$



$n = -1$: Если a вне контура:

По теореме Коши:

$$\int_C (z-a)^{-1} dz = 0$$

ч.т.д.

$$\int_C (z-a)^{-1} dz = \int_0^{2\pi} \frac{i R e^{i\varphi}}{(b-a) + R e^{i\varphi}} d\varphi = \int_0^{2\pi} \frac{d[e^{i\varphi} R + (b-a)]}{(b-a) + R e^{i\varphi}} =$$

$$= \ln[(b-a) + R e^{i\varphi}] \Big|_0^{2\pi} = 0$$

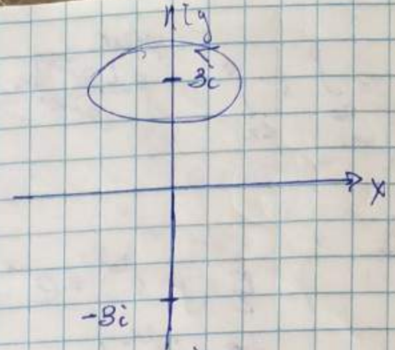
412 (3.27) Вычислить интеграл $\int_C \frac{dz}{z^2+9}$, если:

1) $3i$ внутри C; $-3i$ вне C

По формуле Коши: $\frac{1}{2\pi i} \int_C \frac{f(z)}{z - z_0} dz = f(z_0)$

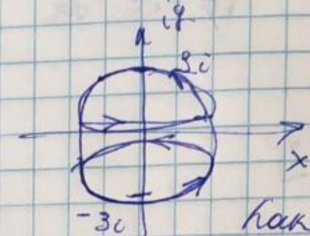
$$\oint_C \frac{dz}{z^2+9} = \oint \frac{1\,dz}{(z-3i)(z+3i)} \quad f(z) =$$

$$= 2\pi i \cdot \frac{1}{3i+3i} = \frac{\pi}{3}$$



2) $-3i$ —//—, $3i$ —//—

$$\oint_C \frac{dz}{z^2+9} = -\frac{\pi}{3}$$



3) $-3i$ и $3i$ внутри C.

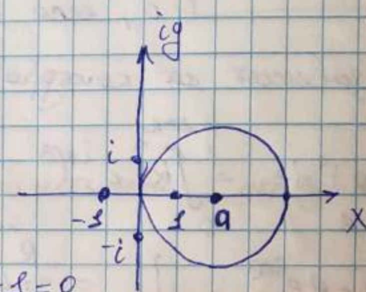
$$I = \frac{\pi}{3} - \frac{\pi}{3} = 0$$

Как сумму интегралов

415(3.30).

$$\oint_{|z-a|=a} \frac{z\,dz}{z^4-1}, \quad a > 1$$

$$\oint_C \frac{z\,dz}{(z^2-1)(z^2+1)}$$



Найдём все точки, которые: $z^4-1=0$

$z^4=1$

$z = \sqrt[4]{1}$; $z = e^{i\frac{2\pi k}{4}}$

Попала только точка 1

$k=0$; $z=1$

$k=1$; $z = e^{i\frac{\pi}{2}} = i\sin\frac{\pi}{2} = i$

$k=2$; $z = e^{i\pi} = -1$

$k=3$; $z = e^{i\frac{3\pi}{2}} = -i$

$$\oint_C \frac{z\,dz}{z^4-1} = \oint_C \frac{z\,dz}{(z-1)(z+1)(z-i)(z+i)} = \frac{2\pi i}{2(1-i)(1+i)} =$$

$$= \frac{2\pi i}{2(1+1)} = \frac{2\pi i}{4} = \frac{\pi i}{2}$$

418(3.33) $\frac{1}{2\pi i}\oint_C \frac{e^z\,dz}{z(1-z)^3}$

1) Если т.0 внутри C, 1 - вне C.

$$\frac{1}{2\pi i}\oint \frac{-e^z\,dz}{z(z-1)\cdot(z-1)^2} = \frac{-1}{1} = 1$$

2) 1 внутри, 0 - вне C

$$\frac{1}{2\pi i}\oint \frac{-e^{z}dz}{z(z-1)(z-1)^2} = -\frac{e}{0}$$

$$\frac{n!}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^{n+1}} = f^{(n)}(z_0)$$

$-e \qquad n=2$

$$f(z) = \frac{-e^{z}}{z}$$

$$f'(z) = -\left(\frac{e^{z}\cdot z - e^{z}}{z^2}\right) = \frac{e^{z}(1-z)}{z^2} = \frac{e^{z}}{z^2} - \frac{e^{z}}{z}$$

$$f''(z) = \frac{e^{z}z^2 - 2ze^{z}}{z^4} - \frac{e^{z}\cdot z - e^{z}}{z^2} = \frac{e^{z}z - 2e^{z} - e^{z}\cdot z^2 + e^{z}z}{z^3} =$$

$$= \frac{2e^{z}z - 2e^{z} - e^{z}\cdot z^2}{z^3}$$

$$f''(1) = \frac{2e - 2e - e}{1} = -e$$

$$I = -\frac{e}{2}$$

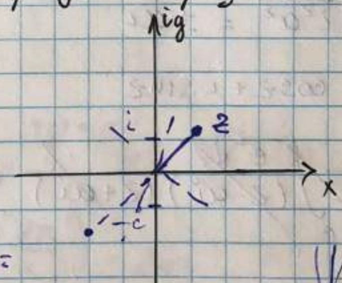
3) 0 и 1 лежат внутри C.

$$I = -\frac{e}{2} + 1 = 1 - \frac{e}{2}$$

Домашняя работа.

3.19. $\displaystyle\int_0^1 \frac{d\xi}{1+\xi^2} = \frac{\pi}{4} + k\pi$, если путь не проходит через точки $\pm i$

$$\frac{1}{1+z^2}$$



$$\int_0^1 \frac{d\xi}{1+\xi^2} = \operatorname{Arctg}(\xi)\Big|_0^1 = \frac{\pi}{4} + \pi k$$

$$= \frac{i}{2}\ln\frac{i+1}{i-1} = \frac{i}{2}\ln\frac{2i}{(-2)} = \frac{i}{2}\ln(-i) =$$

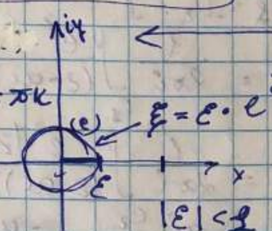
$$= \frac{i}{2}\left(\ln 1 + i\left(-\frac{\pi}{2} + 2\pi k\right)\right) = -\frac{1}{2}\left(-\frac{\pi}{2} + 2\pi k\right) = \frac{\pi}{4} + \pi k$$

$\operatorname{Arg}(u+iv) = \operatorname{arctg}\dfrac{2\varepsilon}{1-\varepsilon^2} + 2\pi k$

$\varepsilon \to 0 \quad \dfrac{2\varepsilon}{1-\varepsilon^2} \to 0 \quad \Rightarrow \quad I_2 = -i(i\pi k) = \pi k$

$\varepsilon \to \infty \quad \dfrac{2\varepsilon}{1-\varepsilon^2} \to 0$

$$I = \int_0^{\varepsilon} + \int_{0\,(c)}^{2\pi k} + \int_{\varepsilon}^{1}$$



$\xi = \varepsilon\cdot e^{i\varphi}$

$|\varepsilon| < 1$

$$I_1 + I_3 = \int_0^{\varepsilon}\frac{dx}{1+x^2} + \int_{\varepsilon}^{1}\frac{dx}{1+x^2} = \operatorname{arctg} x\Big|_0^{\varepsilon} + \operatorname{arctg} x\Big|_{\varepsilon}^{1} =$$

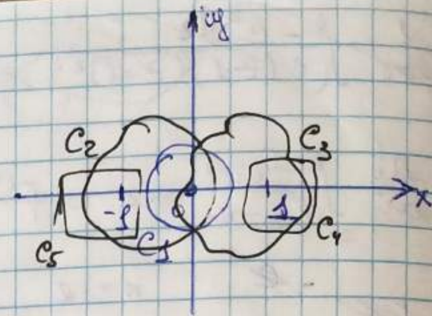
$$= \operatorname{arctg}\varepsilon - \operatorname{arctg} 0 + \operatorname{arctg} 1 - \operatorname{arctg}\varepsilon = \frac{\pi}{4}$$

$$I_2 = \int_0^{2\pi k}\frac{\varepsilon\, de^{i\varphi}}{1+(\varepsilon e^{i\varphi})^2} = \operatorname{Arctg}\varepsilon e^{i2\pi k} = -\frac{i}{2}\ln\frac{1+i\varepsilon e^{i2\pi k}}{1-i\varepsilon e^{i2\pi k}} = -\frac{i}{2}\ln\frac{1+i\varepsilon}{1-i\varepsilon} =$$

$$= -\frac{i}{2}\ln\frac{(1+i\varepsilon)^2}{1+\varepsilon^2} = -\frac{i}{2}\ln\frac{1-\varepsilon^2+2i\varepsilon}{1+\varepsilon^2} \quad \begin{cases} u = \dfrac{1-\varepsilon^2}{1+\varepsilon^2} \\ v = \dfrac{2\varepsilon}{1+\varepsilon^2}\end{cases} \quad \rho = \sqrt{\frac{(1-\varepsilon^2)^2 + 4\varepsilon^2}{(1+\varepsilon^2)^2}} = 1.$$

3.28. $\int\limits_C \frac{dz}{z(z^2-1)} = \int\limits_C \frac{dz}{z(z-1)(z+1)}$

$\frac{1}{2\pi i} \int \frac{f(z)}{(z-z_0)} dz = f(z_0)$



$\oint\limits_{C_1} \frac{dz}{z(z-1)(z+1)} = 2\pi i \cdot (-1) = -2\pi i$

$\oint\limits_{C_2} \frac{dz}{z(z-1)(z+1)} = 2\pi i \cdot \frac{1}{-2} = -\pi i$

$\oint\limits_{C_3} \frac{dz}{z(z-1)(z+1)} = 2\pi i \cdot \frac{1}{2} = \pi i$

$\oint\limits_{C_4} \frac{dz}{z(z-1)(z+1)} = 2\pi i \cdot \frac{1}{2} = \pi i$

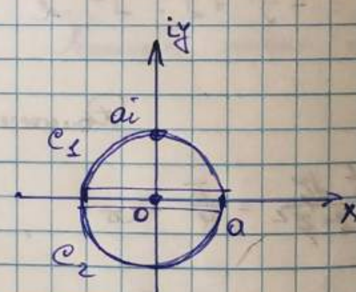
C_6 - содер. -1 и 1

$\oint\limits_{C_6} \frac{dz}{z(z-1)(z+1)} = 2\pi i \cdot 1 = 2\pi i$

C_7 - ничего не содержит $\rightarrow \oint\limits_{C_7} f(z)\,dz = 0$

Ответ: $\{-2\pi i;\ -\pi i;\ 0;\ \pi i;\ 2\pi i\}$

3.31. $\frac{1}{2\pi i} \int\limits_C \frac{e^z dz}{z^2+a^2}$ $\quad C: \ |z| \leq a$



$z^2 = -a^2 = i^2a^2$

$z = \sqrt{i^2a^2} = \pm ai$

$e^z = \cos z + i\sin z$

$\cancel{\frac{1}{2\pi i} \int\limits_C \frac{e^z dz}{(z-ai)(z+ai)}} = \frac{e^{ai}}{2ai} \cancel{= \frac{\cos ai + i\sin ai}{2ai}} = \frac{ch a - sh a}{\cancel{2ai}} =$

$=$

1: $I_1 + I_2 = I$

$I_1 = \frac{1}{2\pi i} \oint\limits_{C_1} \frac{e^z dz}{(z-ai)(z+ai)} = \frac{e^{ai}}{2ai}$

$I_2 = \frac{1}{2\pi i} \oint\limits_{C_2} \frac{e^z dz}{(z-ai)(z+ai)} = \frac{e^{-ai}}{-2ai}$

$I = \frac{e^{ai}}{2ai} - \frac{e^{-ai}}{2ai} = \frac{e^{ai} - e^{-ai}}{2i \cdot a} = \frac{\sin a}{a}$

Ответ: $I = \frac{\sin a}{a}$

3.32. $\frac{1}{2\pi i}\int\limits_C \frac{z\cdot e^z\,dz}{(z-a)^3}$, если $a\in C$

$$\frac{n!}{2\pi i}\int\limits_C \frac{f(z)\,dz}{(z-z_0)^{n+1}} = f^{(n)}(z_0)$$

$n=2$

$f(z) = z\cdot e^z$ $\qquad \frac{1}{2\pi i}\int\limits_C \frac{F(z)dz}{(z-z_0)^{n+1}} = \frac{f^{(n)}(z_0)}{n!}$

$f'_z = e^z + z\cdot e^z = e^z + f(z)$

$f''_z = e^z + f'_z = e^z + e^z + z\cdot e^z = 2e^z + z\cdot e^z = e^z(2+z)$

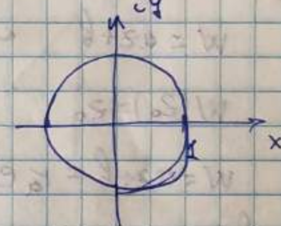
$f''(a) = e^a(2+a)$

$$\Rightarrow I = \frac{e^a(2+a)}{2} = e^a\left(1+\frac{a}{2}\right)$$

Ответ: $I = e^a\left(1+\frac{a}{2}\right)$

3.9 $\int\limits_C \ln z\,dz$, где

а) C - единичная окружность и $\ln i = \frac{\pi i}{2}$



$|z| = 1$

$z = r\cdot e^{i\varphi}$

$\ln z = \ln 1 + i(\varphi + 2\pi n) = i(\varphi + 2\pi n)$

$dz = ie^{i\varphi}d\varphi$

$\int\limits_C \ln z\,dz = i\int\limits_0^{2\pi} i(\varphi+2\pi n)e^{i\varphi}d\varphi = -\int\limits_0^{2\pi}(\varphi+2\pi n)e^{i\varphi}d\varphi$

$\ln|z| = i\varphi$

$dz = ie^{i\varphi}d\varphi$

$\int\limits_0^{2\pi} i\varphi\cdot ie^{i\varphi}d\varphi = -\int\limits_0^{2\pi}\varphi e^{i\varphi}d\varphi = -\left\{\begin{matrix} \varphi = u \to d\varphi = du \\ dv = e^{i\varphi}d\varphi \to v = \frac{e^{i\varphi}}{i}\end{matrix}\right\} =$

$= -\left(\frac{\varphi e^{i\varphi}}{i}\Big|_0^{2\pi} - \underbrace{\int\limits_0^{2\pi}\frac{e^{i\varphi}}{i}d\varphi}_{=0}\right) = -\left(\frac{2\pi}{i}\right) = 2\pi i$

3) C: $|z| = R$; $\ln R = \ln R$

$\ln z = \ln R + i(\varphi + 2\pi n)$

$z = R\cdot e^{i(\varphi+2\pi n)}$

$\int\limits_0^{2\pi}(\ln R + i\varphi + 2\pi n i)iRe^{i(\varphi+2\pi n)}d\varphi = -\int\limits_0^{2\pi}R\varphi e^{i(\varphi+2\pi n)}d\varphi = -R\cdot\left(\frac{e^{i2\pi}\cdot 2\pi}{i}\right) =$

$= 2\pi R i$

14.10.21. 404 (продолж.)

$$-\frac{i}{2}\ln\frac{1+iz}{1-iz} = -\frac{i}{2}\ln\frac{\rho e^{i\varphi_1}}{\rho e^{i\varphi_2}} = -\frac{i}{2}\ln e^{i(\varphi_1-\varphi_2)} = \frac{-i\cdot i}{2}(\varphi_1-\varphi_2)$$

$$z = r\cdot e^{i\varphi}$$

$$1-y+ix \Rightarrow \operatorname{arctg}\frac{x}{1-y}$$

$$1+y-ix \Rightarrow \operatorname{arctg}\frac{-x}{1+y}$$

$$tg(\alpha_1-\alpha_2) = \frac{tg\alpha_1+tg\alpha_2}{1-tg\alpha_1\cdot tg\alpha_2} \to \frac{\frac{x}{1-y}-\frac{x}{1+y}}{1-\frac{x}{1-y}\cdot\frac{x}{1+y}} = \frac{2xy}{1-x^2+y^2}$$

Конформное отображение.

2.2. (194)

$z_0 = 1+2i$

$w(i) = -i$

$w = az+b$ — линейное преобразование.

$w(z_0) = z_0$

$w = az+b = r_a e^{i\alpha}\cdot\rho e^{i\varphi}+b$

$$\begin{cases} 1+2i = a(1+2i)+b \\ -i = ai+b \end{cases}$$

$b = -i-ai$

$1+2i = a+2ai-i-ai$

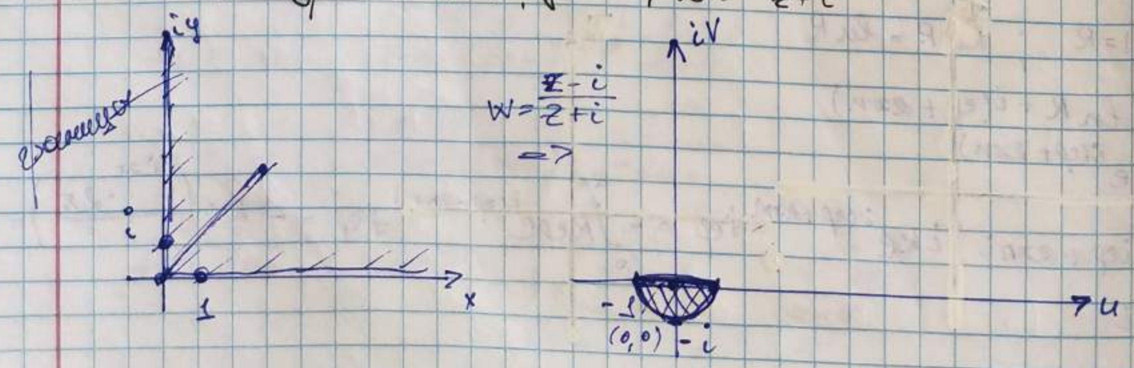
$1+3i = a+ai \to a = \frac{1+3i}{1+i} = \frac{(1+3i)(1-i)}{2} =$

$= \frac{1-i+3i+3}{2} = \frac{4+2i}{2} = 2+i$

$b = -i-i(2+i) = -i-2i+1 = 1-3i$

$w = (2+i)z+(1-3i)$

2.11. (203) Квадрант $x>0,\ y>0$, $w = \frac{z-i}{z+i}$



$\varphi_2)$

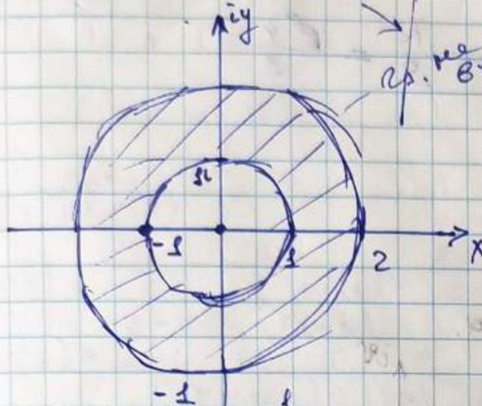
$$w = \frac{1-i}{1+i} = \frac{(1-i)^2}{2} = -i$$

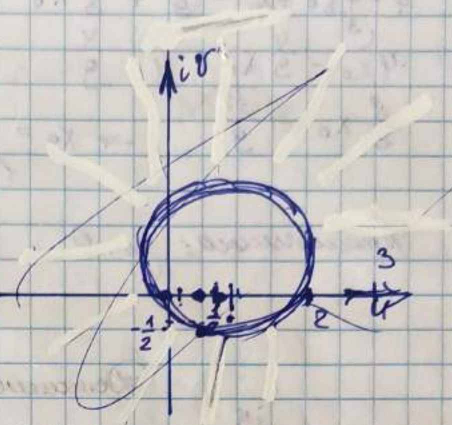
$$w(1+i) = \frac{1+i-i}{1+2i} = \frac{1}{1+2i} = \frac{1-2i}{5} = \frac{1}{5} - \frac{2}{5}i$$

2.15 (207)

Кольцо $1 < |z| < 2$; $w = \frac{z}{z-1}$

гр. не входят





$$w(-1) = \frac{-1}{-2} = \frac{1}{2}$$

$$w(1) \to +\infty$$

$$w(2) = 2 \ ; \ w(-2) = \frac{-2}{-3} = \frac{2}{3}$$

$$w(i) = \frac{i}{i-1} = \frac{i(i+1)}{-2} = \frac{-1+i}{-2} = \frac{1}{2} - \frac{1}{2}i$$

$$w(2i) = \frac{2i}{2i-1} = \frac{2i(2i+1)}{-4-1} = \frac{-4+2i}{-5} = \frac{4}{5} - \frac{2}{5}i$$

$$z = 2e^{i\varphi} \Rightarrow w = \frac{2e^{i\varphi}}{2e^{i\varphi}-1} = \frac{2e^{i\varphi}(-1-2e^{-i\varphi})}{5} = \frac{2}{5}\left(\underset{z_0}{-2} - e^{i\varphi}\right) =$$

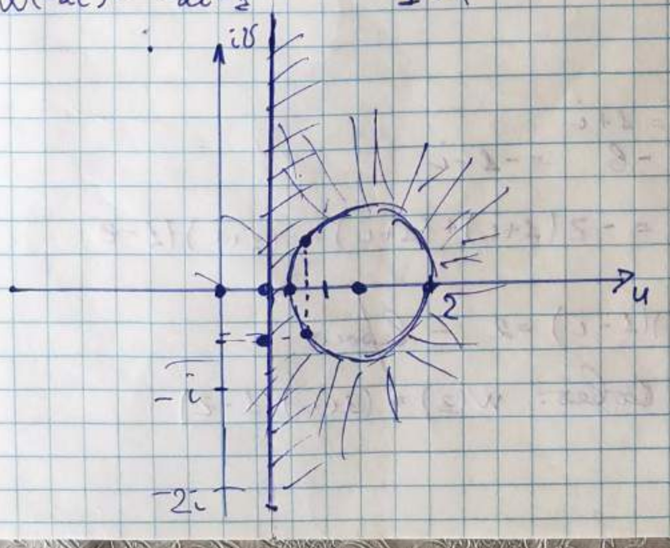
$$= -\frac{4}{5} - \frac{2}{5}e^{i\varphi}$$

$$|z - z_0| = R$$

$$w = (-2) = \frac{2}{3}$$

$$w\left(\frac{3}{2}\right) = \frac{\frac{3}{2}}{\frac{3}{2}-1} = 3$$

$$w(-2i) = \frac{-2i}{-2i-1} = \frac{-2i(+2i-1)}{1+4} = \frac{4+2i}{5} = \frac{4}{5} + \frac{2}{5}i$$



$$-\begin{cases} (2-x_0)^2 + y_0^2 = R^2 \\ \left(\frac{2}{3}-x_0\right)^2 + y_0^2 = R^2 \end{cases}$$

$$(2-x_0)^2 - \left(\frac{2}{3}-x_0\right)^2 = 0$$

$$4 - 4x_0 + \cancel{x_0^2} - \frac{4}{9} + \frac{4}{3}x_0 - \cancel{x_0^2} = 0$$

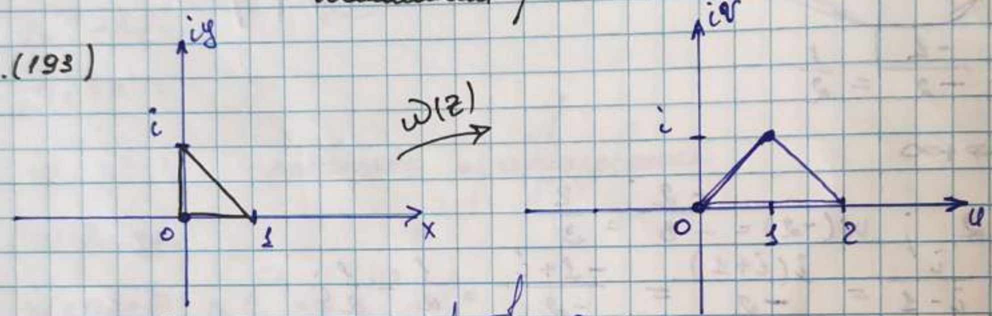
$$4x_0 - \frac{4}{3}x_0 = 4 - \frac{4}{9}$$

$$\frac{8}{3}x_0 = \frac{32}{9} \rightarrow x_0 = \frac{3 \cdot 32}{8 \cdot 9} = \frac{12}{9} = \frac{4}{3}$$

Аналогично: $y_0 = 0$; $R = \frac{2}{3}$

Домашняя работа.

2.1.(193)



$w = az + b$ ~ линейное преобразование.

$w(0) = 0$	$w(0) = 2$	или ...
$w(1) = 2$	$w(1) = 0$	
$w(i) = 1+i$	$w(i) = 1+i$	

$$\begin{cases} 2 = a \cdot 1 + b \\ 1+i = a \cdot i + b \end{cases} \rightarrow b = 2 - a$$

$$1 + i = a \cdot i + 2 - a \rightarrow a - ai = 1 - i$$

$$a(1-i) = 1 - i \rightarrow a = 1, \quad b = 1$$

~~$w = z + 1$, но $w(0) = 1$~~

или $w(0) = 1 + i$

$w(1) = 0$

$w(i) = 2$

$$\begin{cases} 1+i = 0 + b \\ 0 = a \cdot 1 + b \end{cases} \rightarrow b = 1+i, \quad a = -b = -1-i$$

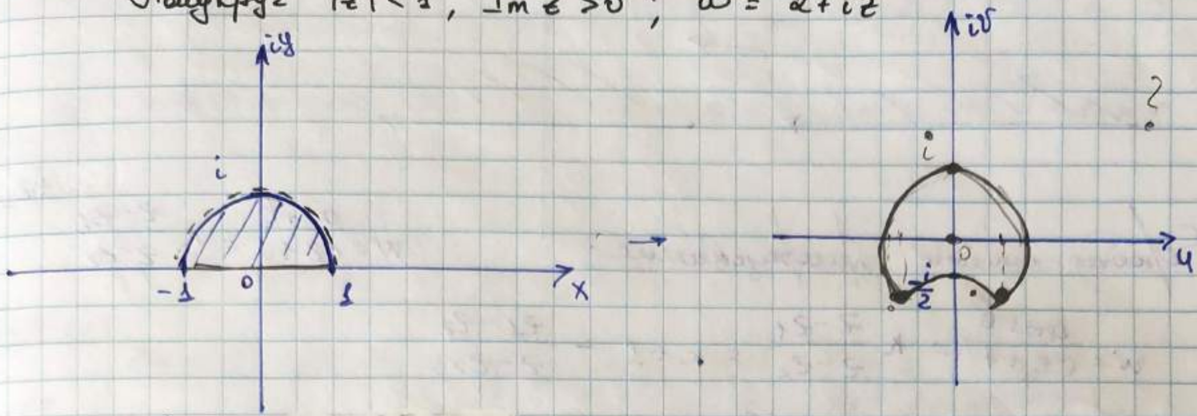
$$w(z) = (-1-i)z + (1+i) = -z(1+i) + (1+i) = (1+i)(1-z)$$

Проверка: $w(0) = 1 + i$

$w(i) = (1+i)(1-i) = 2$ – верно

Ответ: $w(z) = (1+i)(1-z)$

2.12 (204) Полукруг $|z|<1$, $\operatorname{Im} z > 0$; $\omega = \dfrac{2z-i}{2+iz}$



$$\omega = \frac{2z-i}{(2+iz)}$$

$$w(0) = -\frac{i}{2} \quad ; \quad w(i) = \frac{2i-i}{2+i^2} = i$$

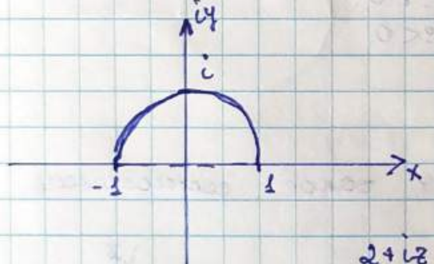
$$w(1) = \frac{2-i}{2+i} = \frac{(2-i)^2}{5} = \frac{4-4i-1}{5} = \frac{3}{5} - \frac{4}{5}i$$

$$w(-1) = \left(\frac{-2-i}{2-i}\right) = \frac{(-2-i)(2+i)}{5} = \frac{-4-2i-2i-i^2}{5} = -\frac{3}{5} - \frac{4}{5}i$$

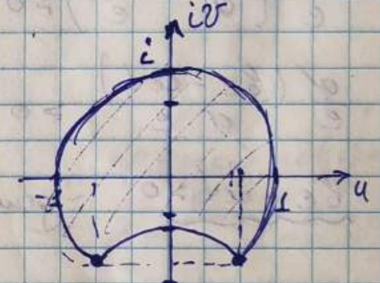
(

$$\begin{cases} |z|<1 \\ \operatorname{Im} z > 0 \end{cases} \Rightarrow \ ? \quad w = \frac{2z-i}{2+iz}$$

21.10.21.



$2+iz=0 \Rightarrow z = -\frac{i}{2}$



1) $z = \pm 1$, $z = i$

$$w(1) = \frac{2-i}{2+i} = \frac{(2-i)^2}{5} = \frac{3-4i}{5}$$

$$w(-1) = \frac{-2-i}{2-i} = \frac{-5(3+4i)}{25} = -\frac{3+4i}{5}$$

$$w(i) = i \quad ; \quad w(0) = -\frac{i}{2}$$

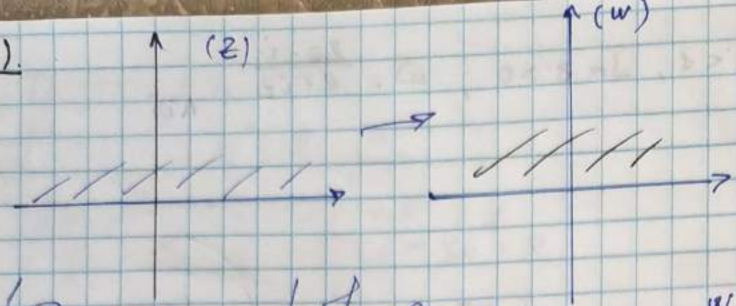
$(x-$

Окр-ть - переходит в окружность

$\boxed{\text{Если прямая не обращается в бесконечность ф-ю и} \rightarrow \text{отобьется в окружность !}}$

Если окр. обращает в/ ∞ $\Rightarrow$ разгибается в прямую.

№213(1).



