02.02.2021. Вычисление частных производных ф-ии многих переменных.

3213 $u = x^4 + y^4 - 4x^2y^2$

$$\frac{\partial u}{\partial x} = 4x^3 + \underbrace{(y^4)'}_{=0} - 8y^2 \cdot x = 4x^3 - 8y^2 x$$

$$\frac{\partial u}{\partial y} = 4y^3 - 8x^2 y$$

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial y}\left(4x^3 - 8xy^2\right) = -16x \cdot y$$

вторая произв. по y

первая произв. по x

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial x}\left(4y^3 - 8x^2y\right) = -16xy$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(4x^3 - 8y^2x\right) = 12x^2 - 8y^2$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(4y^3 - 8x^2y\right) = 12y^2 - 16x^2$$

3214 $u = xy + \frac{x}{y}$

$$\frac{\partial u}{\partial x} = y + \frac{1}{y} \quad ; \quad \frac{\partial u}{\partial y} = x - \frac{x}{y^2}$$

$$\frac{\partial^2 u}{\partial x^2} = 0 \quad ; \quad \frac{\partial^2 u}{\partial y^2} = 2\frac{x}{y^3}$$

$$\frac{\partial^2 u}{\partial x \partial y} = 1 - \frac{1}{y^2}$$

$$\frac{\partial^2 u}{\partial y \partial x} = 1 - \frac{1}{y^2}$$

3220 $u = x^y$

$$\frac{\partial u}{\partial x} = y \cdot x^{y-1}$$

$$\frac{\partial u}{\partial y} = \cancel{e^{\ln x^y}} = \cancel{e^{y\ln x}} \quad x^y \cdot \ln x$$

$$\frac{\partial^2 u}{\partial x^2} = y(y-1)\,x^{y-2}$$

$$\frac{\partial^2 u}{\partial y^2} = x^y \ln^2 x$$

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}(x^y \ln x) = y x^{y-1} \ln x + x^{y-1} = x^{y-1}(y \ln x + 1)$$

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial}{\partial y}\left(y \cdot x^{y-1}\right) = x^{y-1} + x^{y-1} \ln x \cdot y = x^{y-1}(y \ln x + 1)$$

3223.

$$u = \operatorname{arctg} \frac{x+y}{1-xy}$$

$$\frac{\partial u}{\partial x} = \frac{1}{1+\left(\frac{x+y}{1-xy}\right)^2} \cdot \frac{(1-xy)-(x+y)\cdot(-y)}{(1-xy)^2} =$$

$$= \frac{1}{\frac{(1-xy)^2+(x+y)^2}{(1-xy)^2}} \cdot \frac{1-xy+xy+y^2}{(1-xy)^2} =$$

$$= \frac{1+y^2}{(1-xy)^2+(x+y)^2} = \frac{1+y^2}{1+x^2+x^2y^2+y^2} = \frac{1+y^2}{(1+x^2)(1+y^2)} = \frac{1}{1+x^2}$$

$$\frac{\partial u}{\partial y} = \frac{1}{1+y^2}$$ (x и y в задании симметричны

$$\frac{\partial^2 u}{\partial x^2} = -\frac{2x}{(1+x^2)^2} \quad ; \quad \frac{\partial^2 u}{\partial y^2} = -\frac{2y}{(1+y^2)^2}$$

$$\frac{\partial^2 u}{\partial x \partial y} = 0$$

3257. $\frac{\partial^3 u}{\partial x^2 \partial y}$, если $u = x \ln(xy)$

$$\frac{\partial u}{\partial x} = \ln(xy) + \frac{x \cdot y}{xy} = 1 + \ln(xy)$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{y}{xy} = \frac{1}{x}$$

$$\frac{\partial^3 u}{\partial x^2 \partial y} = \frac{\partial}{\partial y}\left(\frac{\partial^2 u}{\partial x^2}\right) = 0$$

(3260)

$$\frac{\partial^3 u}{\partial x \partial y \partial z}, \quad u = e^{xyz}$$

$$\frac{\partial u}{\partial x} = e^{xyz} \cdot yz = yz \cdot e^{xyz}$$

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = z\left(e^{xyz} + e^{xyz} \cdot xzy\right) =$$
$$= z \cdot e^{xyz}(1 + xyz)$$

$$\frac{\partial^3 u}{\partial x \partial y \partial z} = \frac{\partial}{\partial z}\left(\frac{\partial^2 u}{\partial x \partial y}\right) = e^{xyz}(1+xyz) + z(1+xyz)\cdot xy \cdot e^{xyz}$$
$$+ z \cdot e^{xyz} \cdot xy = e^{xyz}\left(1 + xyz + xyz + (xyz)^2 + xyz\right) =$$
$$= e^{xyz}\left(1 + (xyz)^2 + 3xyz\right)$$

(3225)

$$u = \frac{1}{\sqrt{x^2+y^2+z^2}}$$

$$\frac{\partial u}{\partial x} = -\frac{2x}{2\sqrt{x^2+y^2+z^2}} = \frac{-x}{(x^2+y^2+z^2)^{\frac{3}{2}}}$$

$$\frac{\partial u}{\partial y} = -\frac{y}{(x^2+y^2+z^2)^{\frac{3}{2}}}, \quad \frac{\partial u}{\partial z} = -\frac{z}{(x^2+y^2+z^2)^{\frac{3}{2}}}$$

$$\frac{\partial^2 u}{\partial x^2} = -\frac{(x^2+y^2+z^2)^{\frac{3}{2}} - x \cdot \frac{3}{2}\sqrt{x^2+y^2+z^2} \cdot 2x}{(x^2+y^2+z^2)^3} =$$
$$= -\frac{x^2+y^2+z^2-3x^2}{(x^2+y^2+z^2)^{\frac{5}{2}}} = \frac{2x^2-y^2-z^2}{(x^2+y^2+z^2)^{\frac{5}{2}}}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{2y^2-x^2-z^2}{(x^2+y^2+z^2)^{\frac{5}{2}}}$$

$$\frac{\partial^2 u}{\partial z^2} = \frac{2z^2-y^2-x^2}{(x^2+y^2+z^2)^{\frac{5}{2}}}$$

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial y}\left(-\frac{x}{(x^2+y^2+z^2)^{\frac{3}{2}}}\right) = -x \cdot \frac{\partial}{\partial y}\left(\frac{1}{(x^2+y^2+z^2)^{\frac{3}{2}}}\right) =$$
$$= x \cdot \frac{3}{2} \cdot \frac{2y}{(x^2+y^2+z^2)^{\frac{5}{2}}} = \frac{3xy}{(x^2+y^2+z^2)^{\frac{5}{2}}}$$

(3228.)

Д/З:

3228

$$u = x^{y^z}$$

$$\frac{\partial u}{\partial x} = y^z\, x^{y^z - 1} = \frac{y^z}{x}\cdot u$$

$$\frac{\partial u}{\partial y} = x^{y^z}\cdot \ln x \cdot z \cdot y^{z-1} = u \cdot \ln x \cdot z \cdot y^{z-1}$$

$$\frac{\partial u}{\partial z} = x^{y^z}\cdot \ln x \cdot y^z \cdot \ln y = u \cdot y^z \cdot \ln x \cdot \ln y.$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(u\cdot \frac{y^z}{x}\right) = \frac{\partial u}{\partial x}\cdot\frac{y^z}{x} \cancel{4} - u\cdot\frac{y^z}{x^2} =$$

$$= y^z\left(\frac{\frac{\partial u}{\partial x}\cdot x - u}{x^2}\right) = y^z\,\frac{\left(\cancel{x}\,\frac{y^z}{\cancel{x}}\cdot u - u\right)}{x^2} =$$

$$= \frac{y^z\cdot u\,(y^z - 1)}{x^2} = \frac{u\cdot y^z\,(y^z-1)}{x^2}$$

Д/з: ~~[illegible]~~, ~~[illegible]~~, ~~[illegible]~~, ~~[illegible]~~, ~~[illegible]~~, ~~[illegible]~~, ~~[illegible]~~, ~~[illegible]~~, ~~[illegible]~~

Домашняя работа

3215.

$$u = \frac{x}{y^2}$$

$$\frac{\partial u}{\partial x} = \frac{1}{y^2} \; ; \; \frac{\partial u}{\partial y} = -\frac{2x}{y^3}$$

$$\frac{\partial^2 u}{\partial x^2} = 0 \; ; \; \frac{\partial^2 u}{\partial y^2} = \frac{6x}{y^4} \; ; \; \frac{\partial^2 u}{\partial x \partial y} = -\frac{2}{y^3}$$

3217.

$$u = x \cdot \sin(x+y)$$

$$\frac{\partial u}{\partial x} = \sin(x+y) + x \cdot \cos(x+y) \; ; \; \frac{\partial u}{\partial y} = x \cos(x+y)$$

$$\frac{\partial^2 u}{\partial x^2} = \cos(x+y) + \cos(x+y) - x \cdot \sin(x+y) = 2\cos(x+y) - x \cdot \sin(x+y)$$

$$\frac{\partial^2 u}{\partial y^2} = -x \cdot \sin(x+y)$$

$$\frac{\partial^2 u}{\partial x \partial y} = \cos(x+y) - x \cdot \sin(x+y)$$

3218.

$$u = \frac{\cos x^2}{y}$$

$$\frac{\partial u}{\partial x} = \frac{-2x \cdot \sin x^2}{y} \; ; \; \frac{\partial u}{\partial y} = -\frac{\cos x^2}{y^2}$$

$$\frac{\partial^2 u}{\partial x^2} = -\frac{2}{y}\left(\sin x^2 + \cos x^2 \cdot 2x^2\right) = -\frac{2\sin x^2 + 4x^2 \cos x^2}{y}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{2\cos x^2}{y^3}$$

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{4x \sin x^2}{y^2}$$

3221.

$$u = \ln(x+y^2)$$

$$\frac{\partial u}{\partial x} = \frac{1}{x+y^2} \; ; \; \frac{\partial u}{\partial y} = \frac{2y}{x+y^2}$$

$$\frac{\partial^2 u}{\partial x^2} = -\frac{1}{(x+y^2)^2}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{2(x+y^2) - 4y^2}{(x+y^2)^2} = \frac{2x + 2y^2 - 4y^2}{(x+y^2)^2} = \frac{2x - 2y^2}{(x+y^2)^2} = \frac{2(x-y^2)}{(x+y^2)^2}$$

$$\frac{\partial^2 u}{\partial x \partial y} = -\frac{2y}{(x+y^2)^2}$$

3222.

$u = \operatorname{arctg} \frac{y}{x}$

$$\frac{\partial u}{\partial x} = \frac{-y}{\left(1+\frac{y^2}{x^2}\right)x^2} = -\frac{y}{x^2+y^2}$$

$$\frac{\partial u}{\partial y} = \frac{1}{1+\frac{y^2}{x^2}} \cdot \frac{1}{x} = \frac{x}{x^2+y^2}$$

$$\frac{\partial^2 u}{\partial x^2} = + \frac{y \cdot 2x}{(x^2+y^2)^2} = \frac{+2xy}{(x^2+y^2)^2}$$

$$\frac{\partial^2 u}{\partial y^2} = -\frac{2xy}{(x^2+y^2)^2}$$

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{(x^2+y^2) - x \cdot 2x}{(x^2+y^2)^2} = \frac{y^2 - x^2}{(x^2+y^2)^2}$$

3224.

$u = \arcsin\left(\frac{x}{\sqrt{x^2+y^2}}\right)$

$$\frac{\partial u}{\partial x} = \frac{1}{\sqrt{1-\frac{x^2}{x^2+y^2}}} \cdot \frac{\sqrt{x^2+y^2} - x \cdot \frac{\cdot 2x}{2\sqrt{x^2+y^2}}}{x^2+y^2} =$$

$$= \frac{\sqrt{x^2+y^2}}{|y|} \cdot \frac{\sqrt{x^2+y^2} - \frac{x^2}{\sqrt{x^2+y^2}}}{x^2+y^2} = \frac{\sqrt{x^2+y^2}}{|y|} \cdot \frac{y^2}{(x^2+y^2)\sqrt{x^2+y^2}} =$$

$$= \frac{|y|}{x^2+y^2}$$

$$\frac{\partial u}{\partial y} = \frac{\sqrt{x^2+y^2}}{|y|} \cdot \frac{x}{1} \cdot \left(-\frac{1}{2}\right) \frac{\cdot 2y}{(x^2+y^2)^{\frac{3}{2}}} = -\frac{x \operatorname{sgn} y}{x^2+y^2}$$

$$\frac{\partial^2 u}{\partial x^2} = -\frac{2|y|x}{(x^2+y^2)^2}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(-\frac{xy}{|y|(x^2+y^2)}\right) = -\frac{x|y|(x^2+y^2) - xy\left(\frac{|y|}{y}(x^2+y^2) + 2y|y|\right)}{y^2(x^2+y^2)^2} =$$

$$= \frac{2x|y|}{(x^2+y^2)^2}$$

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\frac{y}{|y|}(x^2+y^2) - 2|y| \cdot y}{(x^2+y^2)^2} = \frac{\frac{yx^2}{|y|} + \frac{|y| \cdot |y|}{|y|} - 2|y| \cdot y}{(x^2+y^2)^2} =$$

$$= \frac{\operatorname{sgn} y \cdot x^2 + |y| - 2|y|y}{(x^2+y^2)^2} =$$

$= \ldots$

$$(|x|)' = \frac{x}{|x|} = \operatorname{sgn}(x)$$

3226.

$$u=\left(\frac{x}{y}\right)^{z}$$

$$\frac{\partial u}{\partial x}=\frac{1}{y^{z}}\cdot z\,x^{z-1}=\frac{z}{y^{z}}\,x^{z-1}=\frac{z}{x}\cdot\left(\frac{x}{y}\right)^{z}=u\cdot\frac{z}{x}$$

$$\frac{\partial u}{\partial y}=x^{z}\cdot y^{-z}=-z\cdot x^{z}\cdot y^{-z-1}=-\frac{z}{y}\cdot\left(\frac{x}{y}\right)^{z}=-u\cdot\frac{z}{y}$$

$$\frac{\partial u}{\partial z}=\left(\frac{x}{y}\right)^{z}\cdot\ln\left(\frac{x}{y}\right)=u\cdot\ln\left(\frac{x}{y}\right)$$

$$\frac{\partial^2 u}{\partial x^2}=\frac{z}{x}\cdot\frac{\partial u}{\partial x}+u\cdot\left(-\frac{z}{x^2}\right)=\frac{z^2}{x^2}\cdot u-u\left(\frac{z}{x^2}\right)=$$

$$=u\left(\frac{z^2-z}{x^2}\right)=uz\cdot\left(\frac{z-1}{x^2}\right)$$

$$\frac{\partial^2 u}{\partial y^2}=-\left(\frac{\partial u}{\partial y}\cdot\frac{z}{y}-\frac{z}{y^2}\cdot u\right)=-\left(-u\frac{z^2}{y^2}-u\frac{z}{y^2}\right)=$$

$$=uz\left(\frac{z-1}{y^2}\right)$$

$$\frac{\partial^2 u}{\partial z^2}=\left(\frac{x}{y}\right)^{z}\cdot\ln^2\left(\frac{x}{y}\right)=u\cdot\ln^2\left(\frac{x}{y}\right)$$

$$\frac{\partial^2 u}{\partial x\partial y}=\frac{\partial}{\partial y}\left(\frac{z}{x}\cdot\left(\frac{x}{y}\right)^{z}\right)=\frac{z}{x}\cdot\left(-u\cdot\frac{z}{y}\right)=-\frac{z^2}{xy}\cdot u$$

$$\frac{\partial^2 u}{\partial x\partial z}=\frac{\partial}{\partial z}\left(u\cdot\frac{z}{x}\right)=\left(\frac{\partial}{\partial z}u\right)\cdot\frac{z}{x}+\frac{1}{x}\cdot u=$$

$$=u\cdot\ln\left(\frac{x}{y}\right)\cdot\frac{z}{x}+\frac{u}{x}=\frac{u}{x}\left(z\ln\left(\frac{x}{y}\right)+1\right)$$

$$\frac{\partial^2 u}{\partial y\partial z}=\frac{\partial}{\partial z}\left(-u\cdot\frac{z}{y}\right)=-\left(\frac{\partial u}{\partial z}\cdot\frac{z}{y}+\left(-\frac{z}{y^2}\right)\cdot u\right)=$$

$$=-\left(u\ln\left(\frac{x}{y}\right)\cdot\frac{z}{y}-\frac{zu}{y^2}\right)=$$

$$=\frac{uz}{y}\left(\frac{1}{y}-\ln\left(\frac{x}{y}\right)\right)$$

3256. Найти: $\frac{\partial^4 u}{\partial x^4}$; $\frac{\partial^4 u}{\partial x^3 \partial y}$; $\frac{\partial^4 u}{\partial x^2 \partial y^2}$

$$u = x - y + x^2 + 2xy + y^2 + x^3 - 3x^2 y - y^3 + x^4 - 4x^2 y^2 + y^4$$

$$\frac{\partial u}{\partial x} = 1 + 2x + 2y + 3x^2 - 6xy + 4x^3 - 8y^2 x$$

$$\frac{\partial^2 u}{\partial x^2} = 2 + 6x - 6y + 12x^2 - 8y^2$$

$$\frac{\partial^3 u}{\partial x^3} - 6 + 24x \quad ; \quad \frac{\partial^4 u}{\partial x^4} = 24$$

$$\frac{\partial^4 u}{\partial x^3 \partial y} = 0$$

$$\frac{\partial^4 u}{\partial x^2 \partial y} = -6 - 16y \rightarrow \frac{\partial^4 u}{\partial x^2 \partial y^2} = -16$$