Дробно-линейное преобразование:

$$w=\frac{az+b}{cz+d}=k\,\frac{z-z_1}{z-z_2}\ \pm z_2$$

$$w=\frac{az+b}{cz+d}=k\,\frac{z-z_1}{z-z_2}=k+1-\frac{z_1-z_2}{z-z_2}$$

c и d – действительные числа

k – действительное число

$$=k+1+\frac{\alpha}{z-\beta}$$

– если «–» отобразится на нижнюю пл-ть.

$$w=\frac{az+b}{cz+d}=\frac{a}{c}\cdot\frac{z+\frac{b}{a}\pm\frac{d}{c}}{z+\frac{d}{c}}=\frac{a}{c}\left(\frac{z+\frac{d}{c}}{z+\frac{d}{c}}+\frac{\frac{b}{a}-\frac{d}{c}}{z+\frac{d}{c}}\right)=$$

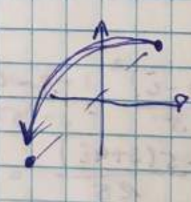
$$=\frac{a}{c}+\frac{a}{c}\cdot\frac{\left(\frac{b}{a}-\frac{d}{c}\right)}{z+\frac{d}{c}}>0$$, иначе развернётся

$$\frac{a}{c}\left(\frac{b}{a}-\frac{d}{c}\right)>0 \quad\to\quad \begin{matrix} a>0 & \text{и} & c>0 \\ a<0 & & c<0 \end{matrix}$$

$bc-ad>0$

$$\frac{\cancel{a}}{c}\,\frac{(bc-ad)}{\cancel{a}\,c}>0$$

$bc-ad>0$ – должно выполняться такое соотношение.



$-1\sim e^{i\pi}$

$z=|z|\cdot e^{i\varphi}$

$-z=|z|\,e^{i(\varphi+\pi)}$

№220(1). $\operatorname{Im} z>0 \Rightarrow |w|<1$, так чтобы $w(i)=0$; $w'(i)=-\frac{\pi}{2}$

по теореме: 7. Дробно-лин. ф-я отображает

$$\left\{ w=k\,\frac{z-\alpha}{z-\bar\alpha} \to |k|=1 \quad \operatorname{Im}\alpha>0 \right\}$$ – выполняет такое отображение опр-ая ф-ла.

1) $w=e^{i\varphi}\,\frac{z-\alpha}{z-\bar\alpha}$;

$$w(i)=e^{i\varphi}\,\frac{i-\alpha}{i-\bar\alpha}=0 \Rightarrow \alpha=i\ ;\ \bar\alpha=-i$$

$$w=e^{i\varphi}\,\frac{z-i}{z+i}$$

2) $W'(i) = e^{i\varphi}\frac{z+i-(z-i)}{(z+i)^2}\Big|_{z=i} = \frac{2i\cdot e^{i\varphi}}{(z+i)^2}\Big|_{z=i} = \frac{2i}{(2i)^2}e^{i\varphi} = \frac{1}{2}\cdot e^{i\varphi}\cdot e^{-i\frac{\pi}{2}} =$

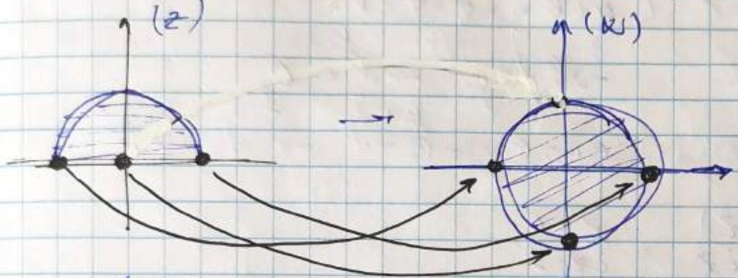
$\Rightarrow \arg w'(i) = \varphi - \frac{\pi}{2} = -\frac{\pi}{2} \Rightarrow \varphi = 0 \Rightarrow k = e^{i\varphi} = e^{0} = 1$

$$\Rightarrow \boxed{W = \frac{z-i}{z+i}}$$

№280 (1.)

Найти ф-ию отобр. полукруга $|z|<1$; $\operatorname{Im} z>0$ на круг при условии:

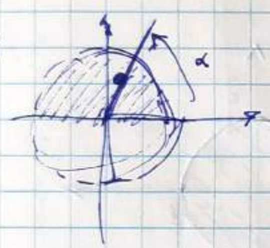
1) $w(\pm 1) = \pm 1$, $w(0) = -i$



Чтобы д.л. отображение полукруг в круг.

$\Rightarrow w = k\frac{z-\alpha}{1-\bar{\alpha} z}$

$|k| = 1$, $|\alpha| < 1$

1) 

$\tilde{w} = z^2$ ~ развернём полукруг.

$\tilde{W} = z^2 = (az)^2 = (e^{i\beta}\cdot z)^2$

$|a| = 1$

$W = k\cdot\frac{z-\alpha}{1-\bar{\alpha} z} = e^{i\varphi}\frac{z-\alpha}{1-\bar{\alpha} z}$, φ - угол поворота.

$W = e^{i\varphi}\frac{z\cdot e^{i\beta}-\alpha}{1-\bar{\alpha} z^2\cdot e^{2i\beta}} \rightarrow 1 = \frac{e^{i\varphi}\cdot e^{2i\beta}-\alpha}{1-\bar{\alpha} e^{2i\beta}}$

$\cancel{-1 = e^{i\varphi}\frac{e^{2i\beta}-\alpha}{1-\bar{\alpha}}}$

$w(0) = e^{i\varphi}\frac{-\alpha}{|\alpha|} = -i$

$|\alpha| = 1$; $e^{i\varphi+\cancel{\pi}+\arg\alpha} =$

$\cancel{-i = e^{i\frac{3\pi}{2}} = e^{i(\varphi+\frac{3\pi}{2})} \to \varphi = 0}$

$-i = e^{i\frac{3\pi}{2}} = e^{i(\varphi+\arg\alpha+\pi)}$

$\cancel{\pi}+\varphi+\arg\alpha = \frac{3\pi}{2}$

$\varphi+\arg\alpha = \frac{\pi}{2}$

$\arg\alpha = \frac{\pi}{2}-\varphi$

$w = e^{i\varphi}\quad z - e^{i(\frac{\pi}{2}-\varphi)}$

Домашняя работа.

2.20. (212) Найти дробно-линейную ф-цию, переводящую точки $-1, 0, 1$ соответственно в точки $1, i, -1$, и выяснить, во что при этом отображении переходит верхняя полуплоскость.

$$w(z) = \frac{a+bz}{c+dz}; \quad \frac{a}{c} \neq \frac{b}{d}$$

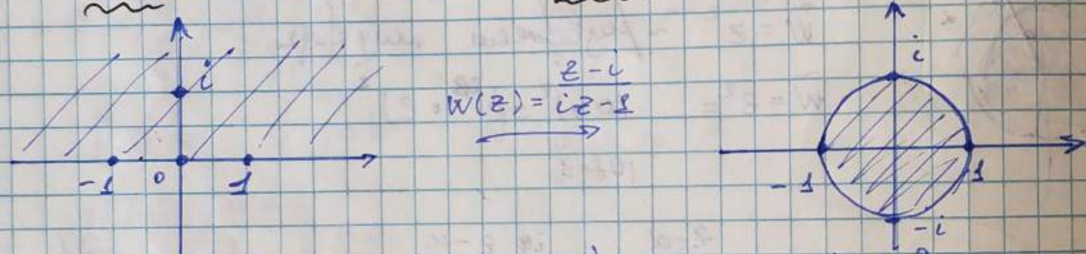
$$w(z) = \lambda \frac{\alpha+z}{\beta+z}, \text{ где } \lambda = \frac{b}{a}, \ \alpha = \frac{a}{b}, \ \beta = \frac{c}{d}, \ \alpha \neq \beta$$

$$w(-1) = \frac{a-b}{c-d} = 1$$
$$w(0) = \frac{a}{c} = i \quad \Rightarrow \begin{cases} a-b = c-d \\ a = ci \\ a+b = -c-d \end{cases}$$
$$w(1) = \frac{a+b}{c+d} = -1$$

$$-2b = 2c \to b = -c$$

$$ci - c = -c - d \to d = -ci$$

$$\Rightarrow w(z) = \frac{ci - cz}{c - ciz} = \frac{(i-z)}{1-iz} = \frac{z-i}{iz-1}$$



$$W(2) = \frac{2-i}{-1+2i} = \frac{(2-i)(-1-2i)}{1-4i^2} = \frac{-2-4i+i+2i^2}{5} =$$

$$= \frac{-4-3i}{5} = -\frac{4}{5} - \frac{3}{5}i \to |W(2)| = \sqrt{\frac{16}{25} + \frac{9}{25}} = 1$$

~ лежит на окр-ти

$$w(i) = \frac{i-i}{i^2-1} = 0$$ ~ в круге.

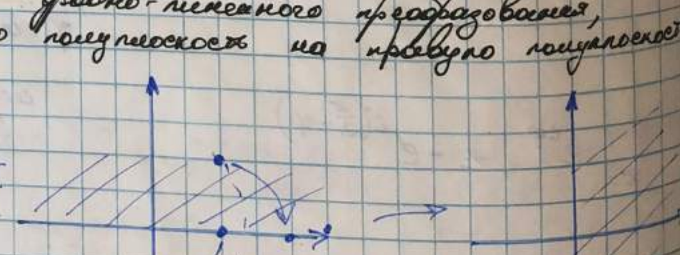
$\Rightarrow$ Полуплоскость перейдёт в единичный круг!

2.21(3) (213) Найти общий вид дробно-линейного преобразования, переводящего верхнюю полуплоскость на правую полуплоскость.

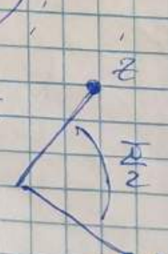
$$w(z) = \frac{az+b}{cz+d} \cdot e^{i\varphi}$$

$$\varphi = \operatorname{Arg} z \to \varphi + \frac{\pi}{2} = \operatorname{Arg} z - \frac{\pi}{2}$$

$$z = |z| \cdot e^{i\varphi} \to |z| \cdot e^{i(\varphi - \frac{\pi}{2})}.$$



$$|z| \cdot e^{i\varphi - i\frac{\pi}{2}} = \frac{|z| \cdot e^{i\varphi}}{e^{i\frac{\pi}{2}}} = \frac{z}{\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}} = \frac{z}{i}$$



$\Rightarrow$ ~~$w(z) = i\frac{az+b}{cz+d}$~~ ~~$w(z) = i\frac{az+b}{cz+d}$~~

из 213(1): $bc - ad > 0$

$\arg w'(z_0) = \varphi$ ~ поворот пространства в т. z_0

$$w' = e^{i\varphi}\frac{a(cz+d) - c(az+b)}{(cz+d)^2} = \frac{\cancel{acz} + ad - \cancel{acz} - cb}{(cz+d)^2} \cdot e^{i\varphi} = \frac{ad - cb}{(cz+d)^2} \cdot e^{i\varphi}$$

$$ad - cb < 0$$

$$w'(0) = \frac{ad - cb}{d^2} \cdot e^{i\varphi}$$

$$\arg w'(0) = \arg e^{i\varphi} = \frac{\pi}{2} = \varphi$$

$$e^{i\varphi} = e^{i\frac{\pi}{2}} = i \qquad \Rightarrow w(z) = i\frac{az+b}{cz+d}$$

?

228 (2) Отобразить верхнюю полуплоскость $\operatorname{Im} z > 0$ на
(220) единичный круг, так, чтобы:

$$w(2i) = 0, \quad \arg w'(2i) = 0$$

Всякое дробно-линейное отображение, переводящее верхнюю полуплоскость на круг $\{w : |w| < 1\}$, осуществляется функцией

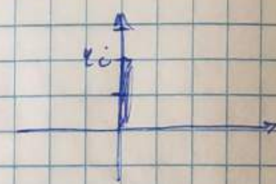
$$w = e^{i\varphi}\frac{z - z_0}{z - \bar{z}_0}$$

$$\Rightarrow \quad 0 = e^{i\varphi}\frac{2i - z_0}{2i - \bar{z}_0} \Rightarrow 2i - z_0 = 0 \Rightarrow z_0 = 2i$$

$$\bar{z}_0 = -2i$$

$$\Rightarrow \quad W = e^{i\varphi}\frac{z-2i}{z+2i}$$

$$w' = \frac{e^{i\varphi}(z+2i) - e^{i\varphi}(z-2i)}{(z+2i)^2} = e^{i\varphi}\left(\frac{z+2i-z+2i}{(z+2i)^2}\right) = \frac{e^{i\varphi}\cdot 4i}{(z+2i)^2}$$



$$w'(2i) = e^{i\varphi}\cdot\frac{4i}{(4i)^2} = \frac{e^{i\varphi}}{4i}$$

$$\arg w'(2i) = \arg(e^{i\varphi}) - \arg(4i) = \varphi - \frac{\pi}{2} = 0$$

$$\Rightarrow \varphi = \frac{\pi}{2}$$

Тогда, $e^{i\varphi} = e^{i\frac{\pi}{2}} = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = i$

$$\Rightarrow \quad w(z) = i\,\frac{z-2i}{z+2i}$$

Ответ: $w(z) = i\,\dfrac{z-2i}{z+2i}$

27.10.21. 213(3).

$w = \dfrac{az+b}{cz+d}$, $ad - bc > 0$, a, b, c, d - действ.

$$\Rightarrow \quad w = \left(\frac{az+b}{cz+d}\right)\cdot e^{-i\frac{\pi}{2}} = -i\,\frac{(az+b)}{cz+d}$$

№ ~~425~~, ~~426~~, ~~427~~ ~~429~~, ~~431~~, 433, 434, ~~435~~, ~~437(1,2,3)~~,
~~440~~, 441, 442, 445

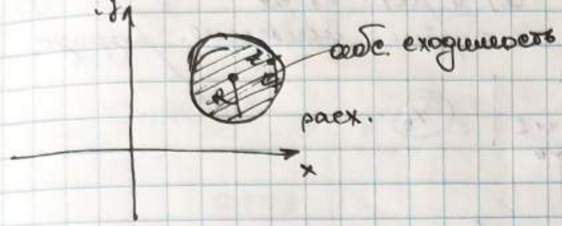
Д/З:

425 Теория: $\sum\limits_{n=0}^{\infty} c_n (z-z_0)^n$

$$R = \lim_{n\to\infty} \left| \frac{c_n}{c_{n+1}} \right|$$

— радиус сходимости.

$$\frac{1}{R} = \overline{\lim_{n\to\infty}} \sqrt[n]{|c_n|}$$



N425. $\sum\limits_{n=1}^{\infty} \frac{z^n}{n} \to R = \lim\limits_{n\to\infty} \left| \frac{n+1}{n} \right| = 1 \qquad c_n = \frac{1}{n}$

N426. $\sum\limits_{n=1}^{\infty} \frac{z^n}{n!} \to R = \lim\limits_{n\to\infty} \left| \frac{(n+1)!}{n!} \right| = \lim\limits_{n\to\infty} |n+1| = \infty$

N427. $\sum\limits_{n=1}^{\infty} n^n z^n \to \frac{1}{R} = \overline{\lim\limits_{n\to\infty}} \sqrt[n]{n^n} = \infty \to R = 0$

N429. $\sum\limits_{n=1}^{\infty} \frac{n!}{n^n} z^n$

$$R = \lim_{n\to\infty} \frac{n!\,(n+1)^{n+1}}{n^n\,(n+1)!} = \cancel{\lim_{n\to\infty} \frac{n}{n+1}} = \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^n = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e$$

N431. $\sum\limits_{n=0}^{\infty} 2^n z^{n!} = 1\cdot z + 2z + 2^2 z^2 + 2^3 z^6 + 2^4 z^{24} + \dots = 3z + 4z^2 + 0z^3 + 0\cdot z^5 +$

R-? $\qquad + 8\cdot z^6 + 0\cdot z^7 + 0\cdot z^8 + \dots$

$C_{n!} = 2^n$; $C_n = 0$

$$\frac{1}{R} = \overline{\lim_{n\to\infty}} \{\sqrt[n!]{2^n}\} = \lim_{n\to\infty} 2^{\overset{\to 0}{\frac{n}{n!}}} = 2^0 = 1$$

$\Rightarrow R = 1$

N433. $\sum\limits_{n=0}^{\infty} [3+(-1)^n]^n \cdot z^n$

$$\to \frac{1}{R} = \overline{\lim_{n\to\infty}} \{ \underset{n=2k}{\sqrt[2k]{4^{2k}}}, \underset{n=2k+1}{\sqrt[2k+1]{2^{2k+1}}} \} = \max\{4, 2\} = 4$$

$\Rightarrow R = \frac{1}{4}$

N434. $\sum\limits_{n=0}^{\infty} \cos(in) z^n = \frac{1}{2}\sum\limits_{n=0}^{\infty} (e^{iin} + e^{-iin}) z^n = \frac{1}{2}\underbrace{\sum\limits_{n=0}^{\infty} e^{-n} z^n}_{(1)} + \frac{1}{2}\underbrace{\sum\limits_{n=0}^{\infty} e^n z^n}_{(2)}$

(1): $R_1 = \lim\limits_{n\to\infty} \dfrac{e^{-n}}{e^{-n-1}} = e$

(2): $R_2 = \lim \dfrac{e^{n}}{e^{n+1}} = \dfrac{1}{e}$

$R = \min\{R_1, R_2\} = R_2 = \dfrac{1}{e}.$

Ряд сходится, когда оба ряда (1) и (2) сх-ся $\Rightarrow$ берём минимальный радиус.

N435. $\sum\limits_{n=0}^{\infty} (n+a^n) z^n \to R = \lim\limits_{n\to\infty} \left| \dfrac{n+a^n}{n+1+a^{n+1}} \right| = (*)$

$(*)$: 1) $|a|<1 \Rightarrow (*) = \lim\limits_{n\to\infty} \left| \dfrac{1+\overbrace{\left(\frac{a^n}{n}\right)}^{\to 0}}{1+\underbrace{\frac{a^{n+1}}{n+1}}_{\to 0}} \right| = 1$

2) $|a| = 1 \Rightarrow (*) = 1$

3) $|a|>1 \Rightarrow (*) = \lim\limits_{n\to\infty} \left| \dfrac{|a|^n \cdot \left(\overbrace{\frac{n}{|a|^n}}^{\to 0} + \overbrace{e^{i\varphi n}}^{\to 1}\right)}{|a|^{n+1} \left(\frac{n+1}{|a|^{n+1}} + \underbrace{e^{i\varphi(n+1)}}_{\to 1}\right)} \right| = \dfrac{1}{|a|}$

$|e^{i\varphi n}| = |\cos\varphi n + i \sin\varphi n| = \sqrt{\cos^2\varphi n + \sin^2\varphi n} = 1$

N437. $\sum c_n z^n$, R - радиус сх-ти этого ряда.

Выразить через R рад. сх-ти:

1) $\sum\limits_{n=0}^{\infty} n^k c_n z^n \to R_1 = \lim\limits_{n\to\infty} \left| \dfrac{n^k c_n}{(n+1)^k c_{n+1}} \right| = \lim\limits_{n\to\infty} \underbrace{\left(\dfrac{n^k}{(n+1)^k}\right)}_{\to 1} \underbrace{\left(\dfrac{c_n}{c_{n+1}}\right)}_{\to R} = R$

$\to R_1 = R$

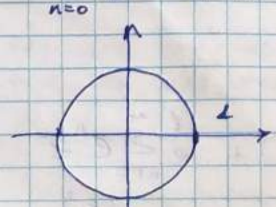
2) $\sum\limits_{n=0}^{\infty} (2^n-1) c_n z^n$

$\to R_2 = \lim\limits_{n\to\infty} \underbrace{\left| \dfrac{2^n-1}{2^{n+1}-1} \right|}_{\to \frac{1}{2}} \cdot \underbrace{\left| \dfrac{c_n}{c_{n+1}} \right|}_{\to R} = \dfrac{R}{2}$

$R_2 = \dfrac{R}{2}$

3) $\sum\limits_{n=0}^{\infty} \dfrac{c_n}{n!} z^n \to R = \lim\limits_{n\to\infty} \underbrace{\left| \dfrac{c_n}{c_{n+1}} \right|}_{\to R} \cdot \left| \dfrac{(n+1)!}{n!} \right| = \infty$

N440 $\sum\limits_{n=0}^{\infty} z^n \Rightarrow R = 1$



$z = e^{i\varphi}$

$0 \le \varphi \le 2\pi$

$\sum\limits_{n=0}^{\infty} e^{i\varphi n} = \sum\limits_{n=0}^{\infty} \cos\varphi n + i \sum\limits_{n=0}^{\infty} \sin\varphi n$

$\sum a_n \to \lim\limits_{n\to\infty} a_n = 0$ ~ необх. усл-е сх-ти.

$\lim\limits_{n\to\infty} \cos\varphi n \;\nexists$, $\lim\limits_{n\to\infty} \sin\varphi n \;\nexists$

$\Rightarrow$ ряд расх-ся.

$\Rightarrow \sum\limits_{n=0}^{\infty} z^n$ - расходится.

N441. $\sum\limits_{n=1}^{\infty} \frac{z^n}{n}$, $R=1$

$\to \sum\limits_{n=1}^{\infty} \frac{\cos\varphi n}{n} + i \sum\limits_{n=1}^{\infty} \frac{\sin\varphi n}{n}$

$\lim\limits_{n\to\infty} \frac{\cos\varphi n}{n} = 0$; $\lim\limits_{n\to\infty} \frac{\sin\varphi n}{n} = 0$

$\sum a_n b_n$

b_n ↓ монотонно убыв. $A_n = \sum\limits_{n=1}^{\infty} a_n$ - огранич.

$\Rightarrow \sum a_n b_n$ - сх-ся

$b_n = \frac{1}{n}$ - монотонно убыв.

$A_N = \sum\limits_{n=1}^{N} a_n = \sum\limits_{n=1}^{N} \cos\varphi n \quad \Big| \cdot 2\sin\varphi$

$2\sin\varphi A_n = \sum\limits_{n=1}^{N} \cos\varphi n \, 2\sin\varphi = \sum\limits_{n=1}^{N} (\sin\varphi(n+1) - \sin\varphi(n-1)) =$

$= \sin 2\varphi - \sin 0 + \sin 3\varphi - \sin\varphi + \sin 4\varphi - \sin 2\varphi + \sin 5\varphi - \sin 3\varphi +$

$+ \ldots + \sin\varphi(N) - \sin\varphi(N-2) + \sin\varphi(N+1) - \sin(\varphi(N-1)) =$

$= -\sin\varphi + \sin\varphi(N) + \sin\varphi(N+1)$.

$|A_n| \leq \left| \frac{-\sin\varphi + \sin\varphi N + \sin\varphi(N+1)}{2\sin\varphi} \right| \leq \frac{3}{2|\sin\varphi|}$ $\quad \varphi \neq 0$, $\varphi \neq \pi$

$\Rightarrow A_n$ - огранич. $\Rightarrow \sum\limits_{n=0}^{\infty} \frac{\cos 4n}{n}$ - сх-ся

(если $\varphi = 0 \Rightarrow \sum\limits_{n=0}^{\infty} \frac{\cos\varphi n}{n}$ - расх-ся.

аналог. с $\frac{\sin\varphi n}{n}$) если $\varphi \neq \frac{\pi}{2}$

при $\varphi = \frac{\pi}{2}$ ~ ряд сходится.

$\varphi = \pi \quad \sum\limits_{n=1}^{\infty} \frac{(-1)^n}{n}$ ~ усл. сх-ся.

$\Rightarrow$ ряд сходится при $\varphi \neq 0$

N442. $\sum\limits_{n=1}^{\infty} \frac{z^n}{n^2}$, $R=1$

$\sum\limits_{n=1}^{\infty} \frac{\cos\varphi n}{n^2} + i \sum\limits_{n=1}^{\infty} \frac{\sin\varphi n}{n^2}$

$\frac{\cos\varphi n}{n^2} \le \frac{1}{n^2}$ ~ обобщ. гармон => сх-ся.

$\frac{\sin\varphi n}{n^2} \le \frac{1}{n^2}$ => сх-ся.

$=> \sum_{n=1}^{\infty} \frac{z^n}{n^2}$ ~ сх-ся абсолютно.

№445 $\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n} z^{3n-1}$

$R = \lim_{n\to\infty} \left| \frac{(-1)^n \ln(n+1)}{\ln(n) \quad (-1)^{n+1}} \right| = 1$

$\sum_{n=2}^{\infty} \frac{(-1)^n \cdot e^{i\varphi n}}{\ln n}$

$e^{i\varphi n}$ - огранич,

$\Downarrow$

по пр-ку Абеля сх-ся.

$\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n}$ - сх-ся по пр.-ку Лейбница, т.к.

1) $\lim_{n\to\infty} \frac{1}{\ln n} = 0$

2) $\frac{1}{\ln(n)} \ge \frac{1}{\ln(n+1)}$ - выполн.

Ряды Тейлора

453. $\operatorname{sh} z = \frac{e^z - e^{-z}}{2} = \frac{1}{2}\left(\sum_{k=0}^{\infty} \frac{z^k}{k!} - \sum_{l=0}^{\infty} \frac{(-1)^l z^l}{l!}\right) = \sum_{l=0}^{\infty} \frac{1}{l!}\left(\frac{1-(-1)^l}{2}\right) z^l =$

$= |l = 2k+1| = \sum_{k=0}^{\infty} \frac{1}{(2k+1)!} z^{2k+1}$

$\frac{1}{R} = \overline{\lim_{k\to\infty}} \sqrt[2k+1]{\frac{1}{(2k+1)!}} = \lim_{k\to\infty} \left(\sqrt[2k+1]{(2k+1)^{(2k+1)+\frac{1}{2}} e^{-(2k+1)}}\right)^{-1} =$

$= \lim_{k\to\infty} \frac{e}{2k+1} = 0 \quad \to R = \infty$

454. $\operatorname{sh}^2 z = \frac{1-\operatorname{ch} 2z}{2} = \frac{1}{2}\left(1 - \sum_{k=0}^{\infty} (-1)^k \frac{(2z)^{2k}}{(2k)!}\right)$?

$\frac{1}{R} = \overline{\lim_{k\to\infty}} \sqrt[2k]{\frac{2^{2k}}{(2k)!}} = \lim_{k\to\infty} \frac{2}{\sqrt[2k]{(2k)^{2k+\frac{1}{2}} e^{-2k}}} = \lim_{k\to\infty} \frac{2 \cdot e}{2k} = 0$

$R = \infty$

456.

$(a+z)^a = a^a \left(1 + \frac{z}{a}\right)^a = a^a \sum_{n=0}^{\infty} \frac{a(a-1)\dots(a-m+1)}{m!} \left(\frac{z}{a}\right)^m$

$R = \lim_{m\to\infty} \left| \frac{a(a-1)\dots(a-m+1)}{m! \, a^m} \frac{(m+1)! \, a^{m+1}}{a(a-1)\dots(a-m+1)(a-m-1+1)} \right| =$

$= \lim_{m\to\infty} \left| \frac{a(m+1)}{a-m} \right| = |a|$

457.
$$\sqrt{z+i} = \frac{1+i}{\sqrt{2}}\left(1+\frac{z}{i}\right)^{\frac{1}{2}} = \frac{1+i}{2}\sum_{m=0}^{\infty}\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\dots\left(\frac{1}{2}-m+1\right)}{m!}\left(\frac{z}{i}\right)^{m}$$

$$R=\lim_{n\to\infty}\left|\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\dots\left(\frac{1}{2}-m+1\right)}{m!\ i^{m}}\cdot\frac{(m+1)!\ i^{m+1}}{\frac{1}{2}\dots\left(\frac{1}{2}-m-1+1\right)}\right| = 1$$

460.
$$\frac{z^2}{(1+z)^2} = z^2(1+z)^{-2} = z^2\sum_{m=0}^{\infty}\frac{-2(-2-1)\dots(-2-m+1)}{m!}z^{m},$$

$$R=1$$

462.
Arctg z

$$(\operatorname{Arctg} z)' = \frac{1}{1+z^2} = \left|\ |z|<1\ \right| = \sum_{k=0}^{\infty}(-z^2)^k = \sum_{k=0}^{\infty}(-1)^k z^{2k}$$

$$\operatorname{Arctg} z = \int_0^z\sum_{k=0}^{\infty}(-1)z^{2k}dz = \sum_{k=0}^{\infty}(-1)^k\frac{z^{2k+1}}{2k+1}$$

463.
$$\operatorname{Arcsh} z = \sum_{n=0}^{\infty}\frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)\dots\left(-\frac{1}{2}-m+1\right)}{n!}\frac{z^{2n+1}}{2n+1}$$

$$(\operatorname{Arcsh} z)' = \frac{1}{\sqrt{z^2+1}} = (1+z^2)^{-\frac{1}{2}} = \left|u=z^2\right| = (1+u)^{-\frac{1}{2}} =$$

$$= \sum_{n=0}^{\infty}\frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)\left(-\frac{1}{2}-2\right)\dots\left(-\frac{1}{2}-n+1\right)}{n!}u^{n}$$

466.
$$\int_0^z\frac{\sin z}{z}dz = \int_0^z\frac{\sin t}{t}dt = \int_0^z\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)!}\frac{t^{2k+1}}{t}dt =$$

$$= \sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)!}\frac{z^{2k+1}}{(2k+1)}$$

468.
$$\frac{z}{z^2-2z+5} = \frac{z}{(z-(1+2i))(z-(1-2i))} = \frac{z}{4i}\left(\frac{1}{z-(1+2i)}-\frac{1}{z-(1-2i)}\right)$$

$z^2-2z+5=0$

$z_{1,2} = 1\pm\sqrt{1-5} = 1\pm 2i$

$$= \frac{z}{4i}\left[-\frac{1}{1+2i}\cdot\frac{1}{1-\frac{z}{1+2i}}+\frac{1}{1-2i}\cdot\frac{1}{1-\frac{z}{1-2i}}\right] =$$

$$= \frac{z}{4i}\left(-\frac{1}{1+2i}\sum_{k=0}^{\infty}\left(\frac{z}{1+2i}\right)^k+\frac{1}{1-2i}\sum_{k=0}^{\infty}\left(\frac{z}{1-2i}\right)^k\right)$$

Домашняя работа.

428. (3.49) $\sum\limits_{n=1}^{\infty} \frac{n}{2^n} z^n$

$$R = \lim_{n\to\infty} \left| \frac{n \cdot 2^{n+1}}{2^n \cdot (n+1)} \right| = \lim_{n\to\infty} \left| \frac{2 n}{n(1+\frac{1}{n})} \right| = \underline{\underline{2}}$$

432 (3.47) $\sum\limits_{n=0}^{\infty} z^{2^n} = 1 + z^2 + z^4 + z^8 + \dots$

$C_n = 1. \ \forall n \Rightarrow \quad R = \underline{\underline{1}}$

436 (3.51)

$$1 + \sum_{n=1}^{\infty} \frac{\alpha(\alpha+1)\dots(\alpha+n-1)\beta(\beta+1)\dots(\beta+n-1)}{n!\,\gamma(\gamma+1)\dots(\gamma+n-1)} z^n$$

$$R = \lim_{n\to\infty} \left| \frac{\alpha(\alpha+1)\dots(\alpha+n-1)\beta(\beta+1)\dots(\beta+n-1)}{n!\,\gamma(\gamma+1)\dots(\gamma+n-1)} \cdot \frac{(n+1)!\,\gamma\dots(\gamma+n)}{\alpha(\alpha+1)\dots(\alpha+n-1)(\alpha+n)\beta(\beta+1)\dots(\beta+n)} \right| =$$

$$= \lim_{n\to\infty} \left| \frac{(n+1)(\gamma+n)}{(\alpha+n)(\beta+n)} \right| = 1$$

$$\Rightarrow \underline{\underline{R = 1}}$$

437. (3.52) $\sum\limits_{n=0}^{\infty} C_n z^n$ → радиус сходимости R.

4) $\sum\limits_{n=1}^{\infty} n^n C_n z^n$

$$\frac{1}{R'} = \overline{\lim_{n\to\infty}} \sqrt[n]{|n^n C_n|} = \lim_{n\to\infty} n \cdot \underbrace{\sqrt[n]{|C_n|}}_{=\frac{1}{R}} = \infty$$

$$\Rightarrow \underline{\underline{R' = 0}}$$

5) $\sum\limits_{n=0}^{\infty} C_n^k z^n$

$$\frac{1}{R'} = \overline{\lim_{n\to\infty}} \sqrt[n]{|C_n^k|} = \lim_{n\to\infty} \Big(\underbrace{\sqrt[n]{|C_n|}}_{=\frac{1}{R}}\Big)^k \to \frac{1}{R'} = \left(\frac{1}{R}\right)^k$$

$$\frac{1}{R'} = \frac{1}{R^k} \Rightarrow \underline{\underline{R' = R^k}}$$

6) $\sum\limits_{n=0}^{\infty} (1 + z_0^n) C_n z^n$

$$\frac{1}{R'} = \overline{\lim_{n\to\infty}} \sqrt[n]{|(1+z_0^n) C_n|} = \overline{\lim_{n\to\infty}} \sqrt[n]{|(1+z_0^n)|} \cdot \sqrt[n]{|C_n|}$$

1) если $0 < z_0 < 1 \Rightarrow \overline{\lim\limits_{n\to\infty}} \sqrt[n]{(1+z_0^n)} \cdot \sqrt[n]{|C_n|}$, $z_0^n \to 0$

$$\Rightarrow \frac{1}{R'} = \frac{1}{R} \Rightarrow \underline{\underline{R' = R}}$$

2) если $z_0 = 1$ $\; - \; \overline{\lim\limits_{n\to\infty}} \sqrt[n]{(1+z_0^n)} \cdot \sqrt[n]{|c_n|}$

$$\sqrt[n]{2} \to 1 \qquad \Rightarrow \underline{R' = R}$$

3) если $z_0 > 1$

$$\lim_{n\to\infty} (1+z_0^n)^{\frac{1}{n}} = \lim_{n\to\infty} e^{\ln(1+z_0^n)^{\frac{1}{n}}} = e^{\lim\limits_{n\to\infty} \frac{1}{n}\cdot\ln(1+z_0^n)} =$$

$$= \{\text{по пр. Лопиталя}\} = e^{\lim\limits_{n\to\infty} \frac{\frac{\ln|z_0|\cdot z_0^n}{1+z_0^n}}{1}} = e^{\lim\limits_{n\to\infty} \frac{\ln|z_0|\cdot z_0^n}{1+z_0^n}} = e^{\ln|z_0|} = z_0$$

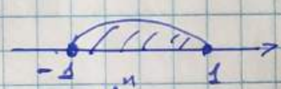
$$\Rightarrow \frac{1}{R'} = \frac{|z_0|}{R} \quad \Rightarrow R' = \frac{R}{|z_0|}$$

$$\Rightarrow R' = \begin{cases} R, & z_0 \le 1 \\ \frac{R}{|z_0|}, & z_0 > 1 \end{cases}$$

443 (3.58) $\sum\limits_{n=1}^{\infty} \frac{(-1)^n}{n}(z^n)$; $c_n = \frac{(-1)^n}{n}$

$$R = \lim_{n\to\infty} \left| \frac{(-1)^n}{n} \frac{(n+1)}{(-1)^{n+1}} \right| = \lim_{n\to\infty} \left| -1 \cdot \frac{n(1+\frac{1}{n})}{n} \right| = 1.$$

$\Rightarrow R = 1$ $\qquad$ -1 и 1?



$z = 1.$ $\quad \to \sum\limits_{n=1}^{\infty} \frac{(-1)^n}{n}$

1) $\lim\limits_{n\to\infty} \frac{1}{n} = 0$

2) $\frac{1}{n} > \frac{1}{n+1}$ $\quad \Rightarrow$ сходится условно по признаку Лейбница.

$z = -1$: $\to \sum\limits_{n=1}^{\infty} \frac{(-1)^{2n}}{n} \sim \sum\limits_{n=1}^{\infty} \frac{1}{n}$ $\sim$ расходится.

444 (3.59) $\sum\limits_{n=1}^{\infty} \frac{z^{pn}}{n}$ (p - натуральное число)

$$\sum_{n=1}^{\infty} \frac{z^{pn}}{n} = z^p + \frac{1}{2} z^{2p} + \frac{1}{3} z^{3p} + \frac{1}{4} z^{4p} + \dots \qquad ?$$

452 (3.67) $\operatorname{ch} z = \dfrac{e^z+e^{-z}}{2} = \dfrac{1}{2}\left(\sum\limits_{k=0}^{\infty}\dfrac{z^k}{k!} + \sum\limits_{\ell=0}^{\infty}\dfrac{(-1)^\ell z^\ell}{\ell!}\right) = \sum\limits_{\ell=0}^{\infty}\dfrac{1}{\ell!}\left(\dfrac{1+(-1)^\ell}{2}\right)z^\ell =$

$= \{\ell = 2n\} = \sum\limits_{n=0}^{\infty}\dfrac{z^{2n}}{(2n)!}$

$\Rightarrow \operatorname{ch} z = \sum\limits_{n=0}^{\infty}\dfrac{z^{2n}}{(2n)!} \qquad c_n = \dfrac{1}{(2n)!}$

$R = \lim\limits_{n\to\infty}\left|\dfrac{(2n+2)!}{(2n)!}\right| = \infty$

455 (3.70) $\operatorname{ch}^2 z = \dfrac{1+\operatorname{ch} 2z}{2} = \dfrac{1}{2}\left(1 + \sum\limits_{n=0}^{\infty}\dfrac{(2z)^{2n}}{(2n)!}\right) =$

$= \dfrac{1}{2} + \sum\limits_{n=0}^{\infty}\dfrac{2^{2n-1}z^{2n}}{(2n)!}$

$R = \lim\limits_{n\to\infty}\left|\dfrac{2^{2n-1}}{(2n)!}\cdot\dfrac{(2n+2)!}{2^{2n+1}}\right| = \infty$

458 (3.73) $\dfrac{1}{az+b} \quad (b\neq 0)$

$\dfrac{1}{az+b} = (az+b)^{-1} = \dfrac{1}{a}\left(z+\dfrac{b}{a}\right)^{-1} = \left(\dfrac{b}{a}\right)^{-1}\cdot\dfrac{1}{a}\left(\dfrac{a}{b}z+1\right)^{-1} =$

$= \dfrac{1}{b}\left(\dfrac{a}{b}z+1\right)^{-1}$

$f(z) = \left(\dfrac{a}{b}z+1\right)^{-1}; \quad f'(z) = -\dfrac{a}{b}\left(\dfrac{a}{b}z+1\right)^{-2}$

$f''(z) = 2\cdot\left(\dfrac{a}{b}\right)^2\left(\dfrac{a}{b}z+1\right)^{-3}$

$f(z) = \sum\limits_{n=0}^{\infty}\dfrac{f^{(n)}(z_0)}{n!}(z-z_0)^n; \quad z_0 = 0$

$f'(z_0) = -\dfrac{a}{b}; \quad f''(z_0) = +2\cdot\left(\dfrac{a}{b}\right)^2 \ \ldots$

$\Rightarrow \dfrac{1}{az+b} = \dfrac{1}{b}\left(\dfrac{a}{b}z+1\right)^{-1} = \dfrac{1}{b}\sum\limits_{n=0}^{\infty}(-1)^n\dfrac{a^n\cancel{(n!)}}{b^n\cancel{(n!)}}z^n =$

$= \sum\limits_{n=0}^{\infty}(-1)^n\dfrac{a^n}{b^{n+1}}z^n$

$R = \lim\limits_{n\to\infty}\left|\dfrac{(-1)^n a^n}{b^{n+1}}\cdot\dfrac{b^{n+2}}{(-1)^{n+1}\cdot a^{n+1}}\right| = \left|\dfrac{b}{a}\right|$

$\Rightarrow \dfrac{1}{az+b} = \sum\limits_{n=0}^{\infty}(-1)^n\dfrac{a^n}{b^{n+1}}z^n$

$R = \left|\dfrac{b}{a}\right|$

461. (3.76) $\ln \frac{1+z}{1-z} = \ln(1+z) - \ln(1-z) = \ln(1+z) - \ln(1+(-z))$

$$\ln(1+z) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} z^n, \quad |z|<1$$

$$\ln(1+(-z))^n = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \cdot (-z)^n = \sum_{n=1}^{\infty} \frac{(-1)^{2n+1}}{n} z^n$$

$$\ln \frac{1+z}{1-z} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} z^n - \sum_{n=1}^{\infty} \frac{(-1)^{2n+1}}{n} z^n =$$

$$= z - \frac{1}{2}z^2 + \frac{1}{3}z^3 - \frac{1}{4}z^4 + \dots - \left(-z - \frac{1}{2}z^2 - \frac{1}{3}z^3 - \dots\right) =$$

$$= 2\sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)}$$

$$\ln \frac{1+z}{1-z} = 2\sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)}$$

$$R = \lim_{n\to\infty} \left| \frac{2n+3}{2n+1} \right| = 1$$

464. (3.79) $\ln(z^2 - 3z + 2)$

$z^2 - 3z + 2 = 0 \quad \to z^2 - 3z + 2 = (z-1)(z-2)$

$z_1 + z_2 = 3 \mid z_1 = 1$

$z_1 \cdot z_2 = 2 \mid z_2 = 2$

$$\ln(z^2 - 3z + 2) = \ln[(z-1)(z-2)] = \ln(z-1) + \ln(z-2)$$

$$\ln(1 + (z-2)) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} (z-2)^n$$

$$\ln(1 + (z-3)) = \sum_{n=1}^{\infty} \frac{(-1)^{n}}{n} (z-3)^n$$

$z_0 = 0$:

$f_1(z) = \ln(z-1)$ $\qquad$ $f_2(z) = \ln(z-2)$

$f_1'(z) = \frac{1}{z-1}$ $\qquad$ $f_2'(z) = \frac{1}{z-2}$

$\Rightarrow f_1'(z_0) = -1$ $\qquad$ $\Rightarrow f_2'(z_0) = -\frac{1}{2}$

$f_1''(z) = -\frac{1}{(z-1)^2}$ $\qquad$ $f_2''(z_0) = -\frac{1}{(z-2)^2} = -\frac{1}{4}$

$\Rightarrow f_1''(z_0) = -1$ $\qquad$ $f_2'''(z_0) = \frac{2}{(z-2)^3} = \frac{2}{8}$

$f_1'''(z_0) = \left(-(z-1)^{-2}\right)' =$

$= \frac{2}{(z-1)^3} = -2$ $\qquad$ $f_2^{IV}(z_0) = -\frac{2\cdot 3}{(z-2)^4} = -\frac{6}{16}$

$\dots$ $\qquad$ $\dots$

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z-z_0)^n$$

$+(-z))$

$$f_1(z) = -\frac{1}{1!}z - \frac{1}{2!}z^2 - \frac{2}{3!}z^3 - \frac{6}{4!}z^4 - \frac{24}{5!}z^5 = \sum_{n=1}^{\infty}\frac{(-1)}{n}z^n$$

$f_1(z_0)$ – непр

$$f_2(z_0) = \ln|z_0 - 2| = \ln 2 \qquad f_1(z_0) = \ln|z_0 - 1| = 0$$

$$f_2(z) = \ln 2 - \frac{1/2}{1!}z - \frac{1/4}{2!}z^2 - \frac{2/8}{3!}z^3 - \frac{6/16}{4!}z^4 =$$

$$= \ln 2 - \sum_{n=1}^{\infty}\frac{1}{2^n \cdot n}z^n$$

$$\Rightarrow f(z) = f_1(z) + f_2(z) = \ln 2 - \sum_{n=1}^{\infty}\left(1 + \frac{1}{2^n}\right)\frac{z^n}{n}$$

$$\Rightarrow \ln(z^2 - 3z + 2) = \ln 2 - \sum_{n=1}^{\infty}\left(1 + \frac{1}{2^n}\right)\frac{z^n}{n}$$

$$R = \lim_{n\to\infty}\left|\frac{(2^n+1)\cdot 2^{n+1}(n+1)}{2^n \cdot n\,(2^{n+1}+1)}\right| = \lim_{n\to\infty}\left|\frac{2^n\left(1+\frac{1}{2^n}\right)2(n+1)}{2^n \cdot 2 \cdot n\left(1+\frac{1}{2^{n+1}}\right)}\right| = 1$$

$$\Rightarrow \ln(z^2 - 3z + 2) = \ln 2 - \sum_{n=1}^{\infty}\left(1 + \frac{1}{2^n}\right)\frac{z^n}{n}$$

$$R = 1.$$

467. (3.82) Разложить в ряд по степеням $(z-1)$ и найти радиус сходимости.

$$\frac{z}{z+2} = \frac{z+2-2}{z+2} = 1 - \frac{2}{z+2} \qquad z_0 = 1$$

$$f(z) = \frac{2}{z+2}; \quad f(z_0) = \frac{2}{3}$$

$$f'(z) = 2\cdot\left(-\frac{1}{(z+2)^2}\right); \quad f'(z_0) = 2\cdot\left(-\frac{1}{3^2}\right) = -\frac{2}{3^2}$$

$$f''(z) = +4\cdot\frac{1}{(z+2)^3}; \quad f''(z_0) = \frac{4}{3^3}$$

$$f'''(z) = -12\cdot\frac{1}{(z+2)^4}; \quad f'''(z_0) = -\frac{12}{3^4}$$

$$f(z) = \sum_{n=0}^{\infty}\frac{f^{(n)}(z_0)}{n!}(z - z_0)^n$$

$$\Rightarrow f(z) = \frac{2}{3} - \frac{2}{3^2\cdot 1!}(z-1) + \frac{4}{3^3\cdot 2!}(z-1)^2 - \frac{12}{3^4\cdot 3!}(z-1)^3 + \dots$$

$$\frac{z}{z+2} = 1 - \frac{2}{3} + \frac{2}{3^2\cdot 1!}(z-1) - \frac{4}{3^3\cdot 2!}(z-1)^2 + \frac{12}{3^4\cdot 3!}(z-1)^3 + \dots =$$

$$= \frac{1}{3} + \sum_{n=1}^{\infty}\frac{2}{3^{n+1}}\cdot(-1)^{n+1}\cdot(z-1)^n = \frac{1}{3} + 2\sum_{n=1}^{\infty}(-1)^{n+1}\frac{(z-1)^n}{3^{n+1}}$$

$$R = \lim_{n\to\infty} \left| \frac{3^{n+2}}{3^{n+1}} \right| = \underline{\underline{3}}$$

465 (3.80).

$$\int_0^z e^{z^2} dz = \sum_{n=0}^{\infty} \int_0^z \frac{z^{2n}}{n!} dz = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)\, n!}$$

$$e^{z^2} = \sum_{n=0}^{\infty} \frac{z^{2n}}{n!}$$

$$R = \lim_{n\to\infty} \left| \frac{(2n+3)(n+1)!}{(2n+1)\; n!} \right| = \infty$$

10.11.21

$$\ln(z^2 - 3z + 2) = \ln(2 - z) + \ln(1 - z) = \ln\left[2\left(1 - \frac{z}{2}\right)\right] + \ln(1 - z) =$$

$$= \ln 2 + \ln\left(1 - \frac{z}{2}\right) + \ln(1 - z) \; (=)$$

$$\ln(1+z) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} z^n}{n}$$

$$(=) \; \ln 2 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}\left(-\frac{z}{2}\right)^n + \sum_{k=1}^{\infty} \frac{(-1)^{k-1}(-z)^k}{k} =$$

$$= \ln 2 - \sum_{n=1}^{\infty} \frac{z^n}{n}\left(\frac{1}{2^n} + 1\right)$$

$$\underline{R = 1.}$$

$$R_1 = \lim_{n\to\infty} \frac{(n+1)\cdot 2^{n+1}}{n \cdot 2^n} = 2 \quad ; \quad R_2 = \lim_{k\to\infty} \left| \frac{k+1}{k} \right| = 1$$

$$|z| < 2 \qquad\qquad |z| < 1$$

$$\Rightarrow \underline{R} = \min(R_1, R_2) = \underline{R_2 = 1}.$$

444. $\displaystyle\sum_{n=1}^{\infty} \frac{z^{np}}{n}$

Если $p = 1$, $R = 1$ $\to \displaystyle\sum_{n=1}^{\infty} \frac{e^{in\varphi}}{n}$ - сходится $\varphi \neq 0$

$0 \leq \varphi < 2\pi$

$\varphi = 0 \to \sum \frac{1}{n}$ - расходится.

$\underline{p = 2}$ $\displaystyle\sum_{n=1}^{\infty} \frac{e^{i2n\varphi}}{n}$

$\varphi = 0 \to \displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$, $\varphi = \pi \to \displaystyle\sum_{n=1}^{\infty} \frac{e^{i2\pi n}}{n} = \sum \frac{1}{n}$ - расх.

$\underline{p = 3}$ $\displaystyle\sum_{n=1}^{\infty} \frac{e^{i3n\varphi}}{n}$ $\quad 3n\varphi = 2\pi$, $\varphi = 0$

$$z = e^{i\frac{2\pi k}{?}} \quad \varphi = \frac{2\pi}{3}$$ — в общем случае будет расходится

Нули аналит-их ф-ий.

507. (3.122)
z^2+9
$\to z^2+9=0$
$z^2=-9$
$z=\pm 3i$

$(z-3i)(z+3i)=z^2+9 \Rightarrow$ нули первого порядка.

509 (3.124)
$z\cdot\sin z = 0$

1) $z=0$
2) $\sin z = 0$
$z=\pi n,\ n\in\mathbb{Z}$

$$z\cdot\sin z = z\sum_{n=0}^{\infty}\frac{(-1)^n z^{2n+1}}{(2n+1)!} = z\left(z-\frac{z^3}{3!}+\frac{z^5}{5!}+\dots\right)=$$
$$= z^2\underbrace{\left(1-\frac{z^2}{3!}+\frac{z^4}{5!}-\dots\right)}_{\varphi(z)}\quad \varphi(0)\neq 0$$

$z=0$ – нуль второго порядка

$z=\pi n,\ n\in\mathbb{Z}$
$t=z-\pi n$
$z=t+\pi n$.

$$z\sin z = (t+\pi n)\sin(t+\pi n) = (-1)^n(t+\pi n)\sin t =$$
$$=(-1)^n(t+\pi n)\sum_{k=0}^{\infty}\frac{(-1)^k t^{2k+1}}{(2k+1)!} = (-1)^n(t+\pi n)\left(t-\frac{t^3}{3!}+\frac{t^5}{5!}+\dots\right)=$$
$$=(-1)^n\underbrace{(t+\pi n)}_{\psi(0)\neq 0}\,t\underbrace{\left(1-\frac{t^2}{3!}+\frac{t^4}{5!}+\dots\right)}_{\varphi(0)\neq 0}$$

$\Rightarrow t=0$ – 1 порядка
$\Rightarrow z=\pi n$ – нуль 1го порядка

$f(z)=(z-z_0)^k\varphi(z),\quad \varphi(z_0)\neq 0$ – опр. нуля ф-ии.

$z=z_0$ – ноль $k^{\text{го}}$ порядка.

511. (3.126)
$1-\cos z$

$1-\cos z=0$
$\cos z = 0$
1) $z=0$
2) $z=2\pi k,\ k\in\mathbb{Z}$

$$1-\cos z = 1-\sum_{n=0}^{\infty}\frac{(-1)^n z^{2n}}{(2n)!} = 1-1+\frac{z^2}{2!}-\frac{z^4}{4!}+\frac{z^6}{6!}-\ldots =$$

$$= z^2\left(\frac{1}{2!}-\frac{z^2}{4!}+\frac{z^4}{6!}-\ldots\right)$$

$$\varphi(0)\neq 0 = \frac{1}{2!} \qquad \Rightarrow z=0 \quad - 2^{го} \text{ порядка}$$

2) $z=2\pi k$, $t=z-2\pi k$.

$$1-\cos(t+2\pi k) = 1-\cos t = \sim \text{предыдущий случай.}$$

$$= t^2\left(\frac{1}{2!}-\frac{t^2}{4!}+\ldots\right)$$

$t=0$ - ноль $2^{го}$ порядка

$\Rightarrow \underline{\underline{z=2\pi k}}$ ~ $2^{го}$ порядка.

513 (3.128)

$$\frac{1-\operatorname{ctg} z}{z} = 0$$

$$\operatorname{ctg} z = 1 \Rightarrow z = \frac{\pi}{4}+\pi k$$

$$t = z-\left(\frac{\pi}{4}+\pi k\right)$$

$$\operatorname{ctg} z = \operatorname{ctg}\left(t+\frac{\pi}{4}+\pi k\right) = \frac{\cos\left(t+\frac{\pi}{4}\right)(-1)^k}{\sin\left(t+\frac{\pi}{4}\right)(-1)^k} =$$

$$\frac{1-\operatorname{ctg} z}{z} = \left(1-\frac{\cos\left(t+\frac{\pi}{4}\right)}{\sin\left(t+\frac{\pi}{4}\right)}\right)\frac{1}{t+\frac{\pi}{4}+\pi k} = 0$$

$$\Rightarrow \frac{\sin\left(t+\frac{\pi}{4}\right)-\cos\left(t+\frac{\pi}{4}\right)}{\sin\left(t+\frac{\pi}{4}\right)\left(t+\frac{\pi}{4}+\pi k\right)} =$$

$$\sin t\cos\frac{\pi}{4}+\cos t\sin\frac{\pi}{4}-\cos t\cos\frac{\pi}{4}+\sin t\sin\frac{\pi}{4} = \sqrt{2}\sin t$$

$$= \frac{\sqrt{2}\sum_{n=0}^{\infty}\frac{(-1)^n t^{2n+1}}{(2n+1)!}}{\psi(t)}$$

$t=0$ - ноль первого порядка

$\Rightarrow \underline{\underline{z=\frac{\pi}{4}+\pi k}}$ - $1^{го}$ порядка

$k\in\mathbb{Z}$

3.130

$$\sin^3 z = \left(\sum_{n=0}^{\infty}(-1)^n\frac{z^{2n+1}}{(2n+1)!}\right)^3 = \left(z-\frac{z^3}{3!}+\frac{z^5}{5!}\ldots\right)^3 = z^3\underbrace{\left(1-\frac{z^2}{3!}+\frac{z^4}{5!}+\ldots\right)}_{\varphi(0)\neq 0}$$

$t=z-\pi k$.

$z=0$ - ноль $3^{го}$ порядка

$$\sin^3(t+\pi k) = (-1)^{3k}\,\underline{\sin^3 t} \qquad t=0 \ - 3^{го} \text{ порядка}$$

$\Rightarrow \underline{z=\pi k}$, $k\in\mathbb{Z}$ - ноль $1^{го}$ порядка

517 (3.132)

$$\sin z^3 = \{u = z^3\} = \sin u = \sum_{n=0}^{\infty} (-1)^n \frac{u^{2n+1}}{(2n+1)!} =$$
$$= u - \frac{u^3}{3!} + \frac{u^5}{5!} \ldots = u\left(1 - \frac{u^2}{3!} + \frac{u^4}{5!} + \ldots\right)$$
$\rightarrow u=0$ – $1^{го}$ порядка.

$u = z^3 \rightarrow z=0$ ноль $3^{го}$ порядка.

$z^3 = \pi k$

$u = \pi k$ – ноль $1^{го}$ порядка

$z^3 = \pi k \rightarrow z = \sqrt[3]{\pi k}$ – ноль $3^{го}$ порядка.

519 (3.134)

$\cos z^3 = 0$

$z^3 = \frac{\pi}{2} + \pi k$

$u = z^3 \rightarrow t = u - \left(\frac{\pi}{2} + \pi k\right)$

$\cos\left(t + \frac{\pi}{2} + \pi k\right) = (-1)^k \cos\left(t + \frac{\pi}{2}\right) = (-1)^{k+1} \sin t \rightarrow t=0$ – „0" $1^{го}$ порядка

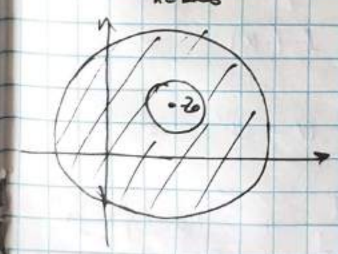
$u = \frac{\pi}{2} + \pi k$

$z^3 = \frac{\pi}{2} + \pi k \rightarrow z = \sqrt[3]{\frac{\pi}{2} + \pi k}$

– „0" $3^{го}$ порядка.

11.11.21

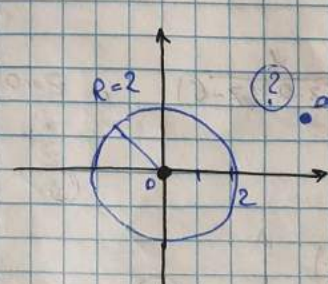
$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z - z_0)^n$$



$$\frac{1}{1-q} = \sum_{n=0}^{\infty} q^n, \quad |q| < 1$$

№ 543. $\frac{1}{z-2}$, в окр-ти $z=0$, $z=\infty$

при $z=2$ – ф-я не существует.



① $z=0$

$$\frac{1}{z-2} = -\frac{1}{2} \frac{1}{1-\frac{z}{2}} = -\frac{1}{2} \sum_{n=0}^{\infty} \left(\frac{z}{2}\right)^n$$

$|z| < 2$ (выносим то, что больше по модулю)

$\left|\frac{z}{2}\right| < 1$

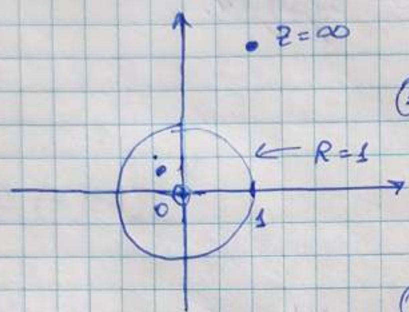
②
$$\frac{1}{z-2} = \frac{1}{z} \frac{1}{1-\frac{2}{z}} = \frac{1}{z} \sum_{n=0}^{\infty} \frac{2^n}{z^n} = \sum_{n=1}^{\infty} \frac{2^n}{z^{n+1}}$$

$|z|>2$ → восхождение → $\left|\frac{2}{z}\right|<1$

546. $\frac{1}{z(z-1)}$, $z=0$, $z=1$, $z=\infty$

z^n (1) $(z-1)^n$ (2) z^n (3)

(1) Область аналитичности:



(3) $z=0$: $\frac{1}{z(1-z)}=\frac{1}{z}\cdot\frac{1}{1-z}=\frac{1}{z}\sum_{n=0}^{\infty}z^n=$

$=\frac{1}{z}+1+z+z^2+\ldots=\frac{1}{z}+\sum_{n=0}^{\infty}z^n$

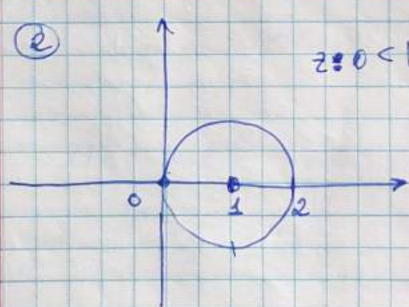
(3) $z=\infty$:

$\frac{1}{z(1-z)}=\frac{1}{z}\cdot\frac{1}{1-z}=-\frac{1}{z^2}\frac{1}{1-\frac{1}{z}}=-\frac{1}{z^2}\sum_{n=0}^{\infty}\frac{1}{z^n}=$

$|z|>1$ $\left|\frac{1}{z}\right|<1$

$=-\sum_{n=0}^{\infty}\frac{1}{z^{n+2}}$

(2)



$z: 0<|z-1|<1$

$u=z-1$; $z=1+u$

$\frac{1}{z(1-z)}=\frac{1}{(1+u)}\cdot\frac{-1}{u}=-\frac{1}{u}\frac{1}{1-(-u)}=$

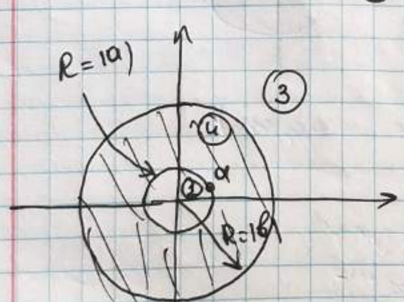
$|u|<1$

$=-\frac{1}{u}\sum_{n=0}^{\infty}(-1)^n u^n=-\frac{1}{u}+1-u+u^2\ldots=$

$=-\frac{1}{u}+\sum_{n=0}^{\infty}u^n\cdot(-1)^n=-\frac{1}{z-1}+\sum_{n=0}^{\infty}(z-1)^n(-1)^n$

546. $\frac{1}{(z-a)(z-b)}$, $z=0$, $z=a$, $z=\infty$; $a<|z|<b$

z^n (1) $(z-a)^n$ (2) z^n (3) z^n (4)



$\frac{1}{(z-a)(z-b)}=\frac{1}{a-b}\left(\frac{1}{z-a}-\frac{1}{z-b}\right)$

(1) $|z|<a$

$\frac{1}{a-b}\left(\frac{1}{z-a}-\frac{1}{z-b}\right)=\frac{1}{a-b}\left(-\frac{1}{a}\frac{1}{1-\frac{z}{a}}+\frac{1}{b}\frac{1}{1-\frac{z}{b}}\right)$

$\left|\frac{z}{a}\right|<1$ $\left|\frac{z}{b}\right|<1$

$$= \frac{1}{a-b}\left(-\frac{1}{a}\sum_{n=0}^{\infty}\left(\frac{z}{a}\right)^n + \frac{1}{b}\sum_{n=0}^{\infty}\left(\frac{z}{b}\right)^n\right) = \frac{1}{a-b}\sum_{n=0}^{\infty}\left(\frac{1}{b^{n+1}} - \frac{1}{a^{n+1}}\right)z^n$$

(3) $$\frac{1}{a-b}\left(\frac{1}{z-a} - \frac{1}{z-b}\right) = \frac{1}{a-b}\left(\frac{1}{z}\,\frac{1}{1-\frac{a}{z}} - \frac{1}{z}\,\frac{1}{1-\frac{b}{z}}\right) =$$

$|a|<|b|<|z|$

$$= \frac{1}{a-b}\cdot\frac{1}{z}\left(\sum_{n=0}^{\infty}\frac{a^n}{z^n} - \sum_{n=0}^{\infty}\frac{b^n}{z^n}\right) = \frac{1}{a-b}\sum_{n=1}^{\infty}\frac{(a^{n-1} - b^{n-1})}{z^n}$$

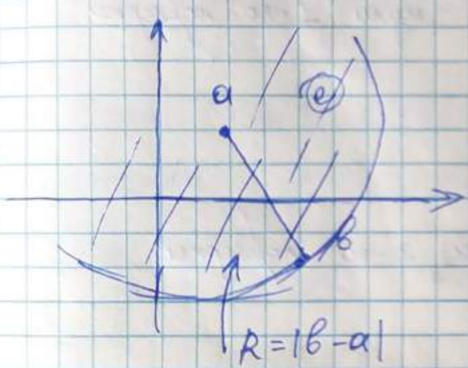
IIII - (4) случай

(4) $|a|<|z|<|b|$

$$\frac{1}{a-b}\left(\frac{1}{z-a} - \frac{1}{z-b}\right) = \frac{1}{a-b}\left(\frac{1}{z}\cdot\frac{1}{1-\frac{a}{z}} + \frac{1}{b}\,\frac{1}{1-\frac{z}{b}}\right) =$$

$$= \frac{1}{a-b}\left(\frac{1}{z}\sum_{n=0}^{\infty}\frac{a^n}{z^n} + \frac{1}{b}\sum_{n=0}^{\infty}\frac{z^n}{b^n}\right) = \frac{1}{a-b}\left(\sum_{n=1}^{\infty}\frac{a^{n-1}}{z^n} + \sum_{n=0}^{\infty}\frac{z^n}{b^{n+1}}\right)$$

(2) Область аналитичности ~ внутри окружности $R=|b-a|$



$u = z-a$

$$\frac{1}{(z-a)(z-b)} = \frac{1}{u(u+a-b)} = \frac{1}{(a-b)}\cdot\frac{1}{u}\,\frac{1}{1+\frac{u}{a-b}} =$$

$$= \frac{1}{(a-b)}\,\frac{1}{u}\sum_{n=0}^{\infty}\frac{(-u)^n}{(a-b)^n} = -\frac{1}{a-b}\cdot\frac{1}{u} + \sum_{n=0}^{\infty}\frac{(-1)^{n+1}u^n}{(a-b)^{n+2}} =$$

$$= -\frac{1}{(a-b)}\cdot\frac{1}{(z-a)} + \sum_{n=0}^{\infty}\frac{(-1)^{n+1}(z-a)^n}{(a-b)^{n+2}}$$

Домашняя работа.