3258. Найти: $\frac{\partial^6 u}{\partial x^3 \partial y^3}$

$$u = x^3 \sin y + y^3 \sin x$$

$$\frac{\partial u}{\partial x} = 3x^2 \sin y + y^3 \cos x$$

$$\frac{\partial^2 u}{\partial x^2} = 6x \sin y - y^3 \sin x$$

$$\frac{\partial^3 u}{\partial x^3} = 6 \sin y - y^3 \cos x$$

$$\frac{\partial^4 u}{\partial x^3 \partial y} = 6 \cos y - 3y^2 \cos x$$

$$\frac{\partial^5 u}{\partial x^3 \partial y^2} = -6 \sin y - 6y \cos x$$

$$\frac{\partial^6 u}{\partial x^3 \partial y^3} = -6 \cos y - 6 \cos x = -6(\cos y + \cos x)$$

04.02.2021 **Дифференцирование сложной функции.**

$w = f(u, v, e)$

$u(x,y)$ $\qquad \dfrac{\partial w}{\partial x} = \dfrac{\partial w}{\partial u}\cdot\dfrac{\partial u}{\partial x} + \dfrac{\partial w}{\partial v}\cdot\dfrac{\partial v}{\partial x} + \dfrac{\partial w}{\partial e}\cdot\dfrac{\partial e}{\partial x}$

$v(x,y)$

$e(x,y)$ $\qquad \dfrac{\partial w}{\partial y} = \dfrac{\partial w}{\partial u}\cdot\dfrac{\partial u}{\partial y} + \dfrac{\partial w}{\partial v}\cdot\dfrac{\partial v}{\partial y} + \dfrac{\partial w}{\partial e}\cdot\dfrac{\partial e}{\partial y}$

3284.

$$u = f(\overset{1}{x}, \overset{2}{\frac{x}{y}})$$

$$\frac{\partial u}{\partial x} = f_1'\cdot\frac{\partial f_1}{\partial x} + f_2'\cdot\frac{\partial f_2}{\partial x} = f_1'\cdot 1 + f_2'\cdot\frac{1}{y}$$

$$\frac{\partial u}{\partial y} = f_1'\cdot 0 + f_2'\cdot\left(-\frac{x}{y^2}\right) = -f_2'\cdot\frac{x}{y^2} \qquad \boxed{f_{12}'' = f_{21}''}$$

$$\frac{\partial^2 u}{\partial x^2} = f_{11}''\cdot 1 + f_{12}''\cdot\frac{1}{y} + \frac{1}{y}\left(f_{21}''\cdot 1 + f_{22}''\cdot\frac{1}{y}\right) =$$

$$= f_{11}'' + \frac{2}{y}f_{12}'' + \frac{1}{y^2}f_{22}''$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(-\frac{x}{y^2}\cdot f_2'\right) = \cancel{\frac{2x}{y^3}\cdot f_2'} \cancel{-\frac{x}{y^2}} = -x\left(-\frac{2}{y^3}\cdot f_2' + \frac{1}{y^2}\cdot\left(f_{21}''\cdot 0 +\right.\right.$$

$$\left.\left.+ f_{22}''\left(-\frac{x}{y^2}\right)\right)\right) = \frac{2x}{y^3}f_2' - \frac{x}{y^4}f_{22}''$$

$$\frac{\partial^2 u}{\partial x\partial y} = \frac{\partial}{\partial y}\left(f_1' + f_2'\cdot\frac{1}{y}\right) = f_{11}''\cdot 0 + f_{12}''\cdot\left(-\frac{x}{y^2}\right) +$$

$$+\left(\left(f_{21}''\cdot 0 + f_{22}''\cdot\left(-\frac{x}{y^2}\right)\right)\cdot\frac{1}{y} + -\frac{1}{y^2}f_2'\right) =$$

$$= -\frac{x}{y^2}f_{12}'' - \frac{x}{y^3}f_{22}'' - \frac{1}{y^2}f_2'$$

3285.

$$u = f(x, xy, xyz) \qquad \frac{\partial u}{\partial x};\ \frac{\partial u}{\partial y};\ \frac{\partial u}{\partial z}$$

$$\frac{\partial^2 u}{\partial x^2};\ \frac{\partial^2 u}{\partial z^2};\ \frac{\partial^2 u}{\partial z^2};\ \frac{\partial^2 u}{\partial x\partial y};\ \frac{\partial^2 u}{\partial x\partial z};\ \frac{\partial^2 u}{\partial y\partial z}$$

$$\frac{\partial u}{\partial x} = f_1'\cdot 1 + f_2'y + f_3'\cdot yz$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(f_1' + f_2'y + f_3'yz\right) = f_{11}''\cdot 1 + f_{12}''\cdot y + f_{13}''\cdot yz +$$

$$+ y(f_{21}''\cdot 1 + f_{22}''\cdot y + f_{23}''\cdot yz) + yz(f_{31}'' + f_{32}''\cdot y + f_{33}''\cdot yz) =$$

$$= f_{11}'' + 2f_{12}''y + 2yz f_{13}'' + 2y^2z f_{23}'' + y^2 f_{22}'' + y^2z^2 f_{33}''$$

$$\frac{\partial u}{\partial y} = f_1' \cdot 0 + f_2' \cdot x + f_3' \cdot xz$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(x f_2' + xz f_3'\right) = x\left(f_{21}'' \cdot 0 + f_{22}'' \cdot x + f_{23}'' \cdot xz\right) +$$

$$+ xz\left(f_{31}'' \cdot 0 + f_{32}'' \cdot x + f_{33}'' \cdot xz\right) =$$

$$= x^2 f_{22}'' + 2x^2 z f_{33}'' + x^2 z^2 f_{33}''$$

$$\frac{\partial u}{\partial z} = f_1' \cdot 0 + f_2' \cdot 0 + f^3 \cdot xy$$

$$\frac{\partial^2 u}{\partial z^2} = xy\left(f_{33}'' \cdot xy\right) = x^2 y^2 \cdot f_{33}''$$

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial y}\left(f_1' + f_2' y + f_3' yz\right) =$$

$$= f_{11}'' \cdot 0 + f_{12}'' \cdot x + f_{13}'' \cdot xz + f_2' + y\left(f_{21}'' \cdot 0 + f_{22}'' \cdot x + f_{23}'' \cdot xz\right) +$$

$$+ z\left(f_3' + y\left(f_{31}'' \cdot 0 + f_{32}'' \cdot x + f_{33}'' \cdot xz\right)\right) =$$

$$= x f_{12}'' + xz f_{13}'' + xy f_{22}'' + 2xyz f_{23}'' + xyz^2 f_{33}'' + f_2' + z f_3'$$

$$\frac{\partial^2 u}{\partial y \partial z} = \frac{\partial}{\partial z}\left(f_2' \cdot x + f_3' \cdot xz\right) =$$

$$= x\left(f_{23}'' \cdot xy\right) + x\left(f_3' + z \cdot f_{33}'' \cdot xy\right) =$$

$$= x^2 y f_{23}'' + x f_3' + x^2 yz \cdot f_{33}''$$

$$\frac{\partial^2 u}{\partial x \partial z} = \frac{\partial}{\partial x \partial z}\left(f_1' + y \cdot f_2' + yz \cdot f_3'\right) =$$

$$= f_{13}'' \cdot xy + y \cdot f_{23}'' \cdot xy + y\left(f_3' + z \cdot f_{33}'' \cdot y\cancel{x}\right) =$$

$$= f_{13}'' xy + xy^2 \cdot f_{23}'' + y f_3' + xyz \cdot f_{33}'' =$$

3283.

$$u = f(x^2 + y^2 + z^2)$$

$$\frac{\partial u}{\partial x} = f' \cdot 2x = 2x f'$$

$$\frac{\partial^2 u}{\partial x^2} = 2f' + 2x f'' \cdot 2x = 4x^2 f'' + 2f'$$

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial y}\left(2x f'\right) = 2x \cdot f'' \cdot 2y = 4xy f''$$

Домашняя работа.

3286 Найти: $\dfrac{\partial^2 u}{\partial x \partial y}$

$u = f(\underset{1}{x+y}, \underset{2}{xy})$

$$\frac{\partial u}{\partial x} = f'_1 \cdot 1 + f'_2 \cdot y$$

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial y}\left(f'_1 \cdot 1 + f'_2 \cdot y\right) =$$

$$= f''_{11} \cdot 1 + f''_{12} \cdot x + f'_2 + (f''_{21} \cdot 1 + f''_{22} \cdot x)\, y =$$

$$= f''_{11} + f''_{12}(x+y) + xy f''_{22} + f'_2$$

3287. Найти: $\Delta u = \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2}$

$u = f(x+y+z,\ x^2+y^2+z^2)$

$$\frac{\partial u}{\partial x} = f'_1 + f'_2 \cdot 2x = f'_1 + 2x f'_2$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(f'_1 + 2x f'_2\right) = f''_{11} + f''_{12} \cdot 2x + 2\left(f'_2 + x(f''_{21} + 2x f''_{22})\right) =$$

$$= f''_{11} + 2x f''_{12} + 2f'_2 + 2x f''_{21} + 4x^2 f''_{22} =$$

$$= f''_{11} + 4x f''_{12} + 4x^2 f''_{22} + 2f'_2.$$

Т.к. x, y, z можно менять местами и всё будет аналогично:

$$\frac{\partial^2 u}{\partial y^2} = f''_{11} + 4y f''_{12} + 4y^2 f''_{22} + 2f'_2$$

$$\frac{\partial^2 u}{\partial z^2} = f''_{11} + 4z f''_{12} + 4z^2 f''_{22} + 2f'_2$$

$$\Rightarrow \Delta u = 3f''_{11} + 4f''_{12}(x+y+z) + 4f''_{22}(x^2+y^2+z^2) + 6f'_2$$

$$\frac{\partial^2 u}{\partial x \partial y} \text{ и } \frac{\partial^3 u}{\partial x \partial z^2}, \text{ если } u = f\left(\frac{x}{y}, \frac{y}{z}\right)$$

$$\frac{\partial u}{\partial x} = f_1' \cdot \frac{1}{y} + f_2' \cdot 0 = f_1' \cdot \frac{1}{y}$$

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial y}\left(f_1' \cdot \frac{1}{y}\right) = -\frac{1}{y^2} \cdot f_1' + \frac{1}{y}\left(f_{11}'' \cdot \left(-\frac{x}{y^2}\right) + f_{12}'' \cdot \frac{1}{z}\right) =$$

$$= -\frac{1}{y^2} f_1' - \frac{x}{y^3} f_{11}'' + \frac{1}{yz} f_{12}''$$

$$\frac{\partial^2 u}{\partial x \partial z} = \frac{\partial u}{\partial z}\left(f_1' \cdot \frac{1}{y}\right) = \frac{1}{y} f_{11}'' \cdot 0 + \frac{1}{y} f_{12}'' \cdot \left(-\frac{y}{z^2}\right) =$$

$$= -\frac{1}{z^2} \cdot f_{12}''$$

$$\frac{\partial^3 u}{\partial x \partial z^2} = \frac{\partial}{\partial z}\left(\left(-\frac{1}{z^2}\right) \cdot f_{12}''\right) = \frac{2}{z^3} \cdot f_{12}'' - \frac{1}{z^2}\left(f_{121}''' \cdot 0 + f_{122}''' \cdot \left(-\frac{y}{z^2}\right)\right) =$$

$$\left(-\frac{1}{z^2}\right)' = -(z^{-2})' = \frac{2}{z^3} \quad = \frac{2 f_{12}''}{z^3} + \frac{y}{z^4} \cdot f_{122}'''$$

9.02.2021.

$$u = f\left(\frac{x}{y}, \frac{y}{z}\right)$$

$$u = \frac{x}{y} + \frac{y}{z}$$

$$\frac{\partial^2 u}{\partial x \partial z} = -\frac{1}{z^2} f_{12}''$$

$$\frac{\partial u}{\partial x} = \frac{1}{y} \quad ; \quad \frac{\partial^2 u}{\partial x \partial z} = 0$$

Вычисление дифференциалов ф-ий многих переменных.

3239.

$u = e^{xy}$ $\quad du,\ d^2u$

$du = e^{xy}\cdot d(xy) = e^{xy}(y\,dx + x\,dy) = y\cdot e^{xy}dx + xe^{xy}dy =$

$= \frac{\partial u}{\partial x}\cdot dx + \frac{\partial u}{\partial y}\cdot dy$

$d^2u = d(du) = d(e^{xy}(y\,dx + x\,dy)) = d(e^{xy})\cdot(y\,dx + x\,dy) +$

$+ e^{xy}\, d(y\,dx + x\,dy) =$

$= e^{xy}(y\,dx + x\,dy)^2 + e^{xy}\cdot(dy\cdot dx + dx\cdot dy) =$

$= e^{xy}(y^2dx^2 + 2xy\,dx\,dy + x^2dy^2 + 2dx\,dy) =$

$= e^{xy}(y^2dx^2 + 2dx\,dy\,(1+xy) + x^2dy^2) =$

$= y^2e^{xy}dx^2 + 2e^{xy}(1+xy)\,dx\,dy + x^2e^{xy}dy^2 =$

$= \frac{\partial^2 u}{\partial x^2}dx^2 + 2\frac{\partial^2 u}{\partial x\partial y}dx\,dy + \frac{\partial^2 u}{\partial y^2}dy^2$

3240

$u = xy + yz + zx$ $\quad ;\ du,\ d^2u$ - ?

$du = (y+z)dx + (x+z)dy + (y+x)dz$

$d^2u = d(du) = (dy+dz)dx + (dx+dz)dy + (dy+dx)dz =$

$= 2(dx\,dy + dx\,dz + dz\,dy)$

3238.

$u = \ln\sqrt{x^2+y^2}$ $\quad ;\ du\ ;\ d^2u$

$u = \frac{1}{2}\ln(x^2+y^2)$

$du = \frac{x}{x^2+y^2}dx + \frac{y}{x^2+y^2}dy$

$\frac{\partial u}{\partial x} = \frac{2x}{2(x^2+y^2)} = \frac{x}{x^2+y^2}\ ;\quad \frac{\partial u}{\partial y} = \frac{y}{x^2+y^2}$

$\frac{\partial^2 u}{\partial x^2} = \frac{x^2+y^2-2x^2}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$

$\frac{\partial^2 u}{\partial y^2} = \frac{x^2-y^2}{(x^2+y^2)^2}$

$\frac{\partial^2 u}{\partial x\partial y} = \frac{\partial}{\partial x}\left(\frac{y}{x^2+y^2}\right) = -\frac{2xy}{(x^2+y^2)^2}$

$$d^2u = \frac{y^2-x^2}{(x^2+y^2)^2}dx^2 - 4\frac{xy}{(x^2+y^2)^2} + \frac{x^2-y^2}{(x^2+y^2)^2}dy^2$$

Диф-лы высших порядков.

3241 $d^{10}u$ - ?

$u = \ln(x+y)$

$$du = \frac{dx}{x+y} + \frac{dy}{x+y} = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy = \frac{1}{x+y}(dx+dy)$$

$$d^2u = -\frac{1}{(x+y)^2}(dx+dy)^2$$

$$d^3u = \frac{2}{(x+y)^3}(dx+dy)^3$$

$$d^4u = -\frac{2\cdot 3}{(x+y)^4}(dx+dy)^4$$

$\dots$ ~~$d^{10}u = -1$~~ $d^n u = (-1)^{n+1}\frac{(n-1)!}{(x+y)^n}(dx+dy)^n$

$$d^{10}u = -\frac{9!}{(x+y)^{10}}(dx+dy)^{10}$$

$$d^n f = \left(\frac{\partial}{\partial x_1}dx_1 + \dots + \frac{\partial}{\partial x_m}dx_m\right)^n f$$

$F = f(x_1, x_2, \dots, x_m)$

символы рассматриваются как множители с дальнейшим применением формулы Бинома - Ньютона.

$$\cancel{d^2f(x,y,z) = \frac{\partial^2 f}{\partial x^2}dx^2}$$

$$d^3f(x,y) = \frac{\partial^3 f}{\partial x^3}dx^3 + 3\frac{\partial^3 f}{\partial x^2\partial y}dx^2\cdot dy + 3\frac{\partial^3 f}{\partial x\partial y^2}dx\cdot dy^2 + \frac{\partial^3 f}{\partial y^3}dy^3$$

Д/З: 3256, 3258, 35, 37, 39, 41, 68, 70, 73, 75

3235.