508. (3.123)

$$w(z)=\frac{z^2+9}{z^4}=0$$

$$z^2+9=0 \quad ; \quad (z^2+9)=(z-3i)(z+3i)$$

$z=\pm 3i$ – нули 1-го порядка

$$w(z)=\frac{1}{z^2}+\frac{9}{z^4}=\frac{1}{z^2}\left(1+\frac{9}{z^2}\right)$$

$\Rightarrow$ $z=\infty$ – нуль второго порядка.

510. (3.125)

$$w(z)=(1-e^z)(z^2-4)^3=0$$

1) $1-e^z=0$

$e^z=1 \to e^{x+iy}=1.$

$$z=\text{Ln}(1)=\ln 1+i(0+2\pi k)=2\pi k i$$

$z=2\pi k i$ $(k\in Z)$ – нули 1-го порядка.

2) $(z^2-4)^3=0$

$z^2=4 \to z=\pm 2$

$$(z^2-4)^3=(z^2-4)(z^2-4)(z^2-4)$$

$\Rightarrow$ $z=\pm 2$ – нули 3-го порядка.

512 (3.127)

$$w(z)=\frac{(z^2-\pi^2)\sin z}{z^7}$$

1) $(z^2-\pi^2)=0$

$z=\pm\pi$

$$(z^2-\pi^2)\sin z=(z^2-\pi^2)\sum_{n=1}^{\infty}\frac{(-1)^{n+1}z^{2n-1}}{(2n-1)!}=(z^2-\pi^2)\cdot\left(z-\frac{z^3}{3!}+\frac{z^5}{5!}-\dots\right.$$

$$=z^3\left(1-\frac{\pi^2}{z^2}\right)\underbrace{\left(1-\frac{z^2}{3!}+\frac{z^5}{5!}-\dots\right)}_{\varphi(z)\neq 0}$$

$\Rightarrow$ $z=\pm\pi$ – ноль $3^{го}$ порядка (лучше через замену)

2) $\sin z=0$

$z=\pi k,\ k\in z \to t=z-\pi k \to z=\pi k+t$

$$\Rightarrow \sin z=\sin(t+\pi k)=(-1)^k\sin t=(-1)^k\sum_{k=1}^{\infty}\frac{(-1)^{k+1}t^{2k-1}}{(2k-1)!}=$$

$$\frac{(z^2-\pi^2)\sin z}{z^7}=\underbrace{\frac{[(t+\pi k)^2-\pi^2]}{(t+\pi k)^7}}\cdot t\underbrace{\left(1-\frac{t^2}{3!}+\frac{t^4}{5!}-\dots\right)}$$

$\to \psi(t=0)\neq 0$

$\Rightarrow$ $z=\pi k$ – ноль $1^{го}$ порядка.

514 (3.129)

$w(z) = e^{tg z} = 0$

$e^{tg z} = 0$

$tg z = \underline{Ln(0)}$ неопр. $\Rightarrow$ нулей нет

516 (3.131)

$w(z) = \dfrac{\sin^3 z}{z}$

$\sin^3 z = 0$

$\sin z = 0$

$z = \pi k, \; k \in Z$

1) $z = 0$ (при $k = 0$)

$$\frac{\sin^3 z}{z} = \frac{1}{z}\left(\sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!}\right)^3 = \frac{1}{z}\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots\right)^3 =$$

$$= z^2\left(1 - \frac{z^2}{3!} + \frac{z^4}{5!} + \dots\right)^3$$

$\Rightarrow z = 0$ – нуль 2^го порядка

2) $z = \pi k$ ($k \in Z$, $k \neq 0$)

$t = z - \pi k \to z = t + \pi k$

$$\frac{\sin^3 z}{z} = \frac{1}{(t+\pi k)} \cdot (-1)^{3k} \sin^3 t$$

$t = 0$ – нуль 3^го порядка

$\Rightarrow z = \pi k$ – нули 3^го порядка

518 (3.133.)

$w(z) = \cos^3 z = 0$

$\cos^3 z = 0$

$\cos z = 0$

$z = \frac{\pi}{2} + \pi k, \; k \in Z = \frac{\pi}{2}(2k+1)$

$t = z - \left(\frac{\pi}{2} + \pi k\right) = z - \frac{\pi}{2} - \pi k \to z = t + \frac{\pi}{2} + \pi k$

$$\cos^3 z = \left(\cos\left(t + \frac{\pi}{2} + \pi k\right)\right)^3 = -\sin^3(t + \pi k) = (-1)^{3k} \cdot (-1) \cdot \sin^3 t =$$

$$= (-1)^{3k+1} \sin^3 t$$

$t = 0$ – нуль 3^го порядка (из 3.130)

$\Rightarrow z = \frac{\pi}{2}(2k+1)$ – нуль 3^го порядка

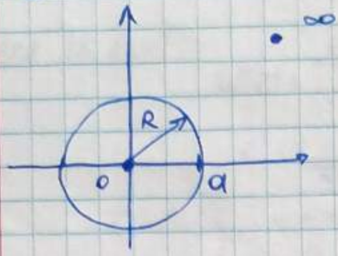
Ряды Лорана.

544 (4.2)

$w(z) = \dfrac{1}{(z-a)^k}$ ($a \neq 0$, $k \in \mathbb{N}$) в окрестности точек $z = 0$ и $z = \infty$

При $z = a$ – ф-я не существует.

① $z=0$



$$\frac{1}{(z-a)^k}=\frac{1}{(-a)^k\left(1-\frac{z}{a}\right)^k}=\left(-\frac{1}{a}\cdot\frac{1}{1-z/a}\right)^k \ominus$$

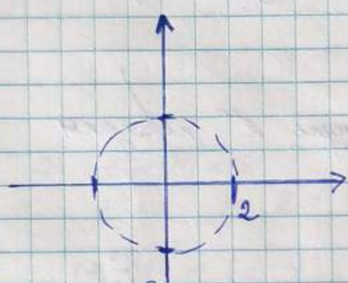
$$|z|<a \rightarrow \left|\frac{z}{a}\right|<1$$

$$\ominus \left(-\frac{1}{a}\right)^k\cdot\left(\sum_{n=0}^{\infty}\left(\frac{z}{a}\right)^n\right)^k=\frac{(-1)^k}{a^k}\sum_{n=0}^{\infty}C_{k-1}^{n+k-1}\left(\frac{z}{a}\right)^n$$

② $z=\infty$

$$\frac{1}{(z-a)^k}=\frac{1}{z^k}\left(\frac{1}{1-\frac{a}{z}}\right)^k=\frac{1}{z^k}\left(\sum_{n=0}^{\infty}\left(\frac{a}{z}\right)^n\right)^k=\frac{1}{z^k}\sum_{n=0}^{\infty}C_{k-1}^{n+k-1}\left(\frac{a}{z}\right)^n$$

547 (4.5) $w(z)=\dfrac{z^2-2z+5}{(z-2)(z^2+1)}$ в окрестности точки $z=2$ и в кольце $1<|z|<2$



① $z=2 \rightarrow (z-2)^n$

$$w(z)=\frac{z^2-2z+5}{(z-2)(z^2+1)}=\frac{1}{z-2}\cdot\frac{z^2-2z+5}{(z^2+1)}=$$

$$=\frac{1}{z-2}\left(1+\frac{4-2z}{(z^2+1)}\right)=\frac{1}{z-2}\left(1+\frac{2(2-z)}{z^2+1}\right)=$$

$$\begin{array}{r|l} z^2-2z+5 & z^2+1 \\ z^2+1 & 1 \\ \hline -2z+4 & \end{array}$$

$$\frac{1}{z-2}-\frac{2}{(z-i)(z+i)} \ominus$$

$$\frac{1}{(z-i)(z+i)}=\frac{1}{z-i}-\frac{1}{z+i}=\frac{z+i-z+i}{(z-i)(z+i)}=\frac{2i}{(z-i)(z+2i)}$$

$$\ominus \frac{1}{z-2}-\frac{\cancel{2}}{\cancel{2}i}\left(\frac{1}{z-i}-\frac{1}{z+i}\right)=\frac{1}{z-2}+i\left(\frac{1}{z-i}-\frac{1}{z+i}\right)=$$

$$=\frac{1}{z-2}+i\left(\frac{1}{z\left(1-\frac{i}{z}\right)}-\frac{1}{z\left(1+\frac{i}{z}\right)}\right)=$$

$$=\frac{1}{z-2}+i\left(\frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{i}{z}\right)^n-\frac{1}{z}\sum_{n=0}^{\infty}\left(-\frac{i}{z}\right)^n\right)=$$

$$=\frac{1}{z-2}+i\left(\sum_{n=0}^{\infty}\frac{i^n}{z^{n+1}}-\sum_{n=0}^{\infty}\frac{(-1)^n i^n}{z^{n+1}}\right)=$$

$$=\frac{1}{z-2}+i\left(\cancel{\frac{1}{z}}+\frac{i}{z^2}-\cancel{\frac{1}{z^3}}+\frac{i^3}{z^4}-\cancel{\frac{1}{z^5}}+\ldots-\cancel{\frac{1}{z}}+\frac{i}{z^2}+\cancel{\frac{1}{z^3}}+\frac{i^3}{z^4}+\cancel{\frac{1}{z^5}}\right)$$

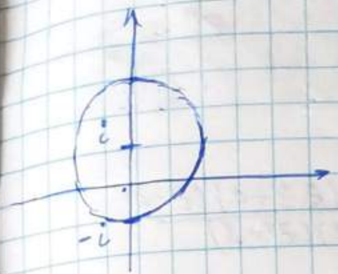
$$=\frac{1}{z-2}+i\left(\frac{2i}{z^2}+\frac{2i^3}{z^4}+\ldots\right)$$

?

548 (4.8) $w(z) = \dfrac{1}{(z^2+1)^2}$ в окрестности $z=i$ и $z=\infty$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \downarrow \qquad\quad \downarrow$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad (z-i)^n \qquad z^n$



1) $z=i \to t=z-i \to z=t+i$

$$\frac{1}{(z+i)^2(z-i)^2} = \frac{1}{(t+2i)^2t^2} = \frac{1}{t^2}\cdot\frac{1}{(2i)^2}\,\frac{1}{\left(1+\left(\frac{t}{2i}\right)^2\right)} =$$

$$= -\frac{1}{4t^2}\left(\sum_{n=0}^{\infty}\left(-\frac{t}{2i}\right)^n\right)^2$$

?

548 (4,6) $w(z) = \frac{1}{(z^2+1)^2}$ в окрестности точек $z=i$ и $z=\infty$

$(z-i)^n \qquad z^n$

1)

№547. $w(z) = \frac{z^2-2z+5}{(z-2)(z^2+1)} = \frac{A}{z-2} + \frac{Bz+C}{z^2+1} = \frac{A(z^2+1)+(Bz+C)(z-2)}{(z-2)(z^2+1)}$

$$\begin{array}{c|l} z^2 & 1 = A+B \\ z & -2 = -2B+C \\ 1 & 5 = A-2C \end{array} \qquad \begin{array}{l} A = 1-B \\ -2 = -2B+C \\ 5 = 1-B-2C \end{array} \qquad \begin{array}{l} B = -2C-4 \\ -2 = -2(-2C-4)+C \\ -3C = -10 \end{array}$$

$\Rightarrow C = \frac{-10}{5} = -2$

$5C = -10$

$B = 0; \; A = 1.$

$$w(z) = \frac{1}{z-2} - \frac{2}{1+z^2}$$

(1) для кольца $1<|z|<2$

$$\Rightarrow w(z) = -\frac{1}{2}\cdot\frac{1}{1-\frac{z}{2}} - \frac{2}{z^2}\,\frac{1}{1+\frac{1}{z^2}} = -\frac{1}{2}\sum_{n=0}^{\infty}\frac{z^n}{2^n} - \frac{2}{z^2}\sum_{k=0}^{\infty}\frac{(-1)^k}{z^{2k}}$$

(2) В окр-ти $z=2$

$u = z-2$

$$w = \frac{(u+2)^2-2(u+2)+5}{u((u+2)^2+1)} = \frac{u^2+2u+5}{u(u^2+4u+5)} = \left\{\text{аналогично пункту (1) только в окрестности т. } u=0\right\}$$

$$= \frac{A}{u} + \frac{B}{u-(-2+i)} + \frac{C}{u-(-2-i)}$$

$|u| < |-2 \pm i|$

...

548. $w(z) = \frac{1}{(z^2+1)^2} = \frac{1}{\left(\frac{1}{u^2}+1\right)^2} = \frac{u^4}{(1+u^2)^2} = -\frac{u^4}{2u}\sum_{n=0}^{\infty}(-1)^n u^{2n} = \frac{1}{2}\sum_{n=0}^{\infty}(-1)^{n+1}u^{2n+2}$

(1) $z=i$, [$z=\infty$, $u = \frac{1}{z}$

$$\frac{1}{(1+u^2)^2} = -\frac{1}{2u}\left(\frac{1}{1+u^2}\right)' = -\frac{1}{2u}\sum_{n=0}^{\infty}(-1)^n u^{2n-1}\cdot 2n$$

(2) $z=i \rightarrow u = z-i$

$$\frac{1}{(z-i)^2(z+i)^2} = \frac{1}{u^2}\cdot\frac{1}{(u+2i)^2} = -\frac{1}{u^2}\left(\frac{1}{u+2i}\right)' = -\frac{1}{u^2}\cdot\frac{1}{2i}\left(\frac{1}{1+\frac{u}{2i}}\right)' =$$

$|u| < |2i| < 2$

$$= -\frac{1}{u^2}\cdot\frac{1}{2i}\left(\sum_{n=0}^{\infty}\frac{(-1)u^n}{(2i)^n}\right)' = -\frac{1}{u^2\cdot 2i}\sum_{n=1}^{\infty}\frac{(-1)^n\cdot n\cdot u^{n-1}}{(2i)^n} =$$

$$= \frac{1}{u^2}\sum_{n=0}^{\infty}\frac{(-1)^n(n+1)u^n}{(2i)^{n+2}} = \frac{1}{u^2}\cdot\frac{1}{(2i)^2} + \frac{-1\cdot 2}{u(2i)^3} + \sum_{n=2}^{\infty}\frac{(-1)^n(n+1)u^{n-2}}{(2i)^{n+2}}$$

$k = n-2$

561. 18.11.21

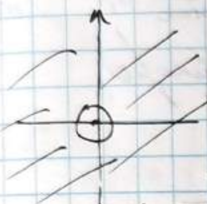
1) $\cos\frac{1}{z}$ — да

$z=0$

$$\cos u = \sum_{n=0}^{\infty}\frac{(-1)^n u^{2n}}{(2n)!}$$

$0<|z|<\infty$

$|u|<\infty$



да

2) $\cos\frac{1}{z} = \cos u \rightarrow z=\infty \quad u=\frac{1}{z}$

$\underline{0<|z|<\infty}$

да.

3) $\sec\frac{1}{z-1} = \frac{1}{\cos\frac{1}{z-1}} = \frac{1}{\cos\frac{1}{u}}$

$z=1$

$u=0$? невозможно построить область аналитичности

$\Rightarrow$ <u>нет</u>

7) $\frac{z}{\sin z - 3}$ ~ не можем построить кольцо аналитичности.

$z=\infty$

$\sin z = 3$

$z = x+iy$

$$\frac{e^{iz}-e^{-iz}}{2i} = 3$$

$e^{2iz}-1 = 6ie^{iz}$

$e^{2iz}-6ie^{iz}-1=0 \quad \rightarrow e^{iz} = 3i+\frac{\sqrt{-32}}{2} = 3i+i\,2\sqrt{2} = i(3+2\sqrt{2})$

$iz = \mathrm{Ln}\, i(3+2\sqrt{2})$

$z = -i\left[\ln(3+2\sqrt{2}) + i\left(\frac{\pi}{2}+2\pi n\right)\right]$

$\Rightarrow$ <u>нет</u>

9) $\ln\frac{1}{1-z} = \cancel{\ln 1} = \left(u=\frac{1}{z}\right) = \ln\frac{u}{1-u} = \ln u -$

неопр. в $u=0$

$\Rightarrow$ нет

№562.

1) $\sqrt{z} = z^{\frac{1}{2}}$

$z=0 \to$ нет - не возможно получить разложение по целым степеням

2) $\sqrt{z(z-1)} = \sqrt{\frac{1}{u}\left(\frac{1}{u}-1\right)} = \frac{1}{\pm u}\cdot\underbrace{\sqrt{1-u}}_{|u|<1} = \pm\frac{1}{u}\sum_{m=0}^{\infty}(-1)^m\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\ldots}{m!}u^m$

$z=\infty$

расклад. в ряд Тейлора $(1+z)^m$

$\Rightarrow$ да

8) $\sqrt{z+\sqrt{z^2-1}} = \sqrt{\frac{1}{u}\pm\frac{1}{u}\sqrt{1-u^2}} = \underline{\underline{\frac{1}{\sqrt{u}}}}\left(1\pm\sqrt{1-u^2}\right)^{\frac{1}{2}}$

$z=\infty$

$\Rightarrow$ нет

13) $\mathrm{Arcsin}\, z \Rightarrow$ да существ. разл. в ряд Тейлора

$z=0$

15) $\mathrm{Arsh}\,(i+z) = \int_0^{z+i}\frac{d\xi}{\sqrt{1+\xi^2}} = \int_0^{z}\frac{d\xi}{\sqrt{1+(\xi+i)^2}}$

$z=0$

$|\xi+i|>1$

№549. $\sqrt{(z-a)(z-b)} = \sqrt{\left(\frac{1}{u}-a\right)\left(\frac{1}{u}-b\right)} = \pm\frac{1}{u}(1-au)^{\frac{1}{2}}(1-bu)^{\frac{1}{2}} =$

$z=\infty,\ u=\frac{1}{z}$

$= \pm\frac{1}{u}\sum_{n=0}^{\infty}\frac{(-1)^n a^n u^n \frac{1}{2}\left(\frac{1}{2}-1\right)\ldots\left(\frac{1}{2}-n+1\right)}{n!}\cdot\sum_{k=0}^{\infty}\frac{(-1)^k b^k u^k\frac{1}{2}\left(\frac{1}{2}-1\right)\ldots\left(\frac{1}{2}-k+1\right)}{k!} =$

$= \pm\frac{1}{u}\left(1+\sum_{n=1}^{\infty}\frac{(-1)^n(-1)^{n-1}(2n-1)!!}{n!\,2^n}u^n\right)\left(1+\sum_{k=1}^{\infty}\frac{(-1)^k(-1)^{k-1}(2k-1)!!}{2^k\,k!}b^k u^k\right) =$

$= \pm\frac{1}{u} + 2\sum_{n=1}^{\infty}\frac{(-1)(2n-1)!!\,(a^n+b^n)}{n!\,2^n}u^{n-1} + \sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{(2n-1)!!\,(2k-1)!!}{2^{k+n}\,k!\,n!}a^n b^k\cdot$

$\cdot\, u^{n+k-1}$

№553. $\cos\frac{z^2-4z}{(z-2)^2} = \cos\frac{(u+2)^2-4(u+2)}{u^2} = \cos\frac{u^2-4}{u^2} = \cos\left(1-\frac{4}{u^2}\right) =$

$z=2;\ z-2=u$

$= \cos 1 \cos\frac{4}{u^2} + \sin 1 \sin\frac{4}{u^2} = \cos 1 \sum\limits_{n=0}^{\infty}\frac{(-1)^n 4^{2n}}{(2n)!\,u^{4n}} + \sin 1 \sum\limits_{n=0}^{\infty}\frac{(-1)^n 4^{2n+1}}{(2n+1)!\,u^{2(2n+1)}}$

и $u = z-2$ → разложение.

№555.

$$e^{z+\frac{1}{z}} = e^{z}\cdot e^{\frac{1}{z}} = \sum_{k=0}^{\infty}\frac{z^k}{k!}\sum_{n=0}^{\infty}\frac{1}{n!\,z^n} = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{1}{k!\,n!}z^{k-n} =$$

ряды сходятся

$m = k-n \to k = n+m$

$$= \sum_{n=0}^{\infty}\sum_{m=-n}^{\infty}\frac{1}{n!\,(n+m)!}z^{m}$$

Домашняя работа.

<u>550</u> (4.8)

$f(z) = \sqrt{\frac{z}{(z-1)(z-2)}}$ в кольце $1<|z|<2$

$$\frac{z}{(z-1)(z-2)} = \frac{A}{z-1} + \frac{B}{z-2} = \frac{A(z-2)+B(z-1)}{(z-1)(z-2)}$$

$A(z-2)+B(z-1) = z$

$Az - 2A + Bz - B = z$

$z(A+B) - (2A+B) = z \rightarrow \begin{cases} A+B=1 \\ 2A+B=0 \end{cases}$

$-A = 1 \rightarrow A = -1 \Rightarrow B = 2$

$$\frac{z}{(z-1)(z-2)} = \frac{2}{z-2} - \frac{1}{z-1}$$

$$\underline{\underline{f(z)}} = \sqrt{\frac{2}{z-2} - \frac{1}{z-1}} = \sqrt{\frac{2}{2\left(\frac{z}{2}-1\right)} - \frac{1}{z\left(1-\frac{1}{z}\right)}} = \sqrt{-\frac{1}{\left(1-\frac{z}{2}\right)} - \frac{1}{z\left(1-\frac{1}{z}\right)}} =$$

$$= \sqrt{-1\cdot\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n - \frac{1}{z}\cdot\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n} = \left[-\sum_{n=0}^{\infty}\frac{z^n}{2^n} - \sum_{n=0}^{\infty}\frac{1}{z^{n+1}}\right]^{\frac{1}{2}}$$

<u>551.</u> (4.9)

$f(z) = z^2 e^{\frac{1}{z}}$ в окрестности точек $z=0$ и $z=\infty$

$$e^{\frac{1}{z}} = \sum_{n=0}^{\infty}\frac{\left(\frac{1}{z}\right)^n}{n!} = \sum_{n=0}^{\infty}\frac{1}{n!\,z^n} = 1 + \frac{1}{z} + \frac{1}{2!\,z^2} + \frac{1}{3!\,z^3} + \dots$$

① $z=0 \rightarrow z^n$

$$f(z) = z^2\cdot\sum_{n=0}^{\infty}\frac{1}{n!\,z^n} = \sum_{n=0}^{\infty}\frac{1}{n!\,z^{n-2}}$$

* Совокупность членов ряда с неотрицательными степенями $\sum\limits_{n=0}^{\infty} C_n(z-z_0)^n$ называется <u>правильной частью ряда Лорана</u>; члены с отрицательными степенями образуют <u>главную часть</u> ряда Лорана $\sum\limits_{n=-\infty}^{-1} C_n(z-z_0)^n$

$$f(z) = z^2\left(1 + \frac{1}{z} + \frac{1}{2!\,z^2} + \dots\right) = z^2 + z + \frac{1}{2} + \sum_{n=3}^{\infty} \frac{1}{n!\,z^n} =$$

$$= z^2 + z + \frac{1}{2} + \sum_{n=-\infty}^{-1} \frac{z^n}{(2-n)!}$$

$$\to u = \frac{1}{z} \quad \to z = \frac{1}{u}$$

② $z = \infty \quad \to z^n$

$$f(u) = \left(\frac{1}{u}\right)^2 \cdot e^u$$

$$f(z) = z^2 + z + \frac{1}{2} + \sum_{n=-\infty}^{-1} \frac{z^n}{(2-n)!} \Leftarrow$$

$$\to f(u) = \frac{1}{u^2} + \frac{1}{u} + \frac{1}{2} + \sum_{n=3}^{\infty} \frac{u^n}{u!}$$

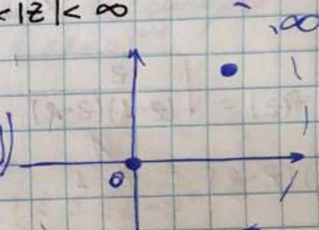
③ $0 < |z| < \infty$

$f(z) = z^2 \cdot e^{\frac{1}{z}}$; Аналогично

$$f(z) = z^2 + z + \frac{1}{2} + \sum_{n=1}^{\infty} \frac{1}{(n+2)!\,z^n} \quad \text{при} \quad 0 < |z| < \infty$$

556 (4.14)

$f(z) = \sin z \cdot \sin\frac{1}{z}$ в области $0 < |z| < \infty$



$$\sin\alpha \cdot \sin\beta = \frac{1}{2}\left(\cos(\alpha-\beta) - \cos(\alpha+\beta)\right)$$

$$\sin z \cdot \sin\frac{1}{z} = \frac{1}{2}\left(\cos\left(z - \frac{1}{z}\right) - \cos\left(z + \frac{1}{z}\right)\right)$$

$$\cos\left(z - \frac{1}{z}\right) = \cos\frac{z^2-1}{z}$$

?

$$\cos\frac{z^2-1}{z} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \cdot \frac{(z^2-1)^{2n}}{z^{2n}}$$

$$\cos\frac{z^2+1}{z} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \frac{(z^2+1)^{2n}}{z^{2n}}$$

$$f(z) = \frac{1}{2}\left(\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \frac{(z^2-1)^{2n}}{z^{2n}} - \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \frac{(z^2+1)^{2n}}{z^{2n}}\right) =$$

$$= \frac{1}{2}\left(\cancel{1} - \frac{(z^2-1)^2}{2!\,z^2} + \frac{(z^2-1)^4}{4!\,z^4} - \frac{(z^2-1)^6}{6!\,z^6} + \dots \right.$$
$$\left. - \cancel{1} + \frac{(z^2+1)^2}{2!\,z^2} - \frac{(z^2+1)^4}{4!\,z^4} + \frac{(z^2+1)^6}{6!\,z^6} - \dots\right) =$$

$$= \frac{1}{2}\left(\frac{(z^2+1)^2 - (z^2-1)^2}{2!\,z^2} + \frac{(z^2+1)^4 - (z^2-1)^4}{4!\,z^4} + \frac{(z^2+1)^6 - (z^2-1)^6}{6!\,z^6} + \dots\right) =$$

$$= \frac{1}{2}\sum_{n=1}^{\infty} \frac{(z^2+1)^n - (z^2-1)^n}{(2n)!\,z^{2n}} \quad ?$$

561. (4.19

4) e

5)

6)

8)

10)

11)

561. (4.19) – 4, 5, 6, 8, 10, 11

4) $ctg\, z$, $z=\infty$

$ctg\, z = \frac{\cos z}{\sin z}$ ~ $\sin z$, $z=\infty$ ~ не раскладывается

=> нет

5) $th \frac{1}{z}$, $z=0$

$th = \frac{\sin\frac{1}{z}}{\cos\frac{1}{z}}$

$\cos\frac{1}{z}$ – раскладывается

$\sin\frac{1}{z}$ в $z=0$ ~ $\sin u$, $u=\infty$ ~ не раскладывается

=> нет

6) $\frac{z^2}{\sin(1/z)}$, $z=0$

$\sin\frac{1}{z}$ в $z=0$ ~ $\sin u$, $u=\infty$ – не раскладывается

=> нет

8) $\ln z$, $z=0$

$z=x+iy$; $f(z)=\ln(x+iy)$

$\ln z$ неопр. в т. $z=0$ => нет

10) $\ln\frac{z-1}{z+1}$, $z=\infty$

$\ln\frac{z-1}{z+1} = \ln(z-1) - \ln(z+1)$

$u=\frac{1}{z}$, $u=0$

$\ln\left(\frac{1}{u}-1\right) - \ln\left(\frac{1}{u}+1\right) = \ln\frac{1-u}{u} - \ln\frac{1+u}{u}$ опр-на в т. $z=0$

=> да.

11) $z^\alpha \left(=e^{\alpha \ln z}\right)$, $z=0$

$z^\alpha = (x+iy)^\alpha$

$\ln z$, в т. $z=0$ → $\ln(z)$ – неопр.

562. (4.20) — 3, 6, 11, 14, 17.

3) $\sqrt{\frac{z}{(z-1)(z-2)}}$, $z=\infty$

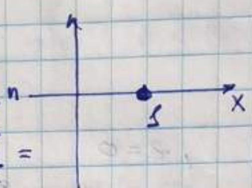
$u=\frac{1}{z} \to z=\frac{1}{u}$

$$\sqrt{\frac{1}{u\left(\frac{1}{u}-1\right)\left(\frac{1}{u}-2\right)}} = \sqrt{\frac{1\cdot u^{\cancel{2}}}{\cancel{u}(1-u)(1-2u)}} = \sqrt{\frac{u}{(1-u)(1-2u)}}$$

$\Rightarrow$ <u>нет</u> $\sqrt{u}$ — не расклад.

6) $\sqrt{1+\sqrt{z}}$, $z=1$

$(1+\sqrt{z})^{\frac{1}{2}} =$ $\qquad z=1$



$= 1+\frac{\frac{1}{2}}{1!}\sqrt{z}+\dots+\frac{\alpha(\alpha-1)\dots(\alpha-n+1)\sqrt{z}^{\,n}}{n!} =$

$= 1+\frac{\frac{1}{2}}{1!}z^{\frac{1}{2}}+\dots+\frac{\frac{1}{2}(\frac{1}{2}-1)\dots(\frac{1}{2}-n+1)}{n!}z^{\frac{n}{2}}$

$z=x=1.$

степени целые не при всех n

$\Rightarrow$ ~~да~~ (?)