$u = x^m y^n$

$\cancel{du = mx^{m-1} y^n dx + ny^{n-1} x^m dy}$

$$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy = y^n \cdot m x^{m-1} dx + x^m \cdot n y^{n-1} dy =$$

$$= x^{m-1} \cdot y^{n-1} (ym\,dx + xn\,dy)$$

$$d^2u = \frac{\partial^2 u}{\partial x^2} dx^2 + \frac{2\partial^2 u}{\partial x \partial y} dx\,dy + \frac{\partial^2 u}{\partial y^2} dy^2 \ominus$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial x}\left(y^n \cdot m x^{m-1}\right) = y^n m(m-1) x^{m-2}$$

$$\frac{\partial^2 u}{\partial y^2} = x^m n(n-1) y^{n-2}$$

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial y}\left(m y^n x^{m-1}\right) = m x^{m-1} \cdot n y^{n-1}$$

$$\ominus\ y^n m(m-1) x^{m-2} \cdot dx^2 + 2m \cdot n x^{m-1} y^{n-1} dx\,dy +$$

$$+ x^m n(n-1) y^{n-2} dy^2$$

3237.

$u = \sqrt{x^2+y^2}$

$$\frac{\partial u}{\partial x} = \frac{2x}{2\sqrt{x^2+y^2}} = \frac{x}{\sqrt{x^2+y^2}}\ ; \quad \frac{\partial u}{\partial y} = \frac{y}{\sqrt{x^2+y^2}}$$

$$\Rightarrow du = \frac{x\,dx + y\,dy}{\sqrt{x^2+y^2}}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\sqrt{x^2+y^2} - x \cdot \frac{\cancel{2}x}{\cancel{2}\sqrt{x^2+y^2}}}{(x^2+y^2)} = \frac{x^2+y^2-x^2}{(x^2+y^2)^{\frac{3}{2}}} = \frac{y^2}{(x^2+y^2)^{\frac{3}{2}}}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{x^2}{(x^2+y^2)^{\frac{3}{2}}}$$

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial y}\left(\frac{x}{\sqrt{x^2+y^2}}\right) = -\frac{x \cdot \cancel{2}y}{\cancel{2}(x^2+y^2)^{\frac{3}{2}}} = \frac{-xy}{(x^2+y^2)^{\frac{3}{2}}}$$

$$d^2u = \frac{y^2 dx^2}{(x^2+y^2)^{\frac{3}{2}}} - \frac{2xy}{(x^2+y^2)^{\frac{3}{2}}} dx\,dy + \frac{x^2 dy^2}{(x^2+y^2)^{\frac{3}{2}}} =$$

$$= \frac{(y\,dx - x\,dy)^2}{(x^2+y^2)^{\frac{3}{2}}}$$

3256. (прошлое д/з!)

3258. (прошлое д/з!)

3239. (в практике!)

3269.

$$u = x^3 + y^3 - 3xy(x-y) = x^3 + y^3 - 3x^2y + 3xy^2$$

симметричны

$$d^3u = 6dx^3 + 6dy^3 + 3\cdot 6\, dx\, dy^2 - 3\cdot 6\, dy\, dx^2 = 6(dx^3 + dy^3 - 3dx^2 dy + 3dx\, dy^2)$$

$$\frac{\partial^3 u}{\partial x \partial y^2} = \frac{\partial}{\partial x \partial y}\left(3y^2 - 3x^2 + 6xy\right) = \frac{\partial}{\partial x}\left(6y - 0 + 6x\right) = 6$$

3270.

$$u = \sin(x^2+y^2) \quad ; \; d^3u - ?$$

$$du = 2x\cos(x^2+y^2)\,dx + 2y\cos(x^2+y^2)\,dy =$$

$$= 2(x\,dx + y\,dy)\cos(x^2+y^2)$$

$$\frac{\partial^3 u}{\partial x^3} = \frac{\partial^2 u}{\partial x^2}\left(2x\cos(x^2+y^2)\right) = \frac{\partial u}{\partial x}\left(2\cos(x^2+y^2) - 4x^2\sin(x^2+y^2)\right) =$$

$$= -4x\sin(x^2+y^2) - 8x\sin(x^2+y^2) - 8x^3\cos(x^2+y^2) =$$

$$= -12x\sin(x^2+y^2) - 8x^3\cos(x^2+y^2)$$

$$\frac{\partial^3 u}{\partial y^3} = -4y\sin(x^2+y^2) - 8y\sin(x^2+y^2) - 8y^3(\cos(x^2+y^2)) =$$

$$= -12y\sin(x^2+y^2) - 8y^3(\cos(x^2+y^2))$$

$$\cancel{\frac{\partial^3 u}{\partial x \partial y^2} = \frac{\partial^2 u}{\partial x}}$$

$$\frac{\partial^3 u}{\partial y \partial x^2} = \frac{\partial u}{\partial y}\left(2\cos(x^2+y^2) - 4x^2\sin(x^2+y^2)\right) =$$

$$= -4y\sin(x^2+y^2) - 8yx^2\cos(x^2+y^2)$$

$$\frac{\partial^3 u}{\partial x \partial y^2} = -4x\sin(x^2+y^2) - 8xy^2\cos(x^2+y^2)$$

$$d^3u = \left(-12x\sin(x^2+y^2) - 8x^3\cos(x^2+y^2)\right)dx^3 +$$

$$+ 3\left(-4x\sin(x^2+y^2) - 8xy^2\cos(x^2+y^2)\right)dx\,dy^2 +$$

$$+ 3\left(-4y\sin(x^2+y^2) - 8yx^2\cos(x^2+y^2)\right)dy\,dx^2 +$$

$$+ \left(-12y\sin(x^2+y^2) - 8y^3\cos(x^2+y^2)\right)dy^3$$

?

3273. $d^3u - ?\quad u = xyz$

$$d^3u = d^3(xyz) = 6\,dx\,dy\,dz$$

$$\frac{\partial^3 u}{\partial x^3} = 0\;;\quad \frac{\partial^3 u}{\partial y^3} = 0\qquad \frac{\partial^3 u}{\partial z^3} = 0$$

$$\frac{\partial^3 u}{\partial x \partial y \partial z} = 1$$

3275. $d^n u - ?\;;\quad u = e^{ax+by}$

$$du = a\cdot e^{ax+by}dx + b\cdot e^{ax+by}dy = e^{ax+by}(a\,dx + b\,dy)$$

$\ldots$

$$d^n u = e^{ax+by}(a\,dx + b\,dy)^n$$

3291. $u = \frac{z}{x^2+y^2}$

$$\frac{\partial u}{\partial x} = -\frac{2zx}{(x^2+y^2)^2}\;;\quad \frac{\partial u}{\partial y} = \frac{-2zy}{(x^2+y^2)^2}\;;\quad \frac{\partial u}{\partial z} = \frac{1}{x^2+y^2}$$

$$du = -\frac{2zx\,dx}{(x^2+y^2)^2} - \frac{2zy\,dy}{(x^2+y^2)^2} + \frac{dz}{x^2+y^2} =$$

$$= -\frac{2z(x\,dx + y\,dy)}{(x^2+y^2)^2} + \frac{dz}{x^2+y^2} = \frac{(x^2+y^2)\,dz - 2z(x\,dx + y\,dy)}{(x^2+y^2)^2}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{-2z(x^2+y^2) - 4x(x^2+y^2)\cdot(-2zx)}{(x^2+y^2)^4} = \frac{8zx^2(x^2+y^2) - 2z(x^2+y^2)}{(x^2+y^2)^4} =$$

$$= \frac{8zx^2 - 2z}{(x^2+y^2)^3}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{8zy^2 - 2z}{(x^2+y^2)^3}\;,\quad \frac{\partial^2 u}{\partial z^2} = 0$$

$$\frac{\partial^2 u}{\partial z \partial x} = 0\;;\quad \frac{\partial^2 u}{\partial z \partial y} = 0$$

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial y}\left(\frac{-2zx}{(x^2+y^2)^2}\right) = \frac{4\cdot 2zxy}{(x^2+y^2)^3}$$

Не сошлось с ответом!

$$\Rightarrow d^3u = \frac{1}{(x^2+y^2)^3}\Big($$

25.02.2021

Дифференцирование неявно заданных функций.

3372.

$$\ln\sqrt{x^2+y^2} = \operatorname{arctg}\frac{y}{x}$$

$$2\sqrt{x^2+y^2}\sqrt{x^2+y^2} \qquad \frac{1}{2}\ln(x^2+y^2) = \operatorname{arctg}\frac{y}{x}$$

$$\frac{1}{2}\cdot\frac{(2x+2yy')}{x^2+y^2} = \frac{1}{1+\frac{y^2}{x^2}}\cdot\frac{y'x-y}{x^2}$$

$$\frac{x+yy'}{x^2+y^2} = \frac{y'x-y}{x^2+y^2}$$

$$x+yy' = y'x-y \quad\rightarrow\quad x+y = y'(x-y) \quad\rightarrow\quad y' = \frac{x+y}{x-y}$$

$$y'' = \frac{(x+y)'(x-y)-(x-y)'(x+y)}{(x-y)^2} = \frac{(1+y')(x-y)-(1-y')(x+y)}{(x-y)^2} =$$

$$= \frac{\left(1+\frac{x+y}{x-y}\right)(x-y)-\left(1-\frac{x+y}{x-y}\right)(x+y)}{(x-y)^2} =$$

$$= \frac{(x-\cancel{y}+x+\cancel{y})-\left(x+y-\frac{(x+y)^2}{(x-y)}\right)}{(x-y)^2} = \frac{2x-x-y+\frac{(x+y)^2}{(x-y)}}{(x-y)^2} =$$

$$= \frac{(x-y)+\frac{(x+y)^2}{(x-y)}}{(x-y)^2} = \frac{(x-y)^2+(x+y)^2}{(x-y)^3} = \frac{x^2-\cancel{2xy}+y^2+x^2+\cancel{2xy}+y^2}{(x-y)^3} =$$

$$= \frac{2(x^2+y^2)}{(x-y)^3}$$

3384. $z^3-3xyz = a^3 \qquad z = z(x,y)$

$$\frac{\partial}{\partial x}: \quad 3z^2z' - 3y(z+z'x) = 0$$

$$3z^2z'_x - 3yz - 3yxz'_x = 0 \quad /:3$$

$$z'_x(z^2-yx) = yz \quad\rightarrow\quad z'_x = \frac{yz}{z^2-yx} = \frac{\partial z}{\partial x}$$

$$\frac{\partial}{\partial y}: \quad 3z^2\frac{\partial z}{\partial y} - 3x\left(z+\frac{\partial z}{\partial y}y\right) = 0$$

$$z^2\frac{\partial z}{\partial y} - xz - \frac{\partial z}{\partial y}yx = 0 \quad\rightarrow\quad \frac{\partial z}{\partial y} = \frac{xz}{z^2-yx}$$

$$\cancel{\frac{\partial^2}{\partial x^2}}\ \cancel{\frac{\partial^2 z}{\partial x^2}} \quad \frac{\partial^2 z}{\partial x^2}(z^2-yx) + \frac{\partial z}{\partial x}\left(2z\frac{\partial z}{\partial x}-y\right) - y\frac{\partial z}{\partial x} = 0$$

$$\frac{\partial^2 z}{\partial x^2}(z^2-yx) + \frac{\partial z}{\partial x}\cdot\left(2z\frac{\partial z}{\partial x}-2y\right) = 0$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{-\frac{\partial z}{\partial x}\left(2z\frac{\partial z}{\partial x}-2y\right)}{z^2-yx}$$

$$\frac{\partial^2 z}{\partial x^2} = -\frac{\frac{2yz}{z^2-yx}\left(\frac{yz^2}{z^2-yx} - y\right)}{(z^2-yx)} = \frac{-2y^3zx}{(z^2-yx)^3}$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{-2x^3zy}{(z^2-yx)^3}$$

$$\frac{\partial^2 z}{\partial x \partial y}(z^2-yx) + \frac{\partial z}{\partial}\left(2z\frac{\partial z}{\partial y} - x\right) - \left(y\frac{\partial z}{\partial y} + z\right) = 0$$

...дальше преобразовать.

3386. $z = \sqrt{x^2-y^2}\,\mathrm{tg}\,\frac{z}{\sqrt{x^2-y^2}}$

$$\frac{\partial z}{\partial x} = \frac{2x}{2\sqrt{x^2-y^2}}\,\mathrm{tg}\,\frac{z}{\sqrt{x^2-y^2}} + \frac{\frac{1}{\cos^2\frac{z}{\sqrt{x^2-y^2}}}}{\cancel{1+\frac{z^3}{x^2-y^2}}} \cdot \frac{\frac{\partial z}{\partial x}\sqrt{x^2-y^2} - \frac{2x}{2\sqrt{x^2-y^2}}\cdot z}{x^2-y^2}\cdot\sqrt{x^2-y^2} =$$

$$= \cancel{\frac{x}{\sqrt{x^2-y^2}}\,\mathrm{tg}\,\frac{z}{\sqrt{x^2-y^2}}}$$

$$1+\mathrm{tg}^2\frac{z}{\sqrt{x^2-y^2}} = \frac{z^2}{x^2-y^2}+1$$

$$= \frac{xz}{x^2-y^2} + \left(\frac{z^2}{x^2-y^2}+1\right)\cdot\frac{\frac{\partial z}{\partial x}(x^2-y^2) - xz}{x^2-y^2}$$

$$\cancel{\frac{\partial z}{\partial x}} = \cancel{\frac{xz}{x^2-y^2}} + \frac{z^2\frac{\partial z}{\partial x}}{x^2-y^2} - \frac{z^3x}{(x^2-y^2)^2} + \cancel{\frac{\partial z}{\partial x}} - \cancel{\frac{xz}{x^2-y^2}}$$

$$\frac{z^2\frac{\partial z}{\partial x}}{x^2-y^2} = \frac{z^3x}{(x^2-y^2)^2} \to \frac{\partial z}{\partial x} = \frac{zx}{x^2-y^2}$$

$$\frac{\partial z}{\partial y} = \frac{-zy}{x^2-y^2}$$

$$\frac{\partial^2 z}{\partial x\partial y} = x\cdot\left(\frac{\frac{\partial z}{\partial y}(x^2-y^2)+2zy}{(x^2-y^2)^2}\right) = x\cdot\left(\frac{-zy+2zy}{(x^2-y^2)^2}\right) = \frac{zyx}{(x^2-y^2)^2}$$

3389. $x^2+2y^2+3z^2+xy-z-9=0$; $x=1,\ y=-2,\ z=1$.

$$2x\,dx + 4y\,dy + 6z\,dz + y\,dx + x\,dy - dz = 0$$

$$dz(6z-1) + dx(2x+y) + dy(4y+x) = 0$$

$$dz = \frac{dx(2x+y)}{(1-6z)} + \frac{dy(4y+x)}{(1-6z)}$$

$$\frac{\partial z}{\partial x} = \frac{2x+y}{1-6z} \quad ; \quad \frac{\partial z}{\partial y} = \frac{4y+x}{1-6z}$$

$$d^2z(6z-1)+6dz^2+2dx^2+dydx+4dy^2+dydx=0$$

$$dz=0+\frac{7}{5}dy$$

$$\frac{49}{\cdot 6}$$

$$d^2z(6z-1)+\frac{6\cdot 49}{25}dy^2+2dx^2+2dydx+4dy^2=0$$

$$d^2z=\frac{\frac{6\cdot 49}{25}dy^2+2dx^2+2dydx+4dy^2}{5}$$

$$\frac{\partial^2 z}{\partial x^2}=-\frac{2}{5}$$

$$\frac{\partial^2 z}{\partial x\partial y}=-\frac{1}{5} \quad ; \quad \frac{\partial^2 z}{\partial y^2}=-\frac{394}{125}$$

Д/З: ~~3391~~, ~~85~~, ~~86~~, ~~89~~, ~~90~~, ~~92~~, 97, 98.