$\Rightarrow$ <u>нет.</u>

11) $Ln[(z-1)(z-2)]$, $z=\infty$

$$Ln\,z=\ln r+i(\varphi+2\pi k)$$

$u=\frac{1}{z} \to z=\infty \to u=0$

$\Rightarrow Ln\left[\left(\frac{1}{u}-1\right)\left(\frac{1}{u}-2\right)\right] = Ln\left(\frac{1}{u}-1\right)-Ln\left(\frac{1}{u}-2\right) = Ln\frac{1-u}{u} - Ln\frac{1-2u}{u}$

$Ln(z-1)-Ln(z-2) = \ln|z-1| + i(Arg(z-1)+2\pi k) - \ln|z-2| -$

$-i(Arg(z-2)+2\pi k) = \underbrace{\ln(z-1)-\ln(z-2)}_{\text{расклад.}} + \underbrace{i(Arg(z-1)+2\pi k)-i(Arg(z-2)+2\pi k)}_{\text{не расклад.}}$

$\Rightarrow$ <u>нет.</u>

14) $Arctg(1+z)$, $z=0$

Есть разложение $Arctg\,z$ и в $z=0$ $Arctg(1+z)$ опр-но

$\Rightarrow$ <u>да.</u>

17) $\sqrt{\frac{\pi}{2}-Arcsin\,z}$, $z=1$.

$f(z)=Arcsin\,z$ ~ раскладывается в ряд Тейлора.

$\sqrt{\frac{\pi}{2}-Arcsin\,z}$ ~ невозможно получить расп. по целым степеням

$\Rightarrow$ <u>нет</u>

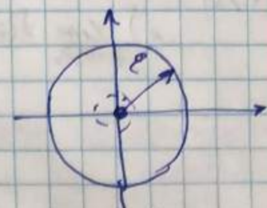
581.

4) $z=\infty$ $ctg\, z = \infty$ $ctg \frac{1}{u} = \frac{\cos\frac{1}{u}}{\sin\frac{1}{u}} = e^{i\frac{1}{u}} -$

$z=\frac{1}{u}$

$\sin\frac{1}{u}=0, \quad u=\frac{1}{\pi k} \quad u\to 0$

$\lim\limits_{k\to\infty} \frac{1}{\pi k}$



$\left|\frac{1}{\pi k}\right| < |\varepsilon|$

$\Rightarrow$ нет.

5) $th\frac{1}{z}$, $z=0$ нет

6) $\frac{z^2}{\sin\frac{1}{z}}$ в т. $z=0$

~~Кольцо аналитичности~~ $\lim\limits_{z\to 0} \sin\frac{1}{z} \nexists \Rightarrow$ нет

8) $\ln z$ в т. $z=0$

ряд расходится.

9) $\ln\frac{1}{z-1} = \ln 1 - \ln(z-1) = -\ln\left(\frac{1}{u}-1\right) = -\ln(1-u) + \ln u$

$z=\infty \to u=\frac{1}{z}$

не можем разложить.

$\Rightarrow$ нет

10) $\ln\left(\frac{z-1}{z+i}\right) = \ln\frac{(1-u)}{(1+ui)} = \ln(1-u) - \ln(1+iu)$ в окр-ти $u=0$ разкл. в ряд.

$u=\frac{1}{z}$

$\Rightarrow$ да

11) z^α α - целое да.

α - дробное $\Rightarrow$ нет

582.

6) $\sqrt{1+\sqrt{z}} = \left(\underbrace{1+(1+u)^{\frac{1}{2}}}_{t}\right)^{\frac{1}{2}} = \{(1+u)^{\frac{1}{2}} = t\} = \sum\limits_n (C_n)\, t^n =$ $(1+t)^{1/2}$

$z=1 \to z-1=u$

Если $|1+u|<1$

$= \sum\limits_n C_n \underbrace{(1+u)^{\frac{n}{2}}}_{=\sum\limits_k} = \sum\limits_n C_n \sum\limits_k C_{nk} u^k$

сх-ся если $|t|<1$

мм ~~...~~ Просто выносим t

$t^{\frac{1}{2}}(t^{-1}+1)^{\frac{1}{2}} \Rightarrow$ не раскладывается только если $|t|=1$

25.11 — **Классификация особых точек.**

№ 565 (4.23)

1) $\lim\limits_{z\to z_0} f(z) = A \qquad f(z_0) = A$ ~ устранимая особ. точка

2) $\lim\limits_{z\to z_0} f(z) = \infty \qquad z_0$ - полюс m-го порядка

z_0 - ноль m-го порядка для $\varphi(z) = \frac{1}{f(z)}$

$$f(z) = \frac{\varphi(z)}{(z-z_0)^m} = \frac{C_{-m}}{(z-z_0)^m} + \ldots + \sum_{n=0}^{\infty} C_n (z-z_0)^n$$

3) Существенная особая точка: $\nexists \lim\limits_{z\to z_0} f(z)$

$$f(z) = \sum_{n=-\infty}^{\infty} C_n (z-z_0)^n$$

$\frac{1}{z-z^3} \to \frac{1}{f} = z - z^3 = z(z-1)(z+1)$

$z = 0;\ z = \pm 1$ - полюса 1-го порядка

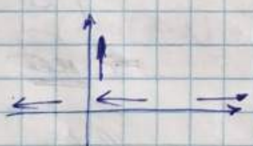
$z = \infty \to u = \frac{1}{z}$

$f = \frac{u^3}{u^2-1}, \quad f(0) = 0 \quad \to \lim\limits_{u\to 0} f(u) = 0$

$\Rightarrow z = \infty$ ~ правильная точка.

№ 569. $f = \frac{e^z}{1+z^2} = \frac{e^z}{(z-i)(z+i)} \qquad z = \pm i$ - полюса 1 п.

$\lim\limits_{z\to\infty} \frac{e^z}{1+z^2} \nexists \Rightarrow z = \infty$ - существен. т.



$z = t,\ t \to +\infty$

$z = t,\ t \to -\infty$

$z = it \quad e^{it} = \cos t + i\sin t \quad \nexists$

по-другому: $z = \frac{1}{u}$

$f = \frac{e^{\frac{1}{u}} u^2}{u^2+1} = u^2 \sum\limits_{n=0}^{\infty} (-1)^n u^{2n} \cdot \sum\limits_{k=0}^{\infty} \frac{1}{k!} \frac{1}{u^k}$

$n = 0 \to u^2 \cdot 1 \cdot \sum\limits_{k=0}^{\infty} \frac{1}{k!}\frac{1}{u^k} = u^2 + u + \frac{1}{2!} + \ldots$

$+ \frac{1}{3!}\frac{1}{u} + \ldots \frac{1}{k!\, u^{k-2}} + \ldots$

$\Rightarrow u = 0$ - сущ. особ. т.

$\Rightarrow z = \infty$ - существ. т.

№ 571. $z \cdot e^{-z} = \{u = \frac{1}{z}\} = \frac{1}{u} e^{-\frac{1}{u}} = \sum\limits_{n=0}^{\infty} (-1)^n \frac{1}{u^{n+1}\, n!}$ ~ существ. особая точка

№ 572. $\frac{1}{e^z - 1} - \frac{1}{z} = \frac{z - e^z + 1}{(e^z-1)z}$

1) $z = 0$

2) $e^z - 1 = 0$

$z=0 \to f = \frac{z - \sum\limits_{n=0}^{\infty}\frac{z^n}{n!} + 1}{z\left(\sum\limits_{n=0}^{\infty}\frac{z^n}{n!} - 1\right)} = \frac{-\frac{z^2}{2!} + \frac{z^3}{3!} + \dots}{z\left(z + \frac{z^2}{2!} + \frac{z^3}{3!} + \dots\right)} =$ z^2 z^2

$= \frac{-\frac{1}{2!} + \frac{z}{3!} + \dots}{1 + \frac{z}{2!} + \dots} \Rightarrow \lim\limits_{z\to 0} f(z) = A = -\frac{1}{2} \Rightarrow z=0$ – устранимая особая точка

$f(0) \sim ?$

2) $e^z = 1 \to z = \ln 1 = i2\pi k \to u = z - i2\pi k.$

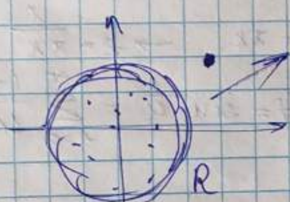
$f = \frac{u + i2\pi k - e^{u+\cancel{i2\pi k}} + 1}{(u+i2\pi k)\left(e^{u+\cancel{i2\pi k}} - 1\right)} = \frac{u + i2\pi k - e^u + 1}{(u+i2\pi k)\left(u + \frac{u^2}{2!} + \dots\right)} =$ $= \varphi(u)$

$= \frac{\varphi(u)}{u}$, но $\varphi(0) \neq 0$ $\varphi(0) \neq \infty$

$u=0$ – полюс 1-го порядка

$z = 2\pi k i$

$\lim\limits_{k\to\infty} 2\pi k i = \infty$



$\to z = \infty$ предельная точка $2\pi k i$

№574 $f = \frac{1-e^z}{2+e^z} = \frac{1 - e^{u + \ln 2 + i(\pi + 2\pi k)}}{2 + e^{u + \ln 2 + i(\pi+2\pi k)}} = \frac{1+2e^u}{2-2e^u} = \frac{1+2e^u}{2(1-e^u)} =$ $= -1$

$2 + e^z = 0 \to e^z = -2 = e^{\ln 2 + i(2\pi k + \pi)}$

$z = \ln 2 + i(\pi + 2\pi k)$

$z - z_k = u$

$= \frac{1+2e^u}{2u\left(1 + \frac{u}{2!} + \dots\right)}$ u – полюс 1-го пор.

$z = \ln 2 + i(\pi + 2\pi k)$

$z = \infty$: $\lim\limits_{k\to\infty} \ln 2 + i(\pi + 2\pi k) = \infty \Rightarrow z = \infty$ – пред. точка.

№579. ~~$f(z) = z \cdot e^{\frac{1}{z}}$~~ $f(z) = e^{\frac{z}{1-z}} = e^{-\frac{u+1}{u}} = e^{-1} e^{-\frac{1}{u}} = e^{-1} \sum\limits_{n=0}^{\infty} (-1)^n \frac{1}{n!} \frac{1}{u^n}$

~~$z=0$~~
~~$z=\infty$~~

1) $z = 1$; $u = z - 1$ $u=0 \to$ сущ. особ. т.

$z = 1$

2) $z = \infty$ $u = \frac{1}{z} \to f = e^{\frac{1}{u\left(1 - \frac{1}{u}\right)}} = e^{\frac{1}{u-1}} \to e^{-1}$

$\Rightarrow z = \infty$ – прав. т-ка

№591. $f = \sin\frac{1}{1-z} = -\sin\frac{1}{u} = -\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\left(\frac{1}{u}\right)^{2n+1}$

$z=1 \to u = z-1 \quad (z = u+1)$

$u=0$ – сущ. о. т. $\to z=1$,
$z=\infty$ – прав. т.-ка.

№596. $f = e^{-z}\cos\frac{1}{z} = \sum_{n=0}^{\infty}(-1)^n\frac{z^n}{n!}\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k)!}\cdot\frac{1}{z^{2k}} = 1\cdot\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k)!}\frac{1}{z^{2k}} + \sum\sum$

$z=0$, $z=\infty$ $\qquad z=0$ – сущ. о. т.

$z=\infty \quad u=\frac{1}{z} \to e^{-\frac{1}{u}}\cos u = \sum_{n=0}^{\infty}\frac{(-1)^n(1)}{u^n n!}\sum_{k=0}^{\infty}(-1)^k\frac{u^{2k}}{(2k)!} =$

$= 1\cdot\sum_{n=0}^{\infty}\frac{1}{u^n}\cdot\frac{(-1)^n}{n!} + \sum\sum$

$\underline{k=0}$

$\Rightarrow z=\infty$ – сущ. о. т.

№599. $f = \sin\left(\frac{1}{\sin\frac{1}{z}}\right)$

$\sin\frac{1}{z} = 0 \to \frac{1}{z} = \pi k \to z = \frac{1}{\pi k}$

$u = \sin\frac{1}{z} \to f = \sin\frac{1}{u} = \sum_{n=0}^{\infty}\frac{(-1)^n 1}{u^{2n+1}(2n+1)!} \to u=0$

$u=0$ – сущ. о. т.

$z=\frac{1}{\pi k}$ – сущ. о. т.

$z=0 \quad \lim_{k\to\infty}\frac{1}{\pi k} = 0 \to \underline{z=0}$ – предельная точка $\frac{1}{\pi k}$

$z=\infty \to u=\frac{1}{z} \Rightarrow f = \sin\left(\frac{1}{\sin u}\right) = \sin\frac{1}{t} = \sum_{k=0}^{\infty}(-1)^k\frac{1}{t^{2k+1}(2k+1)!}$

$t=0 \to$ сущ. о. т.

567 (4.25)

$z=\infty$

570. (4.28)

$z = 0$

$\lim_{u\to}$

566 (4.24)

1

$z=$

1) $n=0$:

$n=1$;

$\Rightarrow$

2) $z=\infty$

Домашняя работа.

567(4.25) $f(z)=\frac{z^5}{(1-z)^2}$ $\to$ $\varphi(z)=\frac{1}{f}=\frac{(1-z)^2}{z^5}=0$

$\to z=1.$

$\Rightarrow$ $z=1$ – полюс 2-го порядка

$z=\infty$

$\lim\limits_{z\to\infty} f(z)=\infty$; $u=\frac{1}{z}$

$\Rightarrow f(u)=\frac{1}{u^5\left(1-\frac{1}{u}\right)^2}=\frac{u^2}{u^5(u-1)^2}=\frac{1}{u^3(u-1)^2}$

$\varphi(u)=u^3(u-1)^2=0$

$\Rightarrow$ $u=0$ ~ полюс 3-го порядка

$\Rightarrow$ $z=\infty$ – полюс 3-го порядка.

570.(4.28) $f(z)=\frac{z^2+1}{e^z}$

$e^z\neq 0$

$z=\infty$; $z=\frac{1}{u}$ $(u=0)$

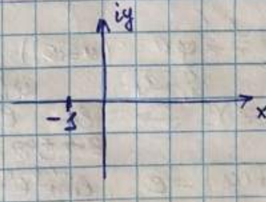
$f(u)=\frac{\left(\frac{1}{u}\right)^2+1}{e^{\frac{1}{u}}}=\frac{1+u^2}{u^2\cdot e^{\frac{1}{u}}}$

$\lim\limits_{u\to0} f(u)=\lim\limits_{u\to0}\frac{1+u^2}{u^2\cdot e^{\frac{1}{u}}}=\lim\limits_{u\to0}\frac{1+u^2}{u^2}\cdot\underbrace{\frac{1}{e^{\frac{1}{u}}}}_{\nexists}\Rightarrow\nexists$ ~ не существует

$\Rightarrow$ $u=0$ ~ сущ. особ. точка

$z=\infty$ ~ сущ. особ. точка.

566(4.24.) $f(z)=\frac{z^4}{1+z^4}$



$1+z^4=0$ $\to$ $z^4=-1$

$z=\sqrt[4]{-1}$

$z=\sqrt[4]{1}\cdot e^{\frac{i(\pi+2\pi n)}{4}}=e^{\frac{i(\pi+2\pi n)}{4}}$

1) $n=0$: $z_0=e^{i\frac{\pi}{4}}=\frac{1}{\sqrt2}+i\cdot\frac{1}{\sqrt2}=\frac{1+i}{\sqrt2}$

$n=1$; $z_1=e^{i\frac{3\pi}{4}}=\frac{-1+i}{\sqrt2}$

$\Rightarrow z_2=\frac{-1-i}{\sqrt2}$; $z_3=\frac{1-i}{\sqrt2}$

$\Rightarrow z_{1,2}=\frac{1\pm i}{\sqrt2}$, $z_{3,4}=\frac{-1\pm i}{\sqrt2}$ ~ полюса 1-го порядка.

2) $z=\infty$; $u=\frac{1}{z}$ $(u=0)$

$$f(u) = \frac{1}{u^4\left(1+\left(\frac{1}{u}\right)^4\right)} = \frac{u^4}{u^4(u^4+1)} = \frac{1}{u^4+1}$$

$\Rightarrow \lim\limits_{u\to 0} f(u) = 1$ $\Rightarrow u=0$ ~ правильная точка

$z=\infty$ ~ правильная точка

579(4.31). $f(z) = \dfrac{e^z}{z(1-e^{-z})} \rightarrow \varphi(z) = \dfrac{1}{f} = \dfrac{z(1-e^{-z})}{e^z} = 0$

1) $1-e^{-z}=0 \;\rightarrow\; e^{-z}=1$

$-z = \mathrm{Ln}(1) = 2k\pi i \quad (k\in\mathbb{Z})$

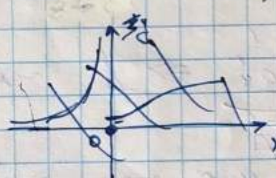
$z = 2k\pi i \quad (k\in\mathbb{Z})$, при $k=0$ $z=0$

$\Rightarrow z=0$ ~ ноль 2-го порядка

$z=2k\pi i$ $(k=\pm1,\pm2,\pm3,\dots)$ ~ ноль 1-го порядка

2) $z=\infty$ $\lim\limits_{k\to\infty} 2k\pi i = \infty$ $\Rightarrow z=\infty$ - предельная для полюсов.

580(4.38.) $f(z) = e^{z-\frac{1}{z}}$



1) $z=0$

$\lim\limits_{z\to+0} e^{z-\frac{1}{z}} = e^{-\infty} = 0$

$\lim\limits_{z\to-0} e^{z-\frac{1}{z}} = e^{+\infty} = +\infty$

$\Rightarrow \lim\limits_{z\to 0} f(z)\ \nexists$ не существует

$z=0$ - сущ. особ. точка.

2) $z=\infty$; $\frac{1}{z}=u$ $(u=0)$

$f(u) = e^{\frac{1}{u}-u}$

$\lim\limits_{u\to+0} e^{\frac{1}{u}-u} = e^{+\infty} = +\infty$

$\lim\limits_{u\to-0} e^{\frac{1}{u}-u} = e^{-\infty} = 0$

$\Rightarrow \lim\limits_{u\to 0} f(u)\ \nexists$ не существует.

$\Rightarrow z=\infty$ - сущ. особ. точка.

584(4.42) $f(z) = \mathrm{tg}(z)$

$\varphi(z) = \dfrac{1}{tg\,z} = \dfrac{\cos z}{\sin z} = 0 \quad \to \cos z = 0$

$z = \dfrac{\pi}{2} + \pi k = \dfrac{\pi}{2}(2k+1),\ k \in Z$

полюс 1-го пор.

$z = \dfrac{\pi}{2}(2k+1)$ ~ ~~сущ. особ. точка~~

$z = \infty \quad ; \quad \lim\limits_{k\to\infty} \dfrac{\pi}{2}(2k+1) = \infty$

$\Rightarrow z = \infty$ ~ предельная для полюсов

593 (4.51.) $\quad f(z) = ctg\,\dfrac{1}{z}$

1) $\quad \varphi(z) = \dfrac{1}{f} = \dfrac{1}{ctg\frac{1}{z}} = \dfrac{\sin\frac{1}{z}}{\cos\frac{1}{z}}$

$\sin\dfrac{1}{z} = 0 \to \dfrac{1}{z} = \pi k,\ k \in Z \to z = \dfrac{1}{\pi k},\ k \in Z$

при $k \to 0 \to z \to \infty$

$\Rightarrow z = \dfrac{1}{\pi k}\ (k = \pm 1, \pm 2, \dots)$ ~ полюсы 1-го порядка.

$\lim\limits_{k\to\infty} \dfrac{1}{\pi k} = 0$

$\Rightarrow z = 0$ ~ предельная точка полюсов

2) $z = \infty, \quad \dfrac{1}{z} = u \quad (u = 0)$

$f(z) = ctg\,u = \dfrac{\cos u}{\sin u} = \dfrac{\sum\limits_{n=0}^{\infty} (-1)^n u^{2n} \cdot \frac{1}{(2n)!}}{\sum\limits_{n=1}^{\infty} \frac{(-1)^{n+1} u^{2n-1}}{(2n-1)!}} =$

$= \dfrac{\left(1 - \frac{u^2}{2!} + \frac{u^4}{4!} - \dots\right)}{\left(u - \frac{u^3}{3!} + \dots\right)} \Rightarrow u = 0$ ~ полюс 1-го порядка

u

$z = \infty$ ~ полюс 1-го порядка.

595 (4.53) $\quad f(z) = \sin\dfrac{1}{z} + \dfrac{1}{z^2}$

$\varphi(z) = \dfrac{1}{\sin\frac{1}{z} + \frac{1}{z^2}} = 0 \quad$ нет полюсов

1) $z = 0$

$\lim\limits_{z\to 0} f(z) = \lim\limits_{z\to 0}\left(\sin\dfrac{1}{z} + \dfrac{1}{z^2}\right) \quad \nexists$ не существует

$z = 0$ ~ сущ. особ. точка.

2) $z = \infty \quad ; \quad u = \dfrac{1}{z}$

$f(u) = \sin u + u^2$

$\lim\limits_{u\to 0} f(u) = 0 \quad \Rightarrow z = \infty$ ~ правильная точка

597. (4.55) $\quad f(z) = e^{ctg\frac{1}{z}} \quad e^{\lim\limits_{z\to 0}\left(\frac{\sin\frac{1}{z}}{\cos\frac{1}{z}}\right)^{-1}} \quad \nexists$

1) $z = 0$? $\quad \lim\limits_{z\to 0} e^{ctg\frac{1}{z}} = e^{\lim\limits_{z\to 0}}$

$\Rightarrow z = 0$ ~ сущ. особ. точка.

2) $\lim\limits_{z\to z_0} e^{ctg\frac{1}{z}} \quad \nexists$, когда?

$$\lim_{z\to z_0} \left(ctg \frac{1}{z}\right) \nexists \to \frac{\cos\frac{1}{z}}{\sin\frac{1}{z}} \nexists \text{, когда } \sin\frac{1}{z}=0$$

$$\Rightarrow \underset{\sim\sim}{z} = \frac{1}{\pi k}, \quad k=\pm 1, \pm 2$$

~ сущ. особ. т.-ка.

$$\lim_{k\to\infty} \frac{1}{\pi k} = 0 \qquad \underset{\sim\sim}{z=0} \text{ ~ предел. сущ. т.-ка}$$

3) $z=\infty$; $u=\frac{1}{z}$ $(u=0)$

$$f(u) = e^{ctg\, u} = e^{\frac{\cos u}{\sin u}}$$

$$\lim_{u\to+0} e^{\frac{\cos u}{\sin u}} = e^{\lim\limits_{u\to+0}\frac{\cos u}{\sin u}} = +\infty$$

$$\lim_{u\to-0} e^{\frac{\cos u}{\sin u}} = -\infty$$

$$\Rightarrow \lim_{u\to 0} f(u) \nexists$$

$\underset{\sim\sim}{z=\infty}$ ~ сущ. особ. точка.

02.12.21 597.

$$f(z) = e^{ctg\frac{1}{z}}$$

$z=\infty$; $\frac{1}{z}=u \Rightarrow u=0$ $\quad e^{ctg\, u} = e^{\frac{\varphi(u)}{u}}$

$$\varphi(0)\neq 0 \quad ctg\, u = \frac{\cos u}{\sin u} = \frac{\cos u}{u\left(1-\frac{u^2}{3!}+\frac{u^4}{5!} \cdots\right)} = \varphi(u)$$

$$e^{\frac{\varphi(u)}{u}} \sim e^{\frac{A}{u}} = \sum_{n=0}^{\infty} \frac{A^n}{u^n \cdot n!} \to \text{беск. ч. отриц. степ.} \to u=0$$

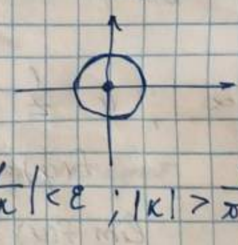
$z=\infty$ - сущ. особ. точка

$$\sin\frac{1}{z} = \pi k \to \frac{1}{z}=\pi k \to z=\frac{1}{\pi k} \Leftrightarrow \frac{1}{z}=\pi k; \quad u=t-\pi k$$

$$f(z) = e^{\frac{\cos(u+\pi k)}{\sin(u+\pi k)}} = e^{\frac{(-1)^k\cos u}{(-1)^k \sin u}} = e^{\frac{\cos u}{\sin u}} \Rightarrow u=0 \text{ сущ. о. т.}$$

$$z=\frac{1}{\pi k} \text{ - сущ. о. т.}$$

$z=0$: $\lim\limits_{k\to\infty}\frac{1}{\pi k}=0$ $\qquad z=0$ - пред. т-ка



$R=\varepsilon$

$$\left|\frac{1}{\pi k}\right| < \varepsilon; \ |k| > \frac{1}{\pi\varepsilon}$$

573.

$$f(z) = \frac{e^z}{z(1-e^{-z})}$$

$z=0$

$$e^{-z}=1$$
$$-z = \ln 1 + i(2\pi k)$$
$$z = -i2\pi k.$$

$$z=\infty; \quad \lim_{k\to\infty}(i2\pi k) = i\infty \quad \to z=\infty \text{ пред. т.}$$

601. $F(z) =$

№603 $f(z) =$

№601 $f(z) = \dfrac{z}{1+\sqrt{z-3}}$

$z=4$

1) $+\sqrt{z-3}$ $\to f(z) = \{u = z-4\} = \dfrac{u+4}{1+\sqrt{u+1}} = \dfrac{u+4}{1+(1+u)^{1/2}}$

2) $-\sqrt{z-3}$

$\to f(0) = \dfrac{4}{1+1} = 2$

$\Rightarrow u=0$ ~ прав. точка.

$z=4$ ~ прав. точка

для данной ветви

$\downarrow$

$f(z) = \left\{ \begin{matrix} u = z-4 \\ u=0 \end{matrix} \right\} =$

$= \dfrac{u+4}{1-(1+u)^{1/2}} =$

$\lim\limits_{u\to 0} \dfrac{u+4}{1-(1+u)^{1/2}} = \infty \to u=0$ полюс.

$\varphi(z) = \dfrac{1-(1+u)^{1/2}}{u+4} \Rightarrow 1-(1+u)^{1/2} = 1 - \sum\limits_{n=0}^{\infty} \dfrac{\frac{1}{2}(\frac{1}{2}-1)\dots(\frac{1}{2}-n+1)}{n!} u^n =$

$= \cancel{1} - (\cancel{1} + \frac{1}{2} u + \sum\limits_{n=2}^{\infty} \dfrac{(-1)^{n-1}(2n-3)!!}{2^n n!} u^n =$

$= u \underbrace{\left(\frac{1}{2} + \sum\limits_{n=2}^{\infty} \dfrac{(-1)^{n-1} \cdot (2n-3)!!}{2^n \, n!} u^{n-1} \right)}_{\varphi(u) \to \varphi(0) \neq 0}$

$\to u$ - ноль $1^{го}$ порядка для $\dfrac{1}{f(u)}$

$\Rightarrow u=0$ ~ полюс $1^{го}$ порядка для f

$z=4$ ~ полюс $1^{го}$ порядка.

№603 $f(z) = \dfrac{2z+3}{1+z-2\sqrt{z}}$,

$z=1$

1) $+\sqrt{z} \to \left\{ \begin{matrix} u = z-1 \\ u=0 \end{matrix} \right\} \Rightarrow f = \dfrac{2u+5}{2+u-2(1+u)^{\frac{1}{2}}} = \dfrac{2u+5}{\cancel{2+u}-2(\cancel{1}+\frac{u}{2} + \sum\limits_{n=2}^{\infty}\frac{(-1)^{n-1}(2n-3)!!}{2^n n!} u^n)} =$

$\lim\limits_{u\to 0} \dfrac{2u+5}{2+u-2(1+u)^{1/2}} = \infty$

$= \dfrac{2u+5}{\underbrace{-2u^2 \sum\limits_{n=2}^{\infty} \frac{(-1)^{n-1} (2n-3)!! \, u^{n-2}}{n! \, 2^n}}_{\varphi(u) \quad \varphi(0) \neq 0}}$

$\Rightarrow u=0$ полюс $2^{го}$ пор.

$z=1$ полюс $2^{го}$ пор.

2) $-\sqrt{z}$

$\to \dfrac{2z+3}{1+z+2\sqrt{z}} \to \lim\limits_{z\to 1} \dfrac{2z+3}{1+z+2\sqrt{z}} = \dfrac{5}{4}$

$\Rightarrow z=1$ ~ прав. точка.

$f(1) = \dfrac{5}{4}$

ф-я опр-на.

№ 606. $f(z) = \dfrac{1}{(2+\sqrt{z})\sin(2-\sqrt{z})}$

$z=4$

1) $+\sqrt{z}$ $\rightarrow f = \left\{ \begin{matrix} u = z-4 \\ u = 0 \end{matrix} \right\} = \dfrac{1}{\underbrace{(2+\sqrt{u+4})}_{\neq 0}\sin(2-\sqrt{u+4})} =$

$\Rightarrow \lim\limits_{u \to 0} \dfrac{1}{(2+\sqrt{u+4})\sin(2-\sqrt{u+4})} = \infty$

$\sin(2-\sqrt{u+4}) = \sin 2\left(1-\left(1+\frac{u}{4}\right)^{\frac{1}{2}}\right) = \sin 2\left(1-\left(1+\frac{1}{2}\frac{u}{4}+\sum\limits_{n=2}^{\infty}\frac{(-1)^{n-1}(2n-3)!!\,u^n}{2^n n!\,4^n}\right)\right)$

$= -\sin 2u\underbrace{\left(\frac{1}{8}+\sum()\right)}_{\varphi(u)\quad \varphi(0)\neq 0} \sim -\sin 2u \cdot A = -\sin\frac{u}{4}$

$A = \varphi(0) = \frac{1}{8}$

$\Rightarrow \dfrac{1}{-4\sin\frac{u}{4}} = -\dfrac{1}{u}\dfrac{1}{\left(\frac{u}{4}-\frac{u^3}{4^3 3!}+\ldots\right)}$ $\Rightarrow u=0 \rightarrow$ 1.1 п.

$z=4 \rightsquigarrow$ полюс 1-го порядка

2) $-\sqrt{z}$

$f_2 = \dfrac{1}{(2-\sqrt{z})\underbrace{\sin(2+\sqrt{z})}_{\neq 0}} = \left| \begin{matrix} u=z-4 \\ u=0 \end{matrix} \right| = \dfrac{1}{2-2\left(1+\frac{u}{4}\right)^{\frac{1}{2}}} \cdot \dfrac{1}{\sin(2+\sqrt{u+4})} \sim$

$\sim \dfrac{1}{2\sin u}\dfrac{1}{1-\left(1+\frac{u}{4}\right)^{\frac{1}{2}}} = \dfrac{1}{2\sin u} \cdot \dfrac{1}{1-\left(1+\frac{1}{2}\frac{u}{4}+\sum\limits_{n=2}^{\infty}\frac{(-1)^{n-1}(2n-3)!!\,u^n}{2^n 4^n \cdot n!}\right)}$

$\Rightarrow z=4 \rightsquigarrow$ полюс 1. пор.