3371.

$$x^2+2xy-y^2=a^2$$

$$2x+2y+2xy'-2yy'=0 \quad /:2$$

$$x+y+xy'-yy'=0 \rightarrow x+y=y'(y-x)$$

$$y'=\frac{y+x}{y-x}$$

$$y''=\frac{(y'+1)(y-x)-(y'-1)(y+x)}{(y-x)^2}=\frac{\left(\frac{y+x}{y-x}+1\right)(y-x)-\left(\frac{y+x}{y-x}-1\right)(y+x)}{(y-x)^2}=$$

$$=\frac{2y-(2x)(y+x)}{(y-x)^2}=\frac{2y-2xy-2x^2}{(y-x)^2}.$$

3383.

$$x^2+y^2+z^2=a^2$$

$$2xdx+2ydy+2zdz=0 \quad /:2 \rightarrow zdz=-xdx-ydy$$

$$dz=-\frac{x}{z}dx-\frac{y}{z}dy$$

$$\Rightarrow \frac{\partial z}{\partial x}=-\frac{x}{z} \quad ; \quad \frac{\partial z}{\partial y}=-\frac{y}{z}$$

$$\frac{\partial^2 z}{\partial x^2}=\frac{\partial z}{\partial(x)}\left(-\frac{x}{z}\right)=-\frac{z-xz'_x}{z^2}=-\frac{z+\frac{x^2}{z}}{z^2}=-\frac{z^2+x^2}{z^3}$$

$$\frac{\partial^2 z}{\partial y^2}=-\frac{z^2+y^2}{z^3}$$

$$\frac{\partial^2 z}{\partial x\partial y}=\frac{\partial z}{\partial y}\left(-\frac{x}{z}\right)=-\frac{0-xz'_y}{z^2}=-\frac{0+\frac{xy}{z}}{z^2}=-\frac{xy}{z^3}$$

3385. $x+y+z=e^z \rightarrow z=\ln(x+y+z)$

$$\frac{\partial z}{\partial x}=\frac{1+z_x'}{x+y+z}$$

$$z_x'=\frac{1+z_x'}{x+y+z}$$

$$z_x'\cdot(x+y+z)-z_x'-1=0$$

$$z_x'=\frac{1}{x+y+z-1}=\frac{\partial z}{\partial y}=\frac{\partial z}{\partial x}$$

$$\frac{\partial^2 z}{\partial x^2}=\frac{-(1+z_x')}{(x+y+z-1)^2}=-\frac{1+\frac{1}{(x+y+z-1)}}{(x+y+z-1)^2}=-\frac{x+y+z}{(x+y+z-1)^3}$$

$$\frac{\partial^2 z}{\partial y^2}=-\frac{x+y+z}{(x+y+z-1)^3}$$

$$\frac{\partial z}{\partial x\partial y}=-\frac{x+y+z}{(x+y+z-1)^3}=\frac{\partial^2 z}{\partial y^2}=\frac{\partial^2 z}{\partial x^2}$$

3387. $x+y+z=e^{-(x+y+z)} \rightarrow -(x+y+z)=\ln(x+y+z)$

$$-x-y-z=\ln(x+y+z)$$

$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial y}$ – т.к., если x и y поменять местами, ничего не изменится

$$-1-z_x'=\frac{1+z_x'}{x+y+z}$$

$$(-1-z_x')(x+y+z)=1+z_x'$$

$$-x-y-z-z_x'x-z_x'y-z_x'z=1+z_x'$$

$$z_x'(x+y+z+1)=-(x+y+z+1) \Rightarrow \frac{\partial z}{\partial x}=\frac{\partial z}{\partial y}=-1$$

Очевидно: $\frac{\partial^2 z}{\partial x^2}=\frac{\partial^2 z}{\partial y^2}=\frac{\partial^2 z}{\partial x\partial y}=0$

3390. $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$

$$\frac{2x}{a^2}dx+\frac{2y}{b^2}dy+\frac{2z}{c^2}dz=0 \quad /:2$$

$$\frac{x\,dx}{a^2}+\frac{y\,dy}{b^2}=-\frac{z\,dz}{c^2}$$

$$dz=-\frac{c^2x}{a^2z}dx-\frac{c^2}{b^2}\frac{y}{z}dy=-\frac{c^2}{z}\left(\frac{x\,dx}{a^2}+\frac{y}{b^2}dy\right)$$

$$\Rightarrow \frac{\partial z}{\partial x}=-\frac{c^2x}{a^2z} \qquad \frac{\partial z}{\partial y}=-\frac{c^2y}{b^2z}$$

$$d^2z = -d\left(\frac{c^2}{z}\right)\cdot\left(\frac{x\,dx}{a^2}+\frac{y\,dy}{b^2}\right)+d\left(\frac{x\,dx}{a^2}+\frac{y\,dy}{b^2}\right)\cdot\left(-\frac{c^2}{z}\right)=$$

$$=\frac{c^2dz}{z^2}\left(\frac{x\,dx}{a^2}+\frac{y\,dy}{b^2}\right)+\left(-\frac{c^2}{z}\right)\left(\frac{1}{a^2}(dx^2+x\,d^2x)+\frac{1}{b^2}(dy^2+y\,d^2y)\right)$$

$$dz=-\frac{c^2}{z}\left(\frac{x\,dx}{a^2}+\frac{y\,dy}{b^2}\right)$$, подставим

$$= -\frac{c^4}{z^3}\left(\frac{x\,dx}{a^2}+\frac{y\,dy}{b^2}\right)^2-\frac{c^2}{z}\left(\frac{dx^2}{a^2}+\frac{x\,d^2x}{a^2}+\frac{dy^2}{b^2}+\frac{y\,d^2y}{b^2}\right)$$

33.92.

$$\frac{x}{z}=\ln\frac{z}{y}+1.$$

$$\frac{dx\cdot z-dz\,x}{z^2}=\frac{y\left(\frac{dz\cdot y-dy\cdot z}{y^2}\right)}{z}+0$$

$$\frac{dx\cdot z-dz\,x}{z^2}=\frac{dz\cdot y-dy\cdot z}{zy}\quad\Big/\cdot z$$

$$\frac{dx\cdot z-dz\,x}{z}=\frac{dz\cdot y-dy\cdot z}{y}\longrightarrow yz\,dx-yx\,dz=zy\,dz-z^2dy$$

$$dz\,(zy+yx)=yz\,dx+z^2dy$$

$$dz=\frac{yz\,dx+z^2dy}{zy+yx}=\frac{z(y\,dx+z\,dy)}{y(z+x)}$$

$$dz=\frac{z(y\,dx+z\,dy)}{y(z+x)}=$$

$$d^2z=\frac{d\big(z(y\,dx+z\,dy)\big)\cdot y(z+x)-z(y\,dx+z\,dy)\cdot d\big(y(z+x)\big)}{y^2(z+x)^2}=$$

$$=\frac{\left[dz(y\,dx+z\,dy)+z(dy\,dx+y\,d^2x+dz\,dy+z\,d^2y)\right]\cdot y(x+z)-z(y\,dx+z\,dy)\left(dy(z+x)+y(dz+dx)\right)}{y^2(z+x)^2}$$

$$=\Big(y^2x\,dz\,dx+2zyx\,dz\,dy+yz^2dz\,dy+y^2xz\,d^2x+y^2z^2d^2x+yxz^2d^2y+yz^3d^2y-$$
$$-z^3dy^2-xz^2dy^2-zy^2dx^2-z^2y\,dx\,dy\Big):\left(y^2(z+x)^2\right)$$

Если подставить $dz=\dfrac{yz\,dx+z^2dy}{y(z+x)}$ и сгруппировать.

$$\rightarrow d^2z=-\frac{z^2(y\,dx-x\,dy)^2}{y^2(z+x)^3}$$

33.97. $F(x,\ x+y,\ x+y+z)=0$

$$F_1'\cdot 1+F_2'\cdot 1+F_3'\left(1+\frac{\partial z}{\partial x}\right)=0$$

$$F_1'+F_2'+F_3'+F_3'\frac{\partial z}{\partial x}=0$$

$$\frac{\partial z}{\partial x} = \frac{-F'_3 - F'_1 - F'_2}{F'_3} = -\left(1 + \frac{F'_1 + F'_2}{F'_3}\right)$$

$$F'_1 \cdot 0 + F'_2 \cdot 1 + F'_3\left(1 + \frac{\partial z}{\partial y}\right) = 0$$

$$\frac{\partial z}{\partial y} = \frac{-F'_2 - F'_3}{F'_3} = -\left(1 + \frac{F'_2}{F'_3}\right)$$

$$\frac{\partial^2 z}{\partial x^2} = -\frac{\left(F''_{11} + F''_{12} + F''_{13}\left(1 + \frac{\partial z}{\partial x}\right) + F''_{21} + F''_{22} + F''_{23}\left(1 + \frac{\partial z}{\partial x}\right)\right) - (F'_1 + F'_2)\left(F''_{31} + F''_{32} + F''_{33}\left(1 + \frac{\partial z}{\partial x}\right)\right)}{(F'_3)^2} =$$

$$= -\frac{\left(F''_{11} + F''_{12} + F''_{13} + F''_{13} \cdot \left(-1 - \frac{F'_1 + F'_2}{F'_3}\right)\right) + F''_{21} + F''_{22} + F''_{23}\left(-\frac{F'_1 + F'_2}{F'_3}\right) - (F'_1 + F'_2)\left(F''_{31} + F''_{32} + F''_{33}\left(-\frac{F'_1 + F'_2}{F'_3}\right)\right)}{(F'_3)^2} =$$

$$= -\frac{\left(F''_{11} + F''_{12} - \frac{F''_{13} F'_1 + F''_{13} F'_2}{F'_3}\right) + F''_{21} + F''_{22} - \frac{F''_{23} F'_1 + F''_{23} F'_2}{F'_3} - (F'_1 + F'_2)\left(F''_{31} + F''_{32} - \frac{F''_{33} F'_1 + F''_{33} F'_2}{F'_3}\right)}{(F'_3)^2}$$

?