№ 607. $F(z) = \dfrac{1}{\sin\left(1+\sqrt{\frac{z}{z-2}}\right)}$

① $z = \dfrac{2(1+\pi k)^2}{(1+\pi k)^2-1}$

② $z=\infty$

① $\dfrac{z}{z-2} = \dfrac{2(1+\pi k)^2}{(1+\pi k)^2-1} \cdot \dfrac{(1+\pi k)^2-1}{2(1+\pi k)^2-2(1+\pi k)^2+2} = (1+\pi k)^2$

$t=(1+\pi k)^2$

1) $+\sqrt{t}$

$\lim\limits_{t\to(\pi k+1)^2} \dfrac{1}{\sin(1+\sqrt{t})} = \dfrac{1}{\sin(1+1+\pi k)} = \dfrac{(-1)^k}{\sin 2} \neq \infty$

$\rightarrow z_k$ - прав. точка

2) $-\sqrt{t}$ $\quad u = t-(\pi k)^2$

$\cancel{f = \dfrac{1}{\sin\left(1+(u+(\pi k)^2)^{\frac{1}{2}}\right)} = \dfrac{1}{\sin\left(1-\pi k\left(1+\frac{u}{(\pi k)^2}\right)^{\frac{1}{2}}\right)}} =$

$\cancel{= \dfrac{1}{\sin}}$

$= \dfrac{}{\sin(1}$

$= \dfrac{(-1)^{n}}{}$

② $z=\infty$

$\dfrac{1}{\sin(1+}$

1) $+\sqrt{}$

2) $-\sqrt{}$

$f(u) =$

$=$

$=$

$$u = t - (1+\pi k)^2$$

$$f = \frac{1}{\sin\left(1-\left(u+(1+\pi k)^2\right)^{\frac{1}{2}}\right)} = \frac{1}{\sin\left(1-(1+\pi k)\left(1+\frac{u}{(1+\pi k)^2}\right)^{\frac{1}{2}}\right)} =$$

$$= \frac{1}{\sin\left(1-(\pi k+1)\sum\limits_{n=0}^{\infty}\frac{(-1)^{n-1}(2n-3)!!}{2^n\, n!}\frac{u^n}{(1+\pi k)^{2n}}\right)} = \frac{1}{\sin\left(1-(\pi k+1)\left(1-\frac{1}{2}\frac{u}{(\pi k+1)^2}+\sum\limits_{n=2}\ldots\right)\right)}$$

$$= (-1)^k\frac{1}{\sin(\pi k+1)\left(-\frac{1}{2}\cdot\frac{u}{(\pi k+1)^2}+\sum\limits_{n=2}^{\infty}\ldots\right)}$$

$\Rightarrow$ ~~$u=0$ ~ полюс $1^{го}$ порядка.~~

$$z_k = \frac{2(1+\pi k)^2}{(1+\pi k)^2-1}$$ ~ полюс $1^{го}$ порядка.

② $z=\infty$

$$\frac{1}{\sin\left(1+\sqrt{\frac{z}{z-2}}\right)}$$

1) $+\sqrt{\ }$ $\to$ $\lim\limits_{z\to\infty}\frac{1}{\sin\left(1+\sqrt{\frac{z}{z-2}}\right)} = \left\{\begin{matrix}\frac{1}{z}=u\\ u=0\end{matrix}\right\} = \lim\limits_{u\to 0}\frac{1}{\sin\left(1+\sqrt{\frac{1}{1-2u}}\right)} = \frac{1}{\sin 2}$

$f(u) = \frac{1}{\sin 2}$

$\Rightarrow z=\infty$ ~ прав. точка.

2) $-\sqrt{\ }$

$$\lim_{z\to\infty}\frac{1}{\sin\left(1-\sqrt{\frac{z}{z-2}}\right)} = \left|\frac{1}{z}=u\right| = \lim_{u\to 0}\frac{1}{\sin\left(1-\frac{1}{\sqrt{1-2u}}\right)} = \infty$$

$$f(u) = \frac{1}{\sin\left(1-(1-2u)^{-\frac{1}{2}}\right)} = \frac{1}{\sin\left(1-\sum\limits_{n=0}^{\infty}\frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)\ldots\left(-\frac{1}{2}-n+1\right)}{n!}(-1)^n u^n 2^n\right)} =$$

$$= \frac{1}{\sin\left(1-\sum\limits_{n=0}^{\infty}\frac{(-1)^n(2n-1)!!\,(-1)^n 2^n u^n}{2^n\, n!}\right)} = \frac{1}{\sin\left(1-\sum\limits_{n=0}^{\infty}\frac{(2n-1)!!}{n!}u^n\right)} =$$

$$= \frac{1}{\sin\left(\cancel{1}-\cancel{1}-u-\sum\limits_{n=2}^{\infty}\ldots\right)} = \frac{1}{\sin u\cdot\varphi(u)}$$

$\varphi(0)\neq 0$

$\to$ $u=0$ ~ полюс $1^{го}$ пор.

$z=\infty$ ~ полюс $1^{го}$ пор.

Домашняя работа.

602 (4.60.)

$$f(z)=\frac{1}{\sqrt{z}+\sqrt[3]{z}}, \quad z=1$$

1) $+\sqrt{z} \to f(u)=\{u=z-1\}=\frac{1}{\sqrt{u+1}+\sqrt[3]{u+1}}$

$$f(0)=\frac{1}{2}$$

$\Rightarrow z=1$ – прав. точка.

2) $-\sqrt{z} \to f(u)=\{u=z-1\}=\frac{1}{-\sqrt{u+1}+\sqrt[3]{u+1}} \ominus$

$$(1+u)^{\frac{1}{2}}=\sum_{n=0}^{\infty}\frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)\dots(\frac{1}{2}-n+1)}{n!}u^n$$

$$\ominus \frac{1}{-\sum\limits_{n=0}^{\infty}\frac{\frac{1}{2}(\frac{1}{2}-1)\dots(\frac{1}{2}-n+1)}{n!}u^n+\sum\limits_{n=0}^{\infty}\frac{\frac{1}{3}(\frac{1}{3}-1)\dots(\frac{1}{3}-n+1)}{n!}u^n}=$$

$$=\frac{1}{-(\cancel{1}+\frac{u}{2}-\frac{u^2}{8}+\dots)+(\cancel{1}+\frac{u}{3}-\frac{u^2}{9}+\dots)}=$$

$$=\frac{1}{-u\underbrace{\left(\left(\frac{1}{2}-\frac{1}{3}\right)-\frac{u}{8}+\frac{u}{9}\dots\right)}_{\varphi(0)\neq 0}}$$

$\Rightarrow u=0$ ~ полюс 1-го порядка

$z=1$ ~ полюс 1-го порядка

604 (4.62.)

$$f(z)=\cos\frac{1}{1+\sqrt{z}}, \quad z=1$$

1) $+\sqrt{z} \to f(u)=\{u=z-1\}=\cos\frac{1}{1+\sqrt{u+1}}$

$$f(0)=\cos\frac{1}{2}$$

$\Rightarrow z=1$ ~ правильная точка.

2) $-\sqrt{z} \to f(u)=\cos\frac{1}{1-\sqrt{u+1}}$

Но! $\lim\limits_{u\to 0} f(u)=\lim\limits_{u\to 0}\cos\frac{1}{1-\sqrt{u+1}}$ $\nexists$

$\Rightarrow u=0$ ~ сущ. особая точка

$\Rightarrow z=1$ ~ сущ. особая точка.

606 (4.64.)

$$f(z)=\operatorname{ctg}\frac{1}{1+\sqrt{z}}, \quad z=\left(1+\frac{1}{\pi k}\right)^2, \quad и \quad z=1$$

1) $+\sqrt{z} \to f(u)=\left\{u=z-\left(1+\frac{1}{\pi k}\right)^2\right\}=\operatorname{ctg}\frac{1}{1+\sqrt{u+\left(1+\frac{1}{\pi k}\right)^2}}$

$$f(0)=\operatorname{ctg}\frac{1}{1+1+\frac{1}{\pi k}}=\operatorname{ctg}\frac{1}{2+\frac{1}{\pi k}}$$ (при $k=\pm 1, \pm 2\dots$ опр-н.

$\Rightarrow z=\left(1+\frac{1}{\pi k}\right)^2$ ~ прав. точка.

Если $z=1$:

$$f(u) = \{u = z-1\} = \operatorname{ctg} \frac{1}{1+\sqrt{u+1}}$$

$$f(0) = \operatorname{ctg} \frac{1}{2} \quad \text{~ определена.}$$

$$\Rightarrow \underset{\sim\sim}{z=1} \text{ ~ прав. точка.}$$

2) $-\sqrt{z} \rightarrow f(u) = \operatorname{ctg} \frac{1}{1-\sqrt{u+\left(1+\frac{1}{\pi k}\right)^2}} \ominus$

$$\operatorname{ctg} \frac{1}{1-\sqrt{u+z}} = \frac{\cos\left(\frac{1}{1-\sqrt{z}}\right)}{\sin\left(\frac{1}{1-\sqrt{z}}\right)}$$

$$\ominus \frac{\cos\left(\frac{1}{1-\sqrt{u+\left(1+\frac{1}{\pi k}\right)^2}}\right)}{\sin\left(\frac{1}{1-\sqrt{u+\left(1+\frac{1}{\pi k}\right)^2}}\right)}$$

$$\lim_{u\to 0} \operatorname{ctg} \frac{1}{1-\sqrt{u+\left(1+\frac{1}{\pi k}\right)^2}} = \operatorname{ctg} \frac{1}{1-1-\frac{1}{\pi k}} = \operatorname{ctg}(-\pi k) \to \infty$$

$$\Rightarrow z = \left(1+\frac{1}{\pi k}\right)^2 \text{ — полюс,}$$

но, какого порядка?

$$\left(1+\frac{1}{\pi k}\right)^2 = t$$

$$f(u) = \operatorname{ctg} \frac{1}{1-\sqrt{u+t}} = \operatorname{ctg} \frac{1}{1-\sqrt{t}\cdot\sqrt{1+\frac{u}{t}}} =$$

$$= \operatorname{ctg} \frac{1}{1-\sqrt{t}\cdot\sum_{n=0}^{\infty}\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\ldots\left(\frac{1}{2}-n+1\right)}{n!}\left(\frac{u}{t}\right)^n} = \operatorname{ctg} \frac{1}{1-\sqrt{t}\cdot\underbrace{\left(1+\frac{u}{2t}-\frac{u^2}{8t^2}+\ldots\right)}} \ominus$$

$$\ominus \operatorname{ctg} \frac{1}{1-1-\frac{1}{\pi k}\cdot\left(1+\frac{u}{2\left(1+\frac{1}{\pi k}\right)^2}\right)}$$

$$f(u) = \operatorname{ctg} \frac{1}{1-\sqrt{u+t}} = \frac{\cos\frac{1}{1-\sqrt{u+t}}}{\sin\frac{1}{1-\sqrt{u+t}}} = \frac{\sum_{n=0}^{\infty}\frac{(-1)^n\cdot\left(\frac{1}{1-\sqrt{u+t}}\right)^{2n+1}}{(2n+1)!}}{\sum_{n=0}^{\infty}\frac{(-1)^n\left(\frac{1}{1-\sqrt{u+t}}\right)^{2n}}{(2n)!}} =$$

$$= \frac{\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!\left(1-\sqrt{u+t}\right)^{2n+1}}}{\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!\left(1-\sqrt{u+t}\right)^{2n}}}$$

Получается если из суммы вынести $\left(1-\sqrt{u+t}\right)^{2n+1}$ и $\left(1-\sqrt{u+t}\right)^{2n}$ в знаменателе

У синуса начинается разложение с $x+\ldots$ $\Rightarrow$ останется $\left(1-\sqrt{u+t}\right)^1$

У косинуса с $1+\ldots$

(т.е. в нашем случае наоборот, т.к. дроби перевёрнуты). $\Rightarrow \underline{z = \left(1+\frac{1}{\pi k}\right)^2}$ ~ полюс первого порядка

$$z = \left(1+\frac{1}{\pi k}\right)^2 \text{ ~ полюс первого порядка.}$$

$$\lim_{k\to\infty}\left(1+\frac{1}{\pi k}\right)^2 = 1 \quad \Rightarrow z=1 \text{ ~ предельная для полюса.}$$

<u>808 (4.66.)</u>

$$f(z) = \sin\frac{1}{1+\sqrt{\frac{z}{z-2}}}, \quad z = \infty$$

1) $+\sqrt{\ } \rightarrow f(u) = \left\{u = \frac{1}{z}\right\} = \sin\frac{1}{1+\sqrt{\frac{1}{u\left(\frac{1}{u}-2\right)}}} = \sin\frac{1}{1+\sqrt{\frac{u}{u(1-u)}}} =$

$$= \sin\frac{1}{1+\sqrt{\frac{1}{1-u}}}$$

$f(0) = \sin\frac{1}{2}$

$\Rightarrow$ $z=\infty$ ~ правильная точка.

2) $-\sqrt{\ }$ $\to f(u) = \sin\dfrac{1}{1-\sqrt{\frac{1}{1-u}}}$

$\lim\limits_{u\to 0} f(u) \nexists$ (т.к. $\sin(\infty)$ неопр.)

$\Rightarrow$ $z=\infty$ ~ сущ. особ. точка

№602. $\dfrac{1}{\sqrt{z}+\sqrt[3]{z}} = \dfrac{1}{\sqrt{1+u}+\sqrt[3]{1+u}} = \dfrac{1}{(1+u)^{1/2}+(1+u)^{1/3}}$

$z=1$

$u=z-1$

$\sqrt[3]{t} = |t|^{1/3} e^{i\frac{\arg t}{3}+\frac{2\pi k}{3}} = z_1 \cdot e^{i\frac{2\pi k}{3}}$

$(1+u)^{1/2} = t^{1/2} = t_1 \cdot e^{i\pi k}$

$k=0 \to z_1$ $e^{i\frac{2\pi}{3}}$

$k=1 \to z_1 e^{i\frac{4\pi}{3}}$

$k=2 \to z_1 \cdot e$

$k=0$, t_1

$k=1$ $= e^{i\pi} t_1 = -t_1$

1) $z=1$ — прав т-ка.

а) $f(1) = \frac{1}{2}$, если $\sqrt{1+u} = t_1$, $\sqrt[3]{1+u} = z_1$ "+" "+"

$u=0$

б) $f(1) = \dfrac{1}{\pm 1 + 1 e^{i\frac{2\pi}{3}}}$ $u=0$ если $\sqrt{1+u} = 1$, $\sqrt[3]{1+u} = 1\cdot e^{i\frac{2\pi}{3}}$

в) $f(1) = \dfrac{1}{\pm 1 + 1\cdot e^{i\frac{\pi 4}{3}}}$ $u=0$; $\sqrt{1+u} = \pm 1$ $\sqrt[3]{1+u} = e^{i\frac{4\pi}{3}}$ 621. (4.79)

т.е. 5 ветвей

г) $z=1$, $\sqrt{1+u} = -1$, $\sqrt[3]{1+u} = 1$

$$\frac{1}{-(1+u)^{1/2}+(1+u)^{1/3}} = \frac{1}{-\left(1+\frac{u}{2}+\sum\limits_{n=2}^{\infty}\frac{\frac{1}{2}(\frac{1}{2}-1)\dots(\frac{1}{2}-n+1)u^n}{n!}\right)+\left(1+\frac{u}{3}+\sum\limits_{n=2}^{\infty}\frac{\frac{1}{3}(\frac{1}{3}-1)\dots(\frac{1}{3}-n+1)u^n}{n!}\right)} =$$

$$= \frac{1}{u\,\varphi(u)}$$

$\varphi(0) \neq 0$ ~ полюс 1-го порядка.

606. $\operatorname{ctg}\dfrac{1}{1+\sqrt{z}} = \dfrac{\cos u}{\sin u}$

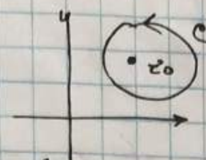
$z = \left[1+\left(\frac{1}{\pi k}\right)\right]^2$ 623.

$1+\sqrt{z} = 1 \pm \left(1+\frac{1}{k\pi}\right)$ $\to u = k\pi$; $\operatorname{ctg} u$.

$u = k\pi$ $\to u = k\pi$ ~ полюс 1-го порядка.

Вычеты.

$$\operatorname*{res}_{z=z_0} f = \int_C f(z)\,dz$$



Способы вычисления вычетов.

1) z_0 – прав. т-ка (уст. т-ка.)

$$\operatorname*{res}_{z=z_0} f = 0$$

2) z_0 – полюс $1^{го}$ порядка

$$\operatorname*{res}_{z=z_0} f = \lim_{z\to z_0}(z-z_0)f(z) = \left\{ \begin{array}{l} f=\dfrac{\varphi(z)}{\psi(z)} \\ \varphi(z_0)\neq 0 \end{array}\right\} = \frac{\varphi(z_0)}{\psi'(z_0)}$$

3) z_0 – полюс m-го порядка.

$$\operatorname{res}[f, z_0] = \frac{1}{(m-1)!}\lim_{z\to z_0}\frac{d^{m-1}}{dz^{m-1}}(z-z_0)^m f(z)$$

4) z_0 – сущ. особ. т-ка:

$$\operatorname*{res}_{z=z_0} f = C_{-1}$$ разложения в ряд Лорана.

5) $$\sum_{k=1}^{n}\operatorname*{res}_{z=z_k} f(z) + \operatorname*{res}_{z=\infty} f = 0$$

621 (4.79) $$f = \frac{1}{z^3 - z^5} = \frac{1}{z^3(1-z^2)}$$

$z=0$ – полюс $3^{го}$ порядка

$z=\pm 1$ – полюса $1^{го}$ порядка

1) $$\operatorname*{res}_{z=\pm 1} f = \left| \begin{array}{l} \varphi(z)=\frac{1}{z^3} \\ \psi(z)=1-z^2 \end{array}\right| = \frac{1}{(\pm 1)^3(-2)(\pm 1)} = -\frac{1}{2}, \quad z=\pm 1.$$

2) $$\operatorname*{res}_{z=0} f = \frac{1}{(3-1)!}\lim_{z\to 0}\frac{d^2}{dz^2}\frac{z^3\cdot 1}{z^3(1-z^2)} = \frac{1}{2!}\lim_{z\to 0}\frac{d}{dz}\frac{+2z}{(1-z^2)^2} =$$

$$= \frac{1}{2}\lim_{z\to 0}\frac{2(1-z^2)^2 + 4z\cdot 2z(1-z^2)}{(1-z^2)^4} = 1.$$

3) $$\operatorname*{res}_{z=\infty} f = -\operatorname*{res}_{z=0} f - \operatorname*{res}_{z=1} f - \operatorname*{res}_{z=-1} f = -1 + \frac{1}{2} + \frac{1}{2} = 0$$

623. $$f = \frac{z^{2n}}{(1+z)^n}$$

$z=-1$ – полюс $n^{-го}$ порядка.

$$\operatorname*{res}_{z=-1} f = \frac{1}{(n-1)!}\lim_{z\to -1}\frac{d^{n-1}}{dz^{n-1}}(1+z)^n\frac{z^{2n}}{(1+z)^n} = \frac{1}{(n-1)!}\lim_{z\to -1} 2n(2n-1)\ldots(2n-(n-1)+1)\cdot z^{2n-(n+1)} \;(=)$$

$$(z^{2n})' = 2n\, z^{2n-1}$$
$$(z^{2n})'' = 2n(2n-1)\, z^{2n-2}$$

$$= \frac{1}{(n-1)!}\,\frac{(2n)!}{(n+1)!}(-1)^{n+1}$$

$$\to \operatorname*{res}_{z=\infty} f = \frac{(2n)!\,(-1)^n}{(n-1)!\,(n+1)!}$$

№628. $f = \frac{\sin 2z}{(z+1)^3} \to \operatorname*{res}_{z=-1} f = \frac{1}{2!}\lim_{z\to -1}\frac{d^2}{dz^2}\frac{(z+1)^3\sin 2z}{(z+1)^3} =$

$z=-1$ – полюс $3^{\text{го}}$ порядка

$$= -\frac{4}{2!}\sin 2(-1) = 2\sin 2$$

$$\operatorname*{res}_{z=\infty} f = -2\sin 2$$

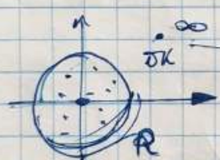
№629.
$$f = \frac{1}{\sin z}$$

$z=\pi k$, ~~$k=\pm1,\pm2,\dots$~~ $k\in\mathbb{Z}$ – полюс $1^{\text{го}}$ порядка

$$\operatorname*{res}_{z=\pi k} f = \frac{1}{\cos z}\Big|_{z=\pi k} = \frac{1}{\cos \pi k} = (-1)^k$$

$z=\infty$; $\lim\limits_{k\to\infty}\pi k = \infty$ – предельная точка.

В ней вычет не вычисляем!



$\overset{\infty}{\pi k}$ – за радиусом окрестности изол. точек.

$$\exists k:\ \pi k > R$$

№632
1) $f = \cos\frac{1}{z-2} = \{u = z-2\} = \cos\frac{1}{u} = \sum\limits_{n=0}^{\infty}\frac{(-1)^n}{(2n)!\,u^{2n}} = 1 - \frac{1}{2!u^2} + \frac{u^4}{4!u^4}\dots$

$z=2$ – сущ. особ. точка.

$\operatorname*{res}_{z=2} f = C_{-1}$ C_{-1} отсутствует.

$$\Rightarrow \operatorname*{res}_{z=2} f = C_{-1} = 0$$

2) $f = z^3\cos\frac{1}{z-2} = \{u=z-2\} = (u+2)^3\cos\frac{1}{u}$

~~$\operatorname*{res}_{z=2} f = C_{-1}$~~

$$= (u^3 + 6u^2 + 12u + 8)\left(1 - \frac{1}{2!u^2} + \frac{1}{4!u^4} - \frac{1}{6!u^6} + \dots\right)$$

$$\operatorname*{res}_{z=2} f = C_{-1} = \left(\frac{1}{4!} - \frac{12}{2!}\right) = \left(\frac{1}{24} - 6\right) = -5\frac{23}{24}$$

$$\to \operatorname*{res}_{z=\infty} f = 5\frac{23}{24}$$

638 $f = z^n\sin\frac{1}{z} = z^n\sum\limits_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)!}\frac{1}{z^{2k+1}} = z^n\left(1 - \frac{z^{-3}}{3!} + \frac{z^{-5}}{5!} - \frac{z^{-7}}{7!}\dots + \frac{(-1)^m}{(2m+1)!\,z^{2m+1}}\right)$

$\lim\limits_{z\to 0} z^n\sin\frac{1}{z}$ $\nexists$

Если $n = 2l+1 \to \frac{z^n}{z^{2m+1}} = z^{2(l-m)}$ – чётная степень

$$\Rightarrow \operatorname*{res}_{z=0} f = 0,\quad C_{-1} = 0$$

$n = 2l = 2m \to C_{-1} = \frac{(-1)^m}{(2m+1)!} = \operatorname{res}_{z=0} f$

$\operatorname{res}_{z=\infty} f = \frac{(-1)^{m-1}}{(2m+1)!}$

№642. 1) $z=1$ ~ полюс 1-го порядка

$\frac{\sqrt{z}}{1-z} \to \pm\sqrt{z} \to \lim_{z\to 1} \frac{\sqrt{z}(z-1)}{1-z} = \begin{cases} -1, & \sqrt{z} \\ +1, & -\sqrt{z} \end{cases}$

№647(1). $f = \operatorname{Ln} z \, \sin\frac{1}{z-1} = \left(i2\pi k + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}u^n}{n}\right) \sin\frac{1}{u} =$

$= \left(i2\pi k + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}u^n}{n}\right) \sum_{l=0}^{\infty} \frac{(-1)^l}{(2l+1)!\, u^{2l+1}} =$

$z = 1$

$\operatorname{Ln} 1 = \ln 1 + i2\pi k$

$u = z - 1$

$\operatorname{Ln} u = \ln(1+u) + i2\pi k$

$= \left(i2\pi k + u - \frac{u^2}{2} + \frac{u^3}{3} + \ldots + \frac{(-1)^{n-1}u^n}{n} + \ldots\right)\left(\frac{1}{u} - \frac{1}{u^3 3!} + \frac{1}{u^5 5!} + \ldots + \frac{(-1)^l}{(2l+1)!\, u^{2l+1}} + \ldots\right)$

$C_{-1} = i2\pi k + \frac{(-1)}{2}\cdot\frac{(-1)}{3!} + \frac{(-1)}{4}\cdot\frac{1}{5!} + \ldots = i2\pi k + \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{2m(2m+1)!}$

Вычисление контурных интегралов с помощью вычетов.

№657. $\oint \frac{dz}{z^4+1} = 2\pi i \sum_k \operatorname{res}_{z_k} f(z)$, z_k внутри C.

$C: x^2+y^2 = 2x$

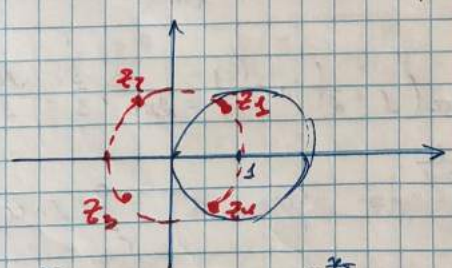
$z^4 + 1 = 0 \to z^4 = -1 = e^{i(\pi + 2\pi k)}$

$z = e^{\frac{i(\pi+2\pi k)}{4}}$; $z_1 = e^{i\frac{\pi}{4}}$, $z_2 = e^{i\frac{3\pi}{4}}$, $z_3 = e^{i\frac{5\pi}{4}}$, $z_4 = e^{i\frac{7\pi}{4}}$

$x^2 + y^2 = 2x$

$x = 2\cos\varphi$, $y = r\sin\varphi$

$r = 2\cos\varphi$



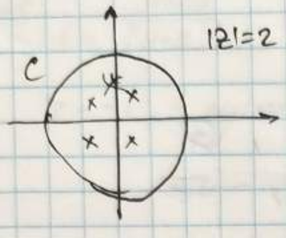
$I = 2\pi i\left[\operatorname{res}(f, z = e^{i\frac{\pi}{4}}) + \operatorname{res}(f, z = e^{i\frac{7\pi}{4}})\right] = 2\pi i\left[\frac{1}{4z^3}\Big|_{z=e^{i\frac{\pi}{4}}} + \frac{1}{4z^3}\Big|_{e^{-\frac{7\pi}{4}i}}\right] =$

$= \frac{2\pi i}{4}\left[e^{-\frac{3\pi}{4}i} + e^{-\frac{21\pi i}{4}}\right] = \frac{\pi i}{2}\left[\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4} + \cos\left(-5\pi - \frac{\pi}{4}\right) - i\sin\left(5\pi + \frac{\pi}{4}\right)\right] =$

$= \frac{\pi i}{2}\left(-\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right) = -\frac{\pi i\sqrt{2}}{2}$

№659. по правилам (теоремам):

$$\int\limits_{\substack{C\\|z|=2}} \frac{dz}{(z-3)(z^5-1)} = 2\pi i \sum_{k=1}^{5} \operatorname*{res}_{z_k:\,|z|=1} f = \left(-\operatorname*{res}_{z=\infty} f - \operatorname*{res}_{z=3} f\right)\cdot 2\pi i \ominus$$



$\Rightarrow$ $z^5-1=0$ в контуре
$z=3$ - вне.

$z=3$ - полюс $1^{го}$ порядка

$$\operatorname*{res}_{z=3} f = \frac{1}{(z^5-1)(z-3)'}\Big|_{z=3} = \frac{1}{3^5-1} = \frac{1}{242}$$

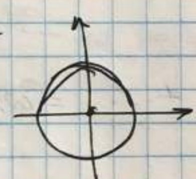
$$\lim_{z\to\infty} \frac{1}{(z-3)(z^5-1)} = \left\{u=\frac{1}{z}\right\} = \lim_{u\to 0}\frac{u^6}{(1-3u)(1-u^5)} = 0$$

$\operatorname*{res}_{z=\infty} f = 0$, т.к. ∞ устранимая точка

$$\ominus \left(-0-\frac{1}{242}\right)2\pi i = -\frac{\pi i}{121}$$

№662. $\displaystyle \frac{1}{2\pi i}\int\limits_{C} \sin\frac{1}{z}\,dz = \frac{1}{2\pi i}\,2\pi i \operatorname*{res}_{z=0} f = 1.$

$|z|=r$

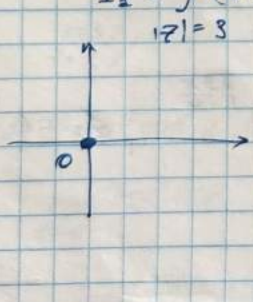


$z=0$ ~ сущ. особ. т. $\quad \operatorname*{res}_{z=0} f = C_{-1}$

$$\sin\frac{1}{z} = \frac{1}{z} - \frac{1}{z^3 3!} + \ldots$$

№665. $\displaystyle \int\limits_{|z|=3} (1+z+z^2)\left(e^{\frac{1}{z}} + e^{\frac{1}{z-1}} + e^{\frac{1}{z-2}}\right) = I_1 + I_2 + I_3$

$$I_1 = \int\limits_{|z|=3} (1+z+z^2)e^{\frac{1}{z}}dz = 2\pi i \operatorname{Res}[f, z=0] = 2\pi i\cdot C_{-1} \ominus$$



$z=0$ ~ сущ. особ. точка.

$$f(z) = (1+z+z^2)\cdot\sum_{k=0}^{\infty}\frac{1}{k!\,z^k} = (1+z+z^2)\left(1+\frac{1}{z}+\frac{1}{z^2 2!}+\frac{1}{z^3 3!}+\ldots\right)$$

$$\ominus\, 2\pi i\left(1+\frac{1}{2!}+\frac{1}{3!}\right) = 2\pi i\cdot\frac{5}{3} = \frac{10}{3}\pi i$$

$$I_2 = \int\limits_{|z|=3}(1+z+z^2)e^{\frac{1}{z-1}}dz = \{u=z-1\} = \int\limits_{C}(1+u+1+(u+1)^2)e^{\frac{1}{u}}du =$$

$z=1$ ~ сущ. особ. точка $\quad = 2\pi i\, C_{-1} \ominus$

$$f = (3+3u+u^2)e^{\frac{1}{u}} = (3+3u+u^2)\left(1+\frac{1}{u}+\frac{1}{u^2 2!}+\frac{1}{u^3 3!}+\ldots\right)$$

$$\ominus\, 2\pi i\left(3+\frac{3}{2!}+\frac{1}{3!}\right) = 2\pi i\cdot 4\frac{2}{3} = \frac{28}{3}\pi i$$

$I_3 = \int\limits_{|z|=3} (1+z+z^2)\, e^{\frac{1}{z-2}}\, dz = \{u = z-2\} = \int\limits_C (1+u+2+(u+2)^2)\, e^{\frac{1}{u}}\, du =$

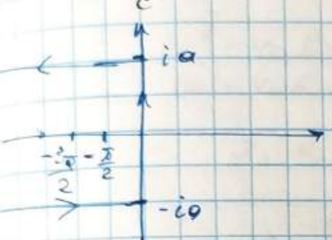
$z=2$ – сущ. особ. т-ка

$= 2\pi i\, c_{-1} = 2\pi i \left(7 + \frac{5}{2!} + \frac{1}{6}\right) = 2\pi i \left(9\frac{2}{3}\right) = \frac{58}{3}\pi i$

$f = (7+5u+u^2)\left(1 + \frac{1}{u} + \frac{1}{u^2 2!} + \frac{1}{u^3 3!} + \dots\right)$

$\Rightarrow I = I_1 + I_2 + I_3 = \frac{10}{3}\pi i + \frac{28}{3}\pi i + \frac{58}{3}\pi i = \frac{96}{3}\pi i = 32\pi i$

№672 $\frac{1}{2\pi i}\int\limits_C \frac{e^z\, dz}{\cos z} = \sum_{k=-1}^{-\infty} res\left[f, \frac{\pi}{2}+\pi k\right] = \sum_{k=-1}^{-\infty} \frac{e^z}{(\cos z)'}\Big|_{z=\frac{\pi}{2}+\pi k} =$



$\cos z = 0$

$z = \frac{\pi}{2} + \pi k,\ k = -1, -2, \dots$ ~ полюса 1-го порядка

$\cos\left(u + \frac{\pi}{2} + \pi k\right) = (-1)^{k+1} \sin u \to u = 0$ ~ полюс 1-го поряд.

$u = z - \left(\frac{\pi}{2} + \pi k\right)$

$= \sum_{k=-1}^{-\infty} \frac{e^{\frac{\pi}{2}+\pi k}}{-\sin\left(\frac{\pi}{2}+\pi k\right)} = e^{\frac{\pi}{2}} \sum_{k=-1}^{-\infty} (-1)^{k+1} e^{\pi k} = \{l = -k\} =$

$= -e^{\frac{\pi}{2}}\left[\underbrace{\sum_{l=1}^{\infty}\left(-e^{-\pi}\right)^l + 1}_{\text{сумма геом. прогрессии}} - 1\right] = -e^{\frac{\pi}{2}}\left[\frac{1}{1+e^{-\pi}} - 1\right] = \frac{e^{-\frac{\pi}{2}}}{1+e^{-\pi}}$

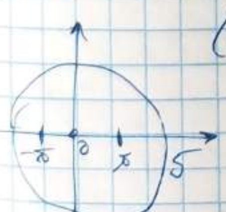
$|e^{-\pi}| < 1$

№666 $f(z) = \int\limits_{|z|=5} \frac{z\, dz}{\sin z\, (1-\cos z)} = I$

1) $z = \pi k \to u = z - \pi k \to \sin(u+\pi k) = (-1)^k \sin u \Rightarrow u = 0$ – п. 1-го порядка

2) $z = 0$

$\frac{z}{\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} \dots\right)\left(\cancel{1} - \cancel{1} + \frac{z^2}{2!} - \frac{z^4}{4!} + \dots\right)} =$



$= \frac{1}{\left(1 - \frac{z^2}{3!} + \dots\right)\left(\frac{1}{2!} - \frac{z^2}{4!} + \dots\right) z^2}$

$\to z = 0$ ~ полюс 2-го поряд.

$\Rightarrow I = 2\pi i\,[Res[f, 0] + Res[f, \pi] + Res[f, -\pi]]$

$Res[f, \pm\pi] = \{$через производную$\} = \left(\frac{z}{1-\cos z}\right)\frac{1}{(\sin z)'}\Big|_{z=\pm\pi} = -\frac{\pm\pi}{2}$

$res[f, 0] = \frac{1}{1!}\lim_{z\to 0}\frac{d}{dz}\left(\frac{z^2 \cdot z}{\sin z(1-\cos z)}\right) = \lim_{z\to 0}$

$= \lim_{z\to 0}\frac{3z^2 \sin z(1-\cos z) - z^3(\cos z - \cos^2 z + \sin^2 z)}{\sin^2(z)\,(1-\cos z)^2} =$

$$= \lim_{z\to 0} \frac{4}{z^4} \cdot \frac{\left(\frac{z^3}{2}\cancel{\cdot 3} - \frac{32z^3}{12} - \frac{15}{4!}z^5\right)}{\ldots} = 0$$

из расположения

$$\cos z - \cos 2z = \cancel{1} - \frac{z^2}{2} + \frac{z^4}{4!} + \ldots - \cancel{1} + \frac{4z^2}{2} - \frac{16z^4}{4!} + \ldots$$

$$\Rightarrow \underline{I = 0}$$

Интегралы от тригоном. функций

№ 673. (4.131) $\displaystyle\int_0^{2\pi} \frac{d\varphi}{a+\cos\varphi} \ominus$

$a > 1$ $\qquad \cos\varphi = \dfrac{e^{i\varphi}+e^{-i\varphi}}{2} \quad \to e^{i\varphi} = z$

$\underline{|z| = 1}$

$$\to \cos\varphi = \frac{z^2+1}{2z}$$

$$dz = ie^{i\varphi}\,d\varphi \to d\varphi = \frac{dz}{iz}$$

$$\ominus \int\limits_{|z|=1} \frac{dz}{iz\left(a+\frac{z^2+1}{2z}\right)} = \frac{1}{i}\int\limits_{|z|=1}\frac{dz}{z^2+2az+1} = \frac{2\pi i}{i}\sum_k \operatorname{Res}[f, z_{k0}] \ominus$$

z_{k0} in C

$$z_{12} = -a \pm \sqrt{a^2-1}$$

$|-a-\sqrt{a^2-1}| < 1$ – не входит

$a + \sqrt{a^2-1} < 1$ против, т.к. $a>1$

$|-a+\sqrt{a^2-1}| < 1$

$a - \sqrt{a^2-1} < 1$

$a - 1 < \sqrt{a^2-1}$

$a^2 - 2a + 1 < a^2 - 1$

$a > 1$ иев.

$$\ominus\ 2\pi \operatorname{Res}[f, -a+\sqrt{a^2-1}] = \star$$

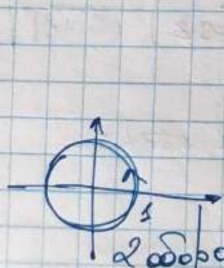
$$\star = 2\pi \left.\frac{1}{2z+2a}\right|_{-a+\sqrt{a^2-1}} = \frac{\pi}{\sqrt{a^2-1}}$$

№ 675. $\displaystyle\int_0^{2\pi} \frac{d\varphi}{(a+b\cos^2\varphi)^2} = \int_0^{2\pi} \frac{d\varphi}{\left(a+b\,\frac{1+\cos 2\varphi}{2}\right)^2} = 4\int_0^{2\pi} \frac{d\varphi}{[(2a+b)+b\cos 2\varphi]^2} =$

$a>0$

$b>0$

$$= \left\{ \begin{array}{l} \cos 2\varphi = \dfrac{e^{2i\varphi}+e^{-2i\varphi}}{2} = \dfrac{z^2+1}{2z} \\ z = e^{2i\varphi} \to d\varphi = \dfrac{dz}{2iz} \end{array} \right\} =$$



2 оборота

$$= 2\cdot 4 \int\limits_{|z|=1} \frac{dz}{2iz\left[(2a+b)+b\,\frac{z^2+1}{2z}\right]^2} = \frac{4\cdot 4}{i}\int \frac{z\,dz}{[bz^2+2(2a+b)z+b]^2}$$

$bz^2+2(2a+b)z+b=0$

$$z_{1,2}=\frac{-(2a+b)\pm\sqrt{(2a+b)^2-b^2}}{b}$$

$$\left|\frac{-(2a+b)-\sqrt{4a^2+4ab}}{b}\right| < 1 \ ?$$ – проверяем попадание корней в круг

$$\cancel{2a+b+\sqrt{4a^2+4ab}<b\,0}$$
понятно.

$$z_2=\frac{-(\ )-\sqrt{\ }}{b}$$ – не входит

$$|-(2a+b)+\sqrt{4a^2+4ab}|<b$$

$$4a^2+4ab<(2a+2b)^2$$
$$4a^2-4ab<4a^2+8ab+4b^2$$

$$z_1=\frac{-(\ )+\sqrt{\ }}{b}$$ – входит.

$$\operatorname*{res}_{z=z_1} f=\frac{1}{1!}\lim_{z\to z_1}\frac{d}{dz}\frac{\cancel{(z-z_1)^2}\,z}{\cancel{(z-z_1)^2}(z-z_2)^2}=\lim_{z\to z_1}\frac{(z-z_2)^{\cancel{2}}-2z\cancel{(z-z_2)}}{(z-z_2)^{\cancel{4}\,3}}=$$

$$=\frac{-(z_1+z_2)}{(z_1-z_2)^3}=\frac{2(2a+b)}{8(4a^2+4ab)^{3/2}}$$

$$I=\frac{16}{i}\,\frac{2(2a+b)}{8(4a^2+4ab)^{3/2}}\cdot 2\pi i=\frac{8\pi(2a+b)}{(4a^2+4ab)^{3/2}}$$

№478.
$$\int_0^{2\pi}\frac{d\varphi}{1-2a\cos\varphi+a^2}=\left\{z=e^{i\varphi}\right\}=\int_{|z|=1}\frac{dz}{iz\left(1-2a\frac{z^2+1}{2z}+a^2\right)}=$$

a ~ компл. число

$$=\frac{1}{i}\int_{|z|=1}\frac{dz}{-az^2+(a^2+1)z-1}=-\frac{1}{i}\int_{|z|=1}\frac{dz}{az^2-(a^2+1)z+a}=★$$

$$az^2-(a^2+1)z+a=0$$

$$z_{1,2}=\frac{a^2+1\pm\sqrt{(a^2+1)^2-4a^2}}{2a}=\frac{a^2+1\pm(a^2-1)}{2a}$$

$$z_1=a\ ;\ z_2=\frac{1}{a}$$

★ 1) $|a|<1$ $$★=-\frac{1}{i}2\pi i\operatorname{res}[f,a]=-2\pi\left.\frac{1}{a\cdot 2z-(a^2+1)}\right|_{z=a}=\frac{-2\pi}{a^2-1}$$

2) $|a|>1$ $$★=-\frac{1}{i}2\pi i\operatorname{res}\left[f,\frac{1}{a}\right]=-2\pi\left.\frac{1}{2az-(a^2+1)}\right|_{z=\frac{1}{a}}=\frac{2\pi}{a^2-1}$$

№479
$$\int_0^{\pi}tg(x+ia)\,dx=\int_0^{\pi}\frac{\sin(x+ia)}{\cos(x+ia)}\,dx=\oint_C ★$$

$$z=e^{i\frac{x}{2}}$$

$$\operatorname{tg}(x+ia) = \frac{\sin(x+ia)}{\cos(x+ia)} = \frac{e^{i(x+ia)} - e^{-i(x+ia)}}{i\left(e^{i(x+ia)} + e^{-i(x+ia)}\right)}$$

Нам нужно замкнуть контур, чтобы z прошлось по всему контуру от $[0; 2\pi]$

$z = e^{i\frac{x}{2}}$

$$z = e^{i(x+ia)} \to C: |z| = \left\{e^{ix-a}\right\} = e^{-a}$$

$$dz = e^{i(x+ia)} i\,dx \to dx = \frac{dz}{iz}$$

$$A = \frac{1}{2i}\oint\limits_{|z|=e^{-a}} \frac{z^2-1}{z^2+1}\cdot\frac{dz}{iz} = -\frac{1}{2}\int\limits_{|z|=e^{-a}} \frac{z^2-1}{z^2+1}\cdot\frac{dz}{z}$$

1) ~~Если $a=0$~~ 1) При $a=0$ – решений нет

2) $a>0 \to e^{-a}<1 \to z=0$ – полюс $1^{го}$ порядка

$$A = -\frac{1}{2}2\pi i\ \mathrm{res}[f,0] = -\pi i \left.\frac{z^2-1}{z^2+1}\right|_{z=0} = \pi i$$

2) $a<0$, $e^{-a}>1 \Rightarrow \left[\begin{array}{l} z=0 \\ z=\pm i \end{array}\right.$ – полюса $1^{го}$ порядка $\Big] =$

$$= -\frac{1}{2}2\pi i\left(-\mathrm{Res}[f,\infty]\right) = \pi i$$

$$\frac{z^2-1}{z(1+z^2)} = \left(1-\frac{1}{z}\right)\sum_{n=0}^{\infty}(-1)^n z^{2n} = \left(1-\frac{1}{z}\right) + \left(1-\frac{1}{z}\right)\sum_{n=1}^{\infty}\ldots$$

$$\Rightarrow C_{-1} = -1 \to \mathrm{Res}[f,\infty] = -C_{-1} = 1$$

624.(4.82)

628.(4.8

634(4.92

Домашняя работа.

622(4.82) $f(z) = \frac{1}{z(1-z^2)} = \frac{1}{z(1-z)(1+z)}$

$z=0,\ z=1,\ z=-1$ ~ полюса 1-го порядка

$z=0$: $\lim\limits_{z\to 0} \frac{z}{z(1-z)(1+z)} = 1 = \operatorname{res}\limits_{z=0} f$

$z=1$: $\lim\limits_{z\to 1} \frac{-(z-1)}{z(1+z)(z-1)} = -\frac{1}{2} = \operatorname{res}\limits_{z=1} f$

$z=-1$: $\lim\limits_{z\to -1} \frac{(z+1)}{z(1+z)(1-z)} = -\frac{1}{2} = \operatorname{res}\limits_{z=-1} f$

$z=\infty$: $\operatorname{res}\limits_{z=\infty} f = -\frac{1}{2} - \frac{1}{2} + 1 = 0$

Ответ: $\operatorname{res}\limits_{z=0} f = 1$; $\operatorname{res}\limits_{z=\pm 1} f = -\frac{1}{2}$; $\operatorname{res}\limits_{z=\infty} f = 0$

628. (4.86)

$f(z) = \operatorname{tg} z$

Из 584 (4.92) $\to$ $\varphi(z) = \frac{1}{\operatorname{tg} z} = \frac{\cos z}{\sin z} = 0 \to \cos z = 0$

$z = \frac{\pi}{2} + \pi k = \frac{\pi}{2}(2k+1),\ k \in \mathbb{Z}$

$z = \frac{\pi}{2}(2k+1)$ ~ полюса 1-го пор.

$z = \infty$ ~ предельная для полюсов (в ней вычет не вычисляем)

$z = \frac{\pi}{2}(2k+1)$: $f(z) = \frac{\sin z}{\cos z}$; $\sin\left(\frac{\pi}{2} + \pi k\right) = (-1)^k \neq 0$

$C_{-1} = 1$

$\varphi(z) = \sin(z)$, $\psi(z) = \cos z$

$\psi'(z) = -\sin z$

$\Rightarrow \operatorname{res}\limits_{z=\frac{\pi}{2}+\pi k} f = -\frac{\sin\left(\frac{\pi}{2}+\pi k\right)}{\sin\left(\frac{\pi}{2}+\pi k\right)} = -1.$

Ответ: $\operatorname{res}\limits_{z=\frac{\pi}{2}+\pi k} f = -1.$

634(4.92)

$f(z) = \sin z \cdot \sin\frac{1}{z}$ $\to \varphi(z) = \frac{1}{f(z)} = \frac{1}{\sin z \cdot \sin\frac{1}{z}}$

$\operatorname{res}\limits_{z=z_0} f = C_{-1}$

$z=0,\ z=\infty$ ~ сущ. особ. точки.

$\sin z \cdot \sin\frac{1}{z} = \left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \ldots\right)\left(\frac{1}{z} - \frac{1}{z^3 3!} + \frac{1}{z^5 5!} - \ldots\right) =$

$C_{-1} = 0$

Ответ: $\operatorname{res}\limits_{z=0,\ z=\infty} f = 0$

639(4.97). $f(z) = \frac{1}{\sin\frac{1}{z}}$

1) $\varphi(z) = \frac{1}{f(z)} = \sin\frac{1}{z}$ $\to \sin\frac{1}{z} = 0$

$\frac{1}{z} = \pi k$, $k \in \mathbb{Z}$, $k \neq 0$

$z = \frac{1}{\pi k}$

$\psi(z) = \sin\frac{1}{z}$ $\psi'(z) = \cos\frac{1}{z}\cdot\left(-\frac{1}{z^2}\right)$

$\psi'(z_0) = \cos\pi k \cdot \frac{-1}{\left(\frac{1}{\pi k}\right)^2} = \frac{(-1)^{k+1}\pi^2}{1}$

$u = z - \frac{1}{\pi k}$

$f(u) = \frac{1}{\sin\frac{1}{u+\frac{1}{\pi k}}}$

$\lim\limits_{u\to 0} f(u)\ \frac{1}{\sin\pi k} = \infty$ $\to z = \frac{1}{\pi k}$ ~ полюс 1-го порядка.

$\operatorname*{res}\limits_{z=\frac{1}{\pi k}} f = \frac{1}{(-1)^{k+1}\pi^2 k^2} = \frac{(-1)^{k+1}}{\pi^2 k^2}$

$k\to\infty$, $z\to\infty$

2) $z=\infty$ ~ сущ. особ. точка

$\operatorname*{res}\limits_{z=\infty} f = -\sum\limits_{k=1}^{\infty}\frac{(-1)^{k+1}}{\pi^2k^2} = \left(\frac{1}{\pi^2} - \frac{1}{4\pi^2} + \frac{1}{9\pi^2} - \ldots\right) = -\frac{1}{6}$

$\sum\limits_{k=1}^{\infty}\left|\frac{(-1)^{k+1}}{\pi^2k^2}\right| = \sum\limits_{k=1}^{\infty}\frac{1}{\pi^2k^2} = \frac{1}{\pi^2}\sum\limits_{k=1}^{\infty}\frac{1}{k^2} = \frac{1}{6}$

Ответ: $\operatorname*{res}\limits_{z=\frac{1}{\pi k}} f = \frac{(-1)^{k+1}}{\pi^2k^2}$

$\operatorname*{res}\limits_{z=\infty} f = -\frac{1}{6}$

643 (4.101) $f(z) = \frac{1}{\sqrt{2-z}+1}$, отн. точки $z=1$

$\sqrt{2-z}+1=0$

$2-z=(-1)^2 \to z=1$

1) $+\sqrt{2-z}$: $\to f(1)=\frac{1}{2}$ $\Rightarrow z=1$ ~ прав. точка

$\operatorname*{res}\limits_{z=1} f = 0$

2) $-\sqrt{2-z}$: $\to z=1$ ~ полюс 1-го порядка

$\psi(z) = -\sqrt{2-z}+1$

$\psi'(z) = -\frac{-1}{2\sqrt{2-z}}$; $\psi'(z_0) = +\frac{1}{2}$

$\Rightarrow \operatorname*{res}\limits_{z=1} f = 2$

Ответ: 0, 2.

547(2). (4.105) $f(z) = \mathrm{Ln}\, z \cos\frac{1}{z-1}$ относительно точки $z=1$ ~ сущ. особ. точка

$u = z-1$

$f(u) = \mathrm{Ln}(u+1)\cos\frac{1}{u} \ominus$

$\mathrm{Ln}(u+1) = \ln(1+u) + i2\pi k.$

$\ominus \left(i2\pi k + \sum\limits_{n=1}^{\infty} \frac{(-1)^{n-1} u^n}{n} \right) \sum\limits_{l=0}^{\infty} \frac{(-1)^l}{(2l)!\, u^{2l}} =$

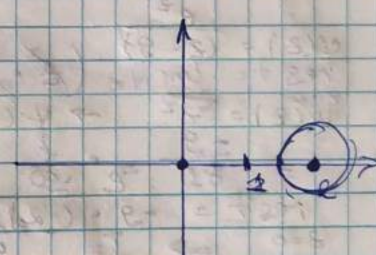
$= \left(i2\pi k + u - \frac{u^2}{2} + \frac{u^3}{3} + \ldots + \frac{(-1)^{n-1} u^n}{n} + \ldots \right) \left(1 - \frac{1}{2!\, u^2} + \frac{1}{4!\, u^4} - \ldots \right.$
$\left. \ldots + \frac{(-1)^l}{(2l)!\, u^{2l}} + \ldots \right)$

$C_{-1} = -\frac{1}{2!\cdot 1} + \frac{1}{3\cdot 4!} - \frac{1}{5\cdot 6!} + \ldots = \sum\limits_{m=1}^{\infty} \frac{(-1)^m}{(2m)!\cdot(2m-1)}$

Ответ: $\operatorname*{res}\limits_{z=1} f = \sum\limits_{m=1}^{\infty} \frac{(-1)^m}{(2m)!\,(2m-1)}$

558. (4.116) $\int\limits_C \frac{z\, dz}{(z-1)(z-2)^2}$, где C - окружность $|z-2| = \frac{1}{2}$

$z=1$ ~ полюс 1^го^ порядка
$z=2$ ~ полюс 2^го^ порядка.



$\operatorname*{res}\limits_{z=2} f = \lim\limits_{z\to 2} \frac{d}{dz}\left((z-2)^2 \cdot \frac{z}{(z-1)(z-2)^2} \right) =$

$= \lim\limits_{z\to 2} \frac{d}{dz}\left(\frac{z}{z-1} \right) = \lim\limits_{z\to 2} \frac{z-1-z}{(z-1)^2} = -\frac{1}{1} = -1$

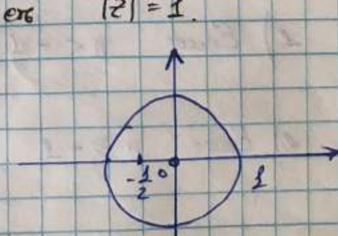
$\Rightarrow I = -1\cdot 2\pi i = -2\pi i$

Ответ: $I = -2\pi i$

560. (4.118) $I = \int\limits_C \frac{z^3\, dz}{2z^4+1}$, где C - окружность $|z| = 1.$

$2z^4 + 1 = 0 \rightarrow z^4 = -\frac{1}{2}$

$|z| = +\frac{1}{2}$



$z = \sqrt[4]{\frac{1}{2}} \cdot e^{\frac{i\pi + 2\pi k}{4}} = \frac{1}{\sqrt[4]{2}} \cdot e^{i\frac{\pi}{4} + \frac{\pi k}{2}}$

$k=0: \quad z_0 = \frac{1}{\sqrt[4]{2}} \cdot e^{i\frac{\pi}{4}}$

$k=1: \quad z_1 = \frac{1}{\sqrt[4]{2}} \cdot e^{i\frac{3\pi}{4}}$

$k=2: \quad z_2 = \frac{1}{\sqrt[4]{2}} \cdot e^{i\frac{5\pi}{4}}$

$k=3: \quad z_3 = \frac{1}{\sqrt[4]{2}} \cdot e^{i\frac{7\pi}{4}}$

Все $\in C$.
полюса 1^го^ порядка.

$$f(z) = \frac{z^3}{2z^4+1}$$

$$z \to \infty \; ; \; u = \frac{1}{z} \quad \to f(u) = \frac{1}{u^3\left(2 \cdot \frac{1}{u^4}+1\right)} = \frac{u^4}{u^3(2+u^4)} = \frac{u}{2+u^4}$$

$$\varphi(z) = z^3$$
$$\psi(z) = 2z^4+1 \quad \to \psi'(z) = 8z^3$$

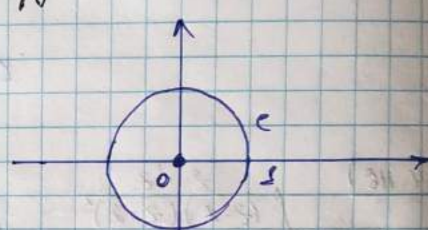
$$\underset{z=z_0}{\operatorname{res}} f = \frac{e^{i\cdot\frac{3\pi}{4}}}{8 \cdot e^{i\frac{3\pi}{4}}} = \frac{1}{8} = \underset{z=z_1}{\operatorname{res}} f = \underset{z=z_2}{\operatorname{res}} f = \underset{z=z_3}{\operatorname{res}} f$$

$$\Rightarrow I = 2\pi i \left(\frac{4}{8}\right) = \pi i$$

Ответ: $I = \pi i$

661. (4.119.) $I = \oint\limits_C \frac{e^z}{z^2(z^2-9)}\,dz$, где C – окружность $|z| = 1$

$$f(z) = \frac{e^z}{z^2(z-3)(z+3)}$$



$z=0$ ~ полюс $2^{\text{го}}$ порядка
$z=3$ ~ полюс $1^{\text{го}}$ порядка
$z=-3$ ~ полюс $1^{\text{го}}$ порядка

$$\underset{z=0}{\operatorname{res}} f = \lim_{z\to 0} \frac{d}{dz}\left(z^2 \cdot \frac{e^z}{z^2(z^2-9)}\right) = \lim_{z\to 0} \frac{-2z \cdot e^z + e^z(z^2-9)}{(z^2-9)^2} =$$

$$= \frac{-9}{81} = -\frac{1}{9}$$

$$\Rightarrow I = -\frac{2\pi i}{9}$$

Ответ: $I = -\frac{2\pi i}{9}$

664. (4.122.) $I = \frac{1}{2\pi i}\oint\limits_C z^n \cdot e^{\frac{2}{z}}\,dz$, где n – целое число, а C – окр-ть $|z| = 1$

$z=0$ ~ сущ. особ. точка.

$$f(z) = z^n \cdot e^{\frac{2}{z}} = z^n \cdot \left(1 + \frac{2}{z} + \frac{2^2}{z^2 \cdot 2!} + \frac{2^3}{z^3 \cdot 3!} + \dots\right)$$

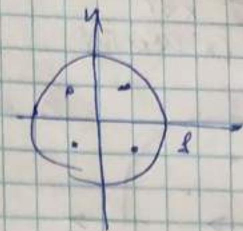
1) Если $n < -1$, то $C_{-1} = 0 \quad \Rightarrow I = \underset{z=0}{\operatorname{res}} f = 0$

2) Если $n \geq -1$, то $C_{-1} = \frac{2^{n+1}}{(n+1)!} = \frac{2^{n+1}}{(n+1)!}$

$$\Rightarrow I = \frac{2^{n+1}}{(n+1)!}$$

Ответ: $I = \begin{cases} \frac{2^{n+1}}{(n+1)!}, & n \geq -1 \\ 0, & n < -1 \end{cases}$

888. $\oint\limits_{|z|=1} \frac{z^3}{2z^4+1}\,dz = 2\pi i \sum\limits_{\substack{k=1\\ z:\,|z|=\sqrt[4]{\frac{1}{2}}}}^{4} \operatorname{res} f = -2\pi i \operatorname{res}\limits_{z=\infty} f$ 16.12.21



$2z^4+1=0$

$z=\sqrt[4]{\left(-\frac{1}{2}\right)}$

$\frac{z^3}{2z^4+1} = \left(u=\frac{1}{z}\right) = \frac{u}{2+u^4}$

$\lim\limits_{u\to 0} f(u) = 0$

Ищем корни: $z_k = e^{i\frac{\pi+2\pi k}{4}}$

$z_1 = e^{i\frac{\pi}{4}}\cdot\frac{1}{\sqrt[4]{2}}$

$z_2 = e^{i\frac{3\pi}{4}}\cdot\frac{1}{\sqrt[4]{2}}$

$z_3 = e^{i\frac{5\pi}{4}}\cdot 1/\sqrt[4]{2}$

$z_4 = e^{i\frac{7\pi}{4}}\, 1/\sqrt[4]{2}$

$\operatorname{res}\limits_{e^{i\frac{\pi}{4}}} f = \frac{z^3}{8z^3} = \frac{1}{8} \Rightarrow I = 2\pi i\cdot\frac{4}{8} = \pi i$

$\frac{z^3}{2z^4+1} = \frac{z^3}{2z^4}\left(\frac{1}{1+\frac{1}{2z^4}}\right) = \frac{1}{2z}\left(1-\frac{1}{2z^4}+\frac{1}{4z^8}-\dots\right)$

$C_{-1} = \frac{1}{2}$

$\operatorname{res}\limits_{\infty} f = -C_{-1} = -\frac{1}{2}$

$\Rightarrow I = \pi i$

889. $f(z) = \frac{1}{\sin\frac{1}{z}}$ $\to \sin\frac{1}{z} \to \frac{1}{z} = \pi n \to z = \frac{1}{\pi n}$

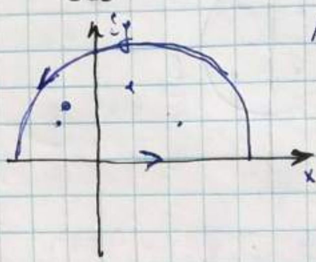
$u = \frac{1}{z} \to \sin u\big|_{u=\pi n} \to u$ – полюс 1-го порядка

$\operatorname{res}\limits_{z=\frac{1}{\pi n}} f = \frac{1}{\cos\frac{1}{z}\left(-\frac{1}{z^2}\right)}\Bigg|_{z=\frac{1}{\pi n}} = \frac{(-1)^{n+1}}{\pi^2 n^2}$

$n=0$? $\lim\limits_{n\to\infty}\frac{1}{\pi n} = 0 \Rightarrow z=0$ – пред. точка

$\operatorname{res}\limits_{z=\infty} f = -\sum\limits_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\frac{(-1)^{n+1}}{\pi^2 n^2} = 2\sum\limits_{n=1}^{\infty}\frac{(-1)^n}{\pi^2 n^2}$

16.12.21 №682. $\int\limits_{-\infty}^{\infty}\frac{x\,dx}{(x^2+4x+13)^2}$



В силу того, что ф-я убывает быстрее, чем $\frac{1}{x}$

$$\int\limits_{-\infty}^{\infty}\frac{x\,dx}{(x^2+4x+13)^2}=\int\limits_{C}\frac{z\,dz}{(z^2+4z+13)^2}=2\pi i\sum_{z:\,\mathrm{Im}\,z>0}\operatorname{res} f =$$

$$z^2+4z+13=0$$

$$z_{1,2}=-2\pm 3i \quad \to z=-2+3i \text{ полюс } 2^{\text{го}}\text{-порядка}$$

$$= 2\pi i\operatorname*{res}_{z=-2+3i} f = 2\pi i\lim_{z\to -2+3i}\frac{d}{dz}\frac{(z-z_1)^2 z}{(z-z_1)^2(z-z_2)^2}=2\pi i\left.\frac{z-z_2-2z}{(z-z_2)^3}\right|_{z=z_1}=$$

$$=-2\pi i\frac{z_1+z_2}{(z_1-z_2)^3}=\frac{-8\pi i}{(6i)^3}=\frac{8\pi}{6^3}=\frac{\pi}{27}$$

684. $\int\limits_{0}^{\infty}\frac{dx}{(x^2+1)^n}=\frac{1}{2}\int\limits_{-\infty}^{\infty}\frac{dx}{(x^2+1)^n}=\frac{1}{2}\int\limits_{C}\frac{dz}{(z^2+1)^n}=\frac{1}{2}\pi i\operatorname*{res}_{z=i} f$

$$\operatorname*{res}_{z=i} f=\frac{1}{(n-1)!}\lim_{z\to i}\frac{d^{n-1}}{dz^{n-1}}\frac{(z-i)^n}{(z-i)^n(z+i)^n}=$$

$$=\frac{1}{(n-1)!}\left.\left((z+i)^{-n}\right)^{(n-1)}\right|_{z=i}=\frac{1}{(n-1)!}(-n)(-(n+1))\dots(-(n+n-2))\cdot(z+i)$$

$= ✡$

$$((z+i)^{-n})'=-n(z+i)^{-(n+1)}\quad ((z+i)^{-n})''=(-n)(-(n+1))(z+i)^{-(n+2)}$$

$$✡=\frac{(-1)^{n-1}(2n-2)!}{(n-1)!\,(n-1)!}\frac{\pi i}{(2i)^{2n-1}}=\frac{(2n-2)!\,\pi}{[(n-1)!]^2\,2^{2n-1}}$$

№685. $\int\limits_{-\infty}^{\infty}\frac{dx}{(x^2+a^2)(x^2+b^2)}=\int\limits_{C}\frac{dz}{(z^2+a^2)(z^2+b^2)}=2\pi i\left[\operatorname*{res}_{z=ia} f+\operatorname*{res}_{z=ib} f\right]=$

$$=\cancel{2}\pi i\left[\left.\frac{1}{(z^2+b^2)\cancel{2}z}\right|_{z=ia}+\left.\frac{1}{(z^2+a^2)\cancel{2}z}\right|_{z=ib}\right]=$$

$$=\frac{2\pi i}{b^2-a^2}\left[\frac{1}{ia}-\frac{1}{ib}\right]=\frac{\pi}{ab(a+b)}$$

№686. $\int\limits_{0}^{\infty}\frac{x^2+1}{x^4+1}dx=\frac{1}{2}\int\limits_{-\infty}^{\infty}\frac{x^2+1}{x^4+1}dx=\frac{1}{2}\int\limits_{C}\frac{z^2+1}{z^4+1}dz=\frac{2\pi i}{2}\left[\operatorname*{res}_{z=z_1} f+\operatorname*{res}_{z=z_2} f\right]=$

$$=\pi i\left[\left.\frac{z^2+1}{4z^3}\right|_{z=e^{i\frac{\pi}{4}}}+\left.\frac{z^2+1}{4z^3}\right|_{z=e^{i\frac{3\pi}{4}}}\right]=$$

$$z^4+1=0\quad\to\quad z_1=e^{i\frac{\pi}{4}};\quad z_2=e^{i\frac{3\pi}{4}}$$

$$= \frac{\pi i}{4}\left[(i+1)e^{-\frac{3i\pi}{4}} + (1-i)e^{-\frac{i3\pi}{4}}\right] = \frac{\pi}{4} i\left[(1+i)\left(\cos\frac{3\pi}{4} - i\sin\frac{3\pi}{4}\right) + (1-i)\left[\cos\frac{\pi}{4} - i\sin\frac{\pi}{4}\right]\right] = \frac{\pi}{4} i\, \frac{\sqrt{2}}{2}\left[(1+i)(-1-i)+(1-i)^2\right] =$$

$$= \frac{\pi\sqrt{2}}{8}\cdot i(-4i) = \frac{\pi\sqrt{2}}{2}$$

Лемма Жордана.

№691

1) $$\int_{-\infty}^{\infty} \frac{x\cos x\, dx}{x^2-2x+10} = \mathrm{Re}\int_{-\infty}^{\infty} \frac{x e^{ix}\, dx}{x^2-2x+10} = \mathrm{Re}\int_C \frac{z\cdot e^{iz}\, dz}{z^2-2z+10} =$$

$$z^2-2z+10=0 \to z=-1\pm 3i \to z_1 = 3i-1$$

$$= \mathrm{Re}\; 2\pi i \operatorname*{res}_{z=-1+3i} f e^{iz} = \mathrm{Re}\cdot 2\pi i \left.\frac{z e^{iz}}{2z-2}\right|_{z=-1+3i} = \mathrm{Re}\; \pi i \frac{(-1+3i)\, e^{i(-1+3i)}}{\cancel{-1+3i-1}} =$$

$$= \frac{\pi}{3} e^{-3}\, \mathrm{Re}\big((-1+3i)(\cos 1 + i\sin 1)\big) = \frac{\pi}{3} e^{-3}(\cos 1 - 3\sin 1)$$

2) $$\int_{-\infty}^{\infty} \frac{x\sin x\, dx}{x^2-2x+10} = \mathrm{Im}\int_{-\infty}^{\infty} \frac{x e^{ix}\, dx}{x^2-2x+10} = \frac{\pi}{3} e^{-3}\, \mathrm{Im}(1+3i)(\cos 1 + i\sin 1) =$$

$$= \frac{\pi}{3} e^{-3}(3\cos 1 + \sin 1)$$

№693

$$\int_0^{\infty} \frac{\cos ax}{x^2+b^2}\, dx = \frac{1}{2}\int_{-\infty}^{\infty} \frac{\cos ax}{x^2+b^2}\, dx = \frac{1}{2}\mathrm{Re}\int_{-\infty}^{\infty} \frac{e^{iax}}{x^2+b^2}\, dx =$$

$$= \frac{1}{2}\mathrm{Re}\int_C \frac{e^{iaz}}{z^2+b^2}\, dz = \left\{ \underset{\text{пол. 1 пор.}}{z = ib} \right\} = \frac{2\pi i}{2}\mathrm{Re} \left.\frac{e^{iaz}}{2z}\right|_{z=ib} =$$

$$= \frac{\pi}{2}\mathrm{Re}\; i\, \frac{e^{ia\cdot ib}}{ib} = \frac{\pi e^{-ab}}{2b}$$

Домашняя работа.

4.132 (674) $\displaystyle\int_0^{2\pi}\frac{d\varphi}{(a+b\cos\varphi)^2} = I \quad (a>b>0)$

$$\cos\varphi = \frac{z^2+1}{2z}\ ; \quad z=e^{i\varphi}$$

$$\Rightarrow dz = ie^{i\varphi}\,d\varphi \ \to \ d\varphi = \frac{dz}{ie^{i\varphi}}$$

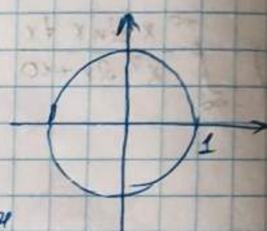
$$I = \oint\limits_{|z|=1} \frac{dz}{ie^{i\varphi}\left(a+b\cdot\frac{z^2+1}{2z}\right)^2} = \oint\limits_{|z|=1}\frac{dz\cdot 4z^{\cancel{2}}}{i\cancel{z}\,(2az+b(z^2+1))^2} =$$

$$= \frac{4}{i}\oint\limits_{|z|=1}\frac{z\,dz}{(2az+b(z^2+1))^2} = \frac{4}{i}\cdot 2\pi i\sum_k Res[f,z_k] = 8\pi\cdot\sum_k Res[f,z_k]$$

$$2az+bz^2+b=0 \quad /:b$$

$$z^2+\frac{2a}{b}z+1=0 \ \to\ D = \frac{4a^2}{b^2}-4 = 4\left(\frac{a^2}{b^2}-1\right) = 4\cdot\frac{a^2-b^2}{b^2}$$

$$z_1 = \frac{-\frac{2a}{b}\pm 2\sqrt{\frac{a^2}{b^2}-1}}{2} = -\frac{a}{b} \pm\frac{\sqrt{a^2-b^2}}{b}$$



$a>b>0$

$$\Rightarrow z_0 = \frac{\sqrt{a^2-b^2}-a}{b}$$ — полюс 2го порядка

$$Res[f,z_0] = \lim_{z\to z_0}\frac{d}{dz}\left(\frac{z}{\cancel{(z-z_0)^2}\left(z+\frac{a+\sqrt{a^2-b^2}}{b}\right)^2}\cdot\cancel{(z-z_0)^2}\right) =$$

$$= \lim_{z\to z_0}\frac{\left(z+\frac{a+\sqrt{a^2-b^2}}{b}\right)^2 - 2\left(z+\frac{a+\sqrt{a^2-b^2}}{b}\right)z}{\left(z+\frac{a+\sqrt{a^2-b^2}}{b}\right)^4} =$$

$$= \lim_{z\to z_0}\frac{\left(z+\frac{a+\sqrt{a^2-b^2}}{b}\right)-2z}{\left(z+\frac{a+\sqrt{a^2-b^2}}{b}\right)^3} = \frac{\left(\frac{\sqrt{a^2-b^2}-\cancel{a}+\cancel{a}+\sqrt{a^2-b^2}}{b}\right) - \frac{2\sqrt{a^2-b^2}-2a}{b}}{\left(\frac{\sqrt{a^2-b^2}-\cancel{a}+\cancel{a}+\sqrt{a^2-b^2}}{b}\right)^3}$$

$$= \frac{\frac{2\sqrt{a^2-b^2}-2\sqrt{a^2-b^2}+2a}{b}}{\left(\frac{2\sqrt{a^2-b^2}}{b}\right)^3} = \frac{2a\cdot b^3}{b\cdot 8(a^2-b^2)^{3/2}} = \frac{ab^2}{4(a^2-b^2)^{\frac{3}{2}}}$$

$$\Rightarrow I = \frac{8\pi\cdot ab^2}{4(a^2-b^2)^{3/2}} = 2\pi\frac{ab^2}{(a^2-b^2)^{3/2}}$$

№ 135 (677) $I=\int_0^{2\pi}\frac{\cos^2 3\varphi\, d\varphi}{1-2a\cos\varphi+a^2}$ (a – комплексное число, $a\neq\pm 1$)

$d\varphi=\frac{dz}{ie^{i\varphi}}=\frac{dz}{iz}$; $\cos\varphi=\frac{z^2+1}{2z}$

$\cos^2 3\varphi=\frac{1+\cos 6\varphi}{2}=\frac{1}{2}+\frac{1}{2}\cos 6\varphi$

$\cos 6\varphi=\frac{e^{i6\varphi}+e^{-i6\varphi}}{2}=\frac{z^6+z^{-6}}{2}=\frac{z^6+\frac{1}{z^6}}{2}=$

$=\frac{z^{12}+1}{2z^6}$

$\Rightarrow \cos^2 3\varphi=\frac{1}{2}+\frac{z^{12}+1}{4z^6}=\frac{z^{12}+2z^6+1}{4z^6}$

$I=\int\limits_{|z|=1}\frac{(z^{12}+2z^6+1)\,dz}{4iz^7\left(1-\frac{2a(z^2+1)}{2z}+a^2\right)}=\int\limits_{|z|=1}\frac{(z^{12}+2z^6+1)\,dz}{4iz^6(z-a(z^2+1)+za^2)}$

$z=0$ ~ полюс 6-го порядка.

$z-a(z^2+1)+za^2=0$

$z-az^2-a+za^2=0$

$az^2-z-za^2+a=0$

$az^2-z(1+a^2)+a=0$

$D=(1+a^2)^2-4a^2=1+2a^2+a^4-4a^2=1-2a^2+a^4=(1-a^2)^2$

$\sqrt{D}=1-a^2$

$z_{1,2}=\frac{+(1+a^2)\pm(1-a^2)}{2a}$

$z_1=\frac{1+\cancel{a^2}+1-\cancel{a^2}}{2a}=\frac{1}{a}$; $z_2=\frac{\cancel{1}+a^2-\cancel{1}+a^2}{2a}=a$

$|a|>1$: $z=\frac{1}{a}$ ~ полюс 1-го порядка

$\operatorname{res}_{z=\frac{1}{a}} f=\lim_{z\to\frac{1}{a}}\frac{z^{12}+2z^6+1}{z^6}=\lim_{z\to\frac{1}{a}}\left(z^6+2+\frac{1}{z^6}\right)=\frac{1}{a^6}+2+a^6=$

$=\frac{\frac{1}{a^{12}}+\frac{2}{a^6}+1}{\frac{1}{a^6}}=\frac{a^{12}+2a^6+1}{a^6}$

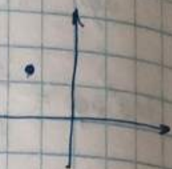
(?)

$\operatorname{res}_{z=0} f=\frac{1}{5!}\lim_{z\to 0}\frac{d^5}{dz^5}\left(\frac{z^{12}+2z^6+1}{z-a(z^2+1)+za^2}\right)$

Лемма Жордана.

№4.150 (692)

$$I=\int_{-\infty}^{\infty}\frac{x\sin x\,dx}{x^2+4x+20}=\operatorname{Im}\int_{-\infty}^{\infty}\frac{z\cdot e^{iz}\,dz}{z^2+4z+20}$$



$$z^2+4z+20=0$$

$$D=16-4\cdot 20=16-80=-64;\quad \sqrt{D}=8i$$

$$z_1=\frac{-4+8i}{2}=-2+4i;\quad z_2=\frac{-4-8i}{2}=-2-4i$$

$$I=\operatorname{Im}\cdot 2\pi i\operatorname*{res}_{z=-2+4i} f\cdot e^{iz}=\operatorname{Im}\cdot 2\pi i\cdot\lim_{z\to-2+4i}\frac{(z+2-4i)\cdot z\cdot e^{iz}}{(z+2-4i)(z+2+4i)}=$$

$$=\operatorname{Im}\,2\pi i\cdot\frac{(-2+4i)\cdot e^{i(-2+4i)}}{(\cancel{-2}+4i\cancel{+2}+4i)}=\operatorname{Im}\,2\pi\cancel{i}\cdot\frac{1}{8\cancel{i}}\cdot(-2+4i)\,e^{-2i-4}=$$

$$=\operatorname{Im}\frac{\pi}{4e^4}(-2+4i)\,(\cos(-2)+i\sin(-2))=$$

$$=\operatorname{Im}\frac{\pi}{4e^4}(-2+4i)\,(\cos 2-i\sin 2)=\operatorname{Im}\left[\frac{\pi}{4e^4}(-2\cos 2+2i\sin 2+4i\cos 2\ldots\right.$$

$$=\frac{\pi}{4e^4}(2\sin 2+4\cos 2)=\frac{\pi}{2e^4}(2\cos 2+\sin 2)$$

№4.152 (694)

$$I=\int_0^{\infty}\frac{x\sin ax}{x^2+b^2}\,dx\qquad (a \text{ и } b \text{ — положительные числа})$$

$$I=\frac12\operatorname{Im}\int_{-\infty}^{\infty}\frac{e^{iax}}{x^2+b^2}=\frac12\operatorname{Im}\oint_C\frac{z e^{iaz}}{z^2+b^2}=\left\{z=ib\ \text{полюс 1-го порядка}\right\}=$$

$$=\operatorname{Im}\frac12\cdot 2\pi i\,\frac{ze^{iaz}}{2z}\Big|_{z=ib}=\operatorname{Im}\,\pi i\,\frac{e^{-ab}}{2ib}\cdot ib=\frac{\pi}{2}e^{-ab}$$

№4.138 (680)

$$I=\int_0^{2\pi}\operatorname{ctg}(x+a)\,dx\qquad (a \text{ — комплексное число и } \operatorname{Im} a\neq 0)$$

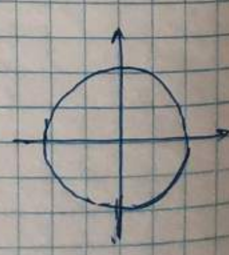
$$\operatorname{ctg}(x+a)=\frac{\cos(x+a)}{\sin(x+a)}=\frac{(e^{i(x+a)}+e^{-i(x+a)})\,i}{(e^{i(x+a)}-e^{-i(x+a)})}=\frac{i(e^{2i(x+a)}+1)}{(e^{2i(x+a)}-1)}$$

$$z=e^{i(x+a)}\ \to\ C:\ |z|=\left\{e^{ia},\ \text{т.к. } a \text{ — комплексное}\right\}=\operatorname{Re}e^{ia}$$

$$dz=i e^{i(x+a)}\,dx\ \to\ dx=\frac{dz}{iz}$$

$$I=\oint_C\frac{i(z^2+1)\,dz}{i(z^2-1)\cdot z}=\oint_C\frac{(z^2+1)\,dz}{(z^2-1)z}=\oint_C\frac{(z^2+1)\,dz}{(z-1)(z+1)z}$$

$z=0,\ z=\pm1$ ~ полюсы 1-го порядка



$|z|=\operatorname{Re}e^{ia}$, $\forall a>1$, кроме $a=0$, но по условию $\operatorname{Im}a\neq0$

1) Если $\operatorname{Im} a>0$: $z_0=0$

$$\operatorname{Res}[f,z_0]=\lim_{z\to0}\frac{z^2+1}{(z^2-1)}=-1$$

2) Если $\operatorname{Im} a < 0$: $z=0$, $z=\pm 1$ — особые точки, вход. в область C

$$\operatorname{Res}[f, z=0] = -1$$

$$\operatorname{Res}[f, z=1] = \lim_{z\to 1} \frac{z^2+1}{z(z+1)} = \frac{2}{1\cdot 2} = 1$$

$$\operatorname{Res}[f, z=-1] = \lim_{z\to -1} \frac{z^2+1}{z(z-1)} = \frac{2}{1\cdot 2} = 1$$

Тогда: $\operatorname{Im} a < 0$

$$I = 2\pi i\cdot(-1+1+1) = 2\pi i$$

$\operatorname{Im} a > 0$

$$I = 2\pi i\,(-1) = -2\pi i$$

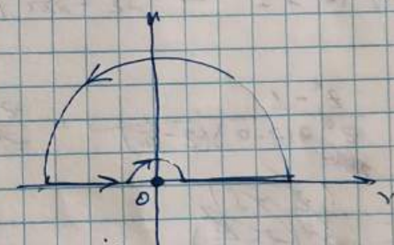
$$\Rightarrow I = -2\pi i\cdot \operatorname{sgn}(\operatorname{Im} a)$$

23.12.21.

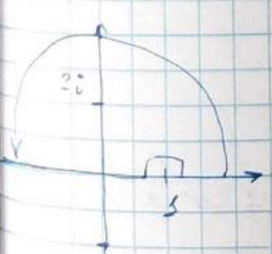
Продолжение леммы Жордана.

№698. $$\int_{-\infty}^{\infty} \frac{e^{itx}}{x}\,dx = \int_C \frac{e^{itz}}{z}\,dz = \pi i \operatorname*{res}_{z=0} f(z) =$$

$$= \pi i\cdot e^0 = \pi i$$



№698. $$\int_{-\infty}^{\infty} \frac{\sin x\,dx}{(x^2+4)(x-1)} = \operatorname{Im}\int_{-\infty}^{\infty}\frac{e^{ix}\,dx}{(x^2+4)(x-1)} = \operatorname{Im}\int_C \frac{e^{iz}\,dz}{(z^2+4)(z-1)} =$$



$$= \operatorname{Im}\left[2\pi i \operatorname*{res}_{z=2i} f(z) + \pi i \operatorname*{res}_{z=1} f(z)\right] =$$

$$= \operatorname{Im}\left[2\pi i\frac{e^{i2i}}{2\cdot 2i(2i-1)} + \pi i\frac{e^{i1}}{1+4}\right] = \frac{\pi}{2}e^{-2}\operatorname{Im}\left(\frac{-1-2i}{1+2^2}\right) +$$

$$+ \operatorname{Im}\frac{\pi}{5}i(\cos 1 + i\sin 1) = \frac{\pi}{2}e^{-2}\cdot\left(-\frac{2}{5}\right) + \frac{\pi}{5}\cos 1$$

№700. $$\int_{-\infty}^{\infty}\frac{\cos tx}{1-x^4}\,dx = \operatorname{Re}\int_{-\infty}^{\infty}\frac{e^{itx}}{1-x^4}\,dx = \operatorname{Re}\int_C\frac{e^{itz}}{1-z^4}\,dz = \operatorname{Re}\left(2\pi i\operatorname*{res}_{z=i} f + \pi i\left[\operatorname*{res}_{z=1} f + \operatorname*{res}_{z=-1} f\right]\right) =$$

$$= \operatorname{Re}\left[2\pi i\frac{e^{it\cdot i}}{-4i^3} + \pi i\left[\frac{e^{it}}{-4} + \frac{e^{-it}}{4}\right]\right] =$$

$$= \frac{\pi}{2}e^{-t} + \frac{\pi}{2}\sin t$$

$z^4 = 1$

$z = \pm 1$; $z = \pm i$

на рёбрах, осях.

$$\frac{1}{2i}\quad \frac{e^{it}}{2i} - \frac{e^{-it}}{2i}$$

677. $\int_0^{2\pi} \frac{\cos^2 3\varphi \, d\varphi}{1-2a\cos\varphi+a^2} = \left\{ \begin{array}{l} z = e^{i\varphi} \\ \cos\varphi = \frac{z^2+1}{2z} \end{array} \right\} =$

$\cos^2 3\varphi = \frac{1+\cos 6\varphi}{2}$

$= \underbrace{\frac{1}{2}\int_0^{2\pi} \frac{d\varphi}{1-2a\cos\varphi+a^2}}_{\text{№ 675}} + \frac{1}{2}\int_0^{2\pi} \frac{\cos 6\varphi \, d\varphi}{1-2a\cos\varphi+a^2}$

$\frac{1}{2}\int_0^{2\pi} \frac{\cos 6\varphi \, d\varphi}{1-2a\cos\varphi+a^2}$

$\cos 6\varphi = \frac{e^{i6\varphi}+e^{-i6\varphi}}{2}$

$= I_1 + \frac{1}{4}\int_{|z|=1} \frac{z^{12}-1}{z^6\left(1-a\frac{z^2+1}{z}+a^2\right)} \frac{dz}{i} =$

$z_1 = a;\ z_2 = \frac{1}{a};\ z = 0$ ~

$z=0$ ~ полюс $6^{го}$ порядка

$= I_1 + \frac{1}{4i}\int_C \frac{z^{12}-1}{z^6(az^2+(a^2+1)z+a)}$

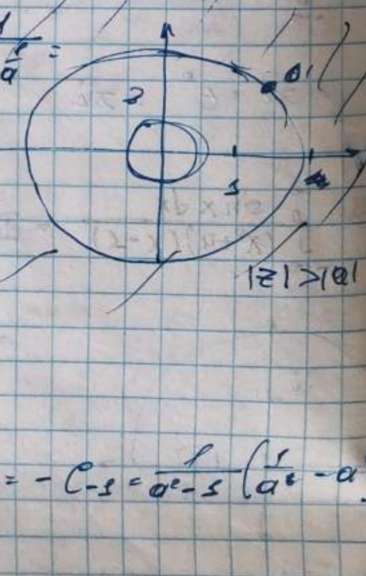
2) $|a|>1 \rightarrow I_2 = 2\pi i\left[\operatorname{res}_{z=\frac{1}{a}} f + \operatorname{res}_{z=0} f(z)\right] = -2\pi i\left[\operatorname{res}_{z=a} f + \operatorname{res}_{z=\infty} f\right]$

$\frac{z^{12}-1}{z^6 a(z-a)(z-\frac{1}{a})} = \frac{z^{12}-1}{z^6 a}\left(\frac{A_1}{z-a} - \frac{A}{z-\frac{1}{a}}\right)\frac{1}{a-\frac{1}{a}} =$

$= \cancel{\frac{(z^{12}-1)}{z^7\left(1-\frac{a}{z}\right)}}$

$= \frac{z^{12}-1}{z^6 a(a-\frac{1}{a})} \cdot \frac{1}{z}\left(\frac{1}{1-\frac{a}{z}} - \frac{1}{1-\frac{1}{az}}\right) =$

$= \frac{z^{12}-1}{(a^2-1)z^7}\left(\sum \frac{a^n}{z^n} - \sum \frac{1}{a^n z^n}\right)$



$\Rightarrow C_{-1} = \frac{1}{a^2-1}\left(a^6 - \frac{1}{a^6}\right) \rightarrow \operatorname{res}_{z=\infty} f = -C_{-1} = \frac{1}{a^2-1}\left(\frac{1}{a^6} - a^6\right)$

... и т.д. сложно.

Интегрирование ф-й комплексного переменного.

<u>Ф-ла Грина</u>: если ф-ии $P(x,y)$ и $Q(x,y)$ непрерывны в замкнутой области $\overline{G}$, ограниченной кусочно-гладким контуром C а их частные производные первого порядка непрерывны в G, то

$$\oint_C P\,dx + Q\,dy = \iint_G \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

<u>Интегральная теорема Коши:</u>

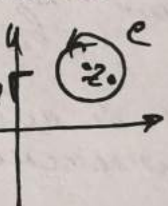
1) Пусть в односвязной области G задана однозначная аналитическая ф-я $f(z)$. Тогда интеграл от этой ф-ии $f(z)$ по любому замкнутому контуру Γ, целиком лежащему в области G, равен нулю.

$$\oint_C f(z)\,dz = 0$$

2) $\int_{z_1}^{z_2} f(z)\,dz$, когда ф-я аналит; непрерывна, то не зависит от формы кривой.

3) Если ф-я аналитична и непрерывна на контуре или внутри него:

$$\frac{1}{2\pi i}\int_C \frac{f(z)}{z-z_0}\,dz = f(z_0) \rightarrow \begin{cases} \text{если внутри} \\ 0, \text{вне} \end{cases}$$



...

$$\frac{n!}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^{n+1}} = f^{(n)}(z_0)$$

Ряды

$$R = \lim_{n\to\infty}\left|\frac{C_n}{C_{n+1}}\right|$$ ~ радиус сходимости.

$$\frac{1}{R} = \overline{\lim_{n\to\infty}}\sqrt[n]{|C_n|}$$

Необходимое условие сх-ти: $\sum_{k=0}^{\infty} W_k \rightarrow \lim_{k\to\infty} W_k = 0$

Критерий Коши и Даламбера для абс. сх-ти:

$$\lim_{k\to\infty}\left|\frac{W_{k+1}}{W_k}\right| = q \quad \text{или} \quad \lim_{k\to\infty}\sqrt[k]{|W_k|} = q \qquad \begin{cases} q<1 & \text{сх-ся} \\ q>1 & \text{расх-ся} \\ q=1 & ? \end{cases}$$

Сх-ся абсолютно, если сх-ся ряд из модулей,
условно, если просто сх-ся.

Признак Абеля: Функц. ряд вида: $\sum\limits_{n=1}^{\infty} a_n \cdot b_n$.

1) $\{a_n\}$ - монотонна и ограничена

2) $\{b_n\}$ - равномерно сх-ся.

$\Rightarrow \sum\limits_{n=1}^{\infty} a_n b_n$ сх-ся.

Признак Лейбница (для знакоперемен.)

$\sum\limits_{n=1}^{\infty}(-1)^{n+1} b_n$ - знакочередующийся ряд. ($b_n > 0$)

Если 1) b_n монотонно убывает, т.е.

$\forall n > n_0,\ b_n > b_{n+1}$.

2) $\lim\limits_{n\to\infty} b_n = 0$, тогда $\sum\limits_{n=1}^{\infty}(-1)^{n+1} b_n$ сходится.

Признак Дирихле: $\sum\limits_{n=1}^{\infty} a_n b_n$.

1) Если пос-ть $\{b_n\}$ не возрастающая и беск. малая, $b_n \geq b_{n+1}$, $\lim\limits_{n\to\infty} b_n = 0$

2) Пос-ть частичных сумм для ряда $\sum\limits_{n=1}^{\infty} a_n$ ограничена, т.е. $|S_n| = |a_1 + a_2 + \ldots| \leq M$

$\Rightarrow \sum\limits_{n=1}^{\infty} a_n b_n$ - сх-ся.

Ряды Тейлора

Если $f(z)$ - аналит.

$\Rightarrow f(z) = \sum\limits_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n$ ~ разложение в ряд Тейлора

Ряды Фурье

1) $f(x),\ x\in(a,b)\to(-l;l)$

$$f(x)\sim\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos\frac{2\pi nx}{b-a}+b_n\sin\frac{2\pi nx}{b-a}$$

$t=x-\frac{a+b}{2}$ ~ смещение к симметричному интервалу.

$$\Rightarrow f(x)\sim\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos\frac{\pi nx}{l}+b_n\sin\frac{\pi nx}{l},$$ где

$$a_n=\frac{1}{l}\int_{-l}^{l}f(x)\cos\frac{\pi nx}{l}dx\ ;\quad b_n=\frac{1}{l}\int_{-l}^{l}f(x)\sin\frac{\pi nx}{l}dx$$

$$a_0=\frac{1}{l}\int_{-l}^{l}f(x)dx$$

Если $f(x)$ ~ нечётная $a_n=0$ нечёт • нечёт = чётн.

$f(x)$ ~ чётная $b_n=0$

2) Равенство Ляпунова:

$$\frac{a_0^2}{2}+\sum_{n=1}^{\infty}(a_n^2+b_n^2)=\frac{1}{l}\int_{-l}^{l}f^2(x)dx$$

Элементарные трансцендентные функции.

1) $e^z=e^x(\cos y+i\sin y)$

$$\cos z=\frac{e^{iz}+e^{-iz}}{2};\quad \sin z=\frac{e^{iz}-e^{-iz}}{2i};\quad tg z=\frac{\sin z}{\cos z},\quad ctg z=\frac{\cos z}{\sin z}$$

2) $$ch z=\frac{e^z+e^{-z}}{2};\quad sh z=\frac{e^z-e^{-z}}{2};\quad th z=\frac{sh z}{ch z},\quad cth z=\frac{ch z}{sh z}.$$

$\sin(iz)=i\,sh z$ | $\cos(iz)=ch(z)$ | $tg(iz)=i\,th z$

$ctg(iz)=-i\,cth z$

3) $\sin z=\sin x\,ch y+i\cos x\,sh y$

$\cos z=\cos x\,ch y-i\sin x\,sh y$

Комплексный логарифм.

$Ln z=\ln r+i(\varphi+2\pi k)$ $\to$ $Ln(1)=2k\pi i$

$Ln(-1)=(2k+1)i\pi$

$Ln(i)=i\frac{4k+1}{2}\pi$

$a^{\alpha}=exp\{\alpha Ln a\}$ ~ для любых (a) - комплек. чисел

$a^{\alpha}=e^{\alpha Ln a}$

1) $\operatorname{Arccos} z = -i\, \operatorname{Ln}(z+\sqrt{z^2-1})$

$\operatorname{Arcsin} z = -i\, \operatorname{Ln} i(z+\sqrt{z^2-1})$

$\operatorname{Arctg} z = \frac{i}{2} \operatorname{Ln} \frac{i+z}{i-z} = \frac{1}{2i} \operatorname{Ln} \frac{1+iz}{1-iz}$

$\operatorname{Arcctg} z = \frac{i}{2} \operatorname{Ln} \frac{z-i}{z+i}$ | $\operatorname{Ar\,ch} z = \operatorname{Ln}(z+\sqrt{z^2-1})$

$\operatorname{Ar\,sh} z = \operatorname{Ln}(z+\sqrt{z^2+1})$ | $\operatorname{Arth} z = \frac{1}{2} \operatorname{Ln} \frac{1+z}{1-z}$

$\operatorname{Arcth} z = \frac{1}{2} \operatorname{Ln} \frac{z+1}{z-1}$

$\bar{z} \cdot z = |z|^2$

$z = x+iy \rightarrow \operatorname{Re} z = x,\ \operatorname{Im} z = y$

$\bar{z} = x-iy \rightarrow \operatorname{Arg} z = \varphi \rightarrow -\pi < \arg z \leq \pi$

$\operatorname{Arcsin} z + \operatorname{Arccos} z = \frac{\pi}{2}$ | $\operatorname{Arctg} z + \operatorname{Arcctg} z = \frac{\pi}{2}$

$$\varphi = \operatorname{Arg} z = \operatorname{arctg}\left(\frac{\operatorname{Im} z}{\operatorname{Re} z}\right)$$

Условие Коши - Римана.

1) $w = f(z) = u(x,y) + i\,v(x,y)$ ~ ф-я $w=f(z)$ аналит., $\Leftrightarrow$

$$\boxed{\begin{aligned} u'_x &= v'_y \\ v'_x &= -u'_y \end{aligned}}$$

2) В полярных координатах: $z = r \cdot e^{i\varphi}$

$f(z) = u(r,\varphi) + i\,v(r,\varphi)$

$$\Rightarrow \boxed{\begin{aligned} u'_r &= \frac{1}{r} v'_\varphi \\ v'_r &= -\frac{1}{r} u'_\varphi \end{aligned}}$$

Геометрический смысл модуля и аргумента производной

$w = f(z)$

$|w'(z_0)| = \alpha$ - коэф-т растяжения пространства в т. z_0

$\arg w'(z_0) = \varphi$ - поворот пространства в т. z_0

Если $\alpha < 1$ - сжимается, $\alpha > 1$ - растяжение.