3398. 1) $\frac{\partial^2 z}{\partial x^2}$, если $F(xz, yz) = 0$

$\frac{\partial z}{\partial x}$ – ? : $F'_1\left(z + \frac{\partial z}{\partial x} \cdot x\right) + F'_2 \, y\frac{\partial z}{\partial x} = 0$

$$zF'_1 + F'_1 \, x\frac{\partial z}{\partial x} + yF'_2 \frac{\partial z}{\partial x} = 0$$

$$\frac{\partial z}{\partial x}(F'_1 + F'_2) = \frac{-zF'_1}{1} \Rightarrow \frac{\partial z}{\partial x} = \frac{-zF'_1}{F'_1 + F'_2}$$

$$\frac{\partial^2 z}{\partial x^2} = -\frac{\partial z}{\partial x} F'_1 + z\left(F''_{11}\left(z + \frac{\partial z}{\partial x} \cdot x\right) + F''_{12} \cdot y\frac{\partial z}{\partial x}\right) - \left(F''_{11}\left(z + \frac{\partial z}{\partial x} \cdot x\right) + F''_{12} \cdot y\frac{\partial z}{\partial x} + F''_{21}\left(z + \frac{\partial z}{\partial x} \cdot x\right) + F''_{21} \cdot y\frac{\partial z}{\partial x}\right)$$

Если подставить $\frac{\partial z}{\partial x}$

$$= -\left(\left(-\frac{zF_1'^2}{F'_1 + F'_2} + z^2 F''_{11} + zx \cdot \left(\frac{-zF'_1}{F'_1 + F'_2}\right) + \frac{yF''_{12}(-zF'_1)}{F'_1 + F'_2} - \left(F''_{11} z + xF''_{11}\left(\frac{-zF'_1}{F'_1 + F'_2}\right) + F''_{12} \cdot y \cdot \frac{(-zF'_1)}{F'_1 + F'_2} + F'_{21} z + xF'_{21} \cdot \frac{(-z)F'_1}{F'_1 + F'_2} + F''_{21} \cdot \frac{y(-z)F'_1}{F'_1 + F'_2}\right)\right) : (F'_1 + F'_2)^2 =$$

$$= -\frac{y^2 z^2 (F_2'^2 F''_{11} - 2F'_1 \cdot F'_2 \cdot F''_{12} + F_1'^2 F''_{22})}{(xF'_1 + yF'_2)^3} - 2z(xF'_1 + yF'_2)F_1'^2$$

?

2) $d^2 z$; если а) $F(x+z, y+z) = 0$

$$F_1'(dx+dz) + F_2'(dy+dz) = 0 \to F_1' dx + F_1' dz + F_2' dy + F_2' dz = 0 \to dz = -\frac{F_1' dx + F_2' dy}{F_1' + F_2'}$$

$$\left(F_{11}''(dx+dz) + F_{12}''(dy+dz)\right)(dx+dz) + F_1'(d^2x + d^2z) + \left(F_{21}''(dx+dz) + F_{22}''(dy+dz)\right)(dy+dz) +$$
$$+ F_2'(d^2y + d^2z) = 0$$

$$(F_{11}'' dx + F_{11}'' dz + F_{12}'' dy + F_{12}'' dz)(dx+dz) = F_{11}'' dx^2 + F_{11}'' dx dz + F_{12}'' dx dy + F_{12}'' dx dz +$$
$$+ F_{11}'' dx dz + F_{11}'' dz^2 + F_{12}'' dy dz + F_{12}'' dz^2 =$$
$$= F_{11}''(dx^2 + dx dz + dx dz + dz^2) + F_{12}''(dx dy + dx dz + dy dz + dz^2) =$$
$$= F_{11}''(dx+dz)^2 + F_{12}''\left(dx(dy+dz) + dz(dy+dz)\right)$$

$$(F_1'(d^2x + d^2z)) = F_1' d^2x + F_1' d^2z$$

$$F_2'(d^2y + d^2z) = F_2' d^2y + F_2' d^2z$$

$$\left(F_{21}''(dx+dz) + F_{22}''(dy+dz)\right)(dy+dz) = (F_{21}'' dx + F_{21}'' dz + F_{22}'' dy + F_{22}'' dz)(dy+dz) =$$
$$= F_{21}'' dx dy + F_{21}'' dx dz + F_{21}'' dz dy + F_{21}'' dz^2 + F_{22}'' dy^2 + F_{22}'' dy dz + F_{22}'' dz dy + F_{22}'' dz^2 =$$
$$= F_{21}''(dx dy + dx dz + dz dy + dz^2) + F_{22}''(dy^2 + 2 dy dz + dz^2) =$$
$$= F_{22}''(dy+dz)^2 + F_{21}''\left(dx(dy+dz) + dz(dy+dz)\right)$$

$$d^2 z = -\frac{d(F_1' dx + F_2' dy)(F_1' + F_2') - d(F_1' + F_2')(F_1' dx + F_2' dy)}{(F_1' + F_2')^2}$$

Дифференцирование систем неявно заданных функций.

3402.(а) а) $\frac{dx}{dz}$; $\frac{dy}{dz}$, $\frac{d^2x}{dz^2}$ и $\frac{d^2y}{dz^2}$, при $x=1$, $y=-1$, $z=2$, если

$$\begin{cases} x^2+y^2=\frac{1}{2}z^2 \\ x+y+z=2 \end{cases} \qquad \frac{dx}{dz} \to \quad \begin{matrix} x=x(z) \\ y=y(z) \end{matrix}$$

$$\begin{cases} 2x\frac{dx}{dz}+2y\frac{dy}{dz}=z \\ \frac{dx}{dz}+\frac{dy}{dz}+1=0 \end{cases} \qquad \begin{cases} 2\frac{dx}{dz}-2\frac{dy}{dz}=2 \;\Big|:2 \\ \frac{dx}{dz}+\frac{dy}{dz}=-1 \end{cases}$$

$$+\begin{cases} \frac{dx}{dz}-\frac{dy}{dz}=1 \\ \frac{dx}{dz}+\frac{dy}{dz}=-1 \end{cases} \to 2\frac{dx}{dz}=0 \to \frac{dx}{dz}=0$$

$$-2\frac{dy}{dz}=2 \Rightarrow \frac{dy}{dz}=-1.$$

$$\begin{cases} x\frac{dx}{dz}+y\frac{dy}{dz}=\frac{z}{2} \\ \frac{dx}{dz}+\frac{dy}{dz}=-1 \end{cases} \to \begin{cases} \left(\frac{dx}{dz}\right)^2+x\frac{d^2x}{dz^2}+\left(\frac{dy}{dz}\right)^2+y\cdot\frac{d^2y}{dz^2}=\frac{1}{2} \\ \frac{d^2x}{dz^2}+\frac{d^2y}{dz^2}=0 \end{cases}$$

$$\begin{cases} 0+\frac{d^2x}{dz^2}+1-\frac{d^2y}{dz^2}=\frac{1}{2} \\ \frac{d^2x}{dz^2}+\frac{d^2y}{dz^2}=0 \end{cases} ;\; +\begin{cases} \frac{d^2x}{dz^2}-\frac{d^2y}{dz^2}=-\frac{1}{2} \\ \frac{d^2x}{dz^2}+\frac{d^2y}{dz^2}=0 \end{cases}$$

$$2\frac{d^2x}{dz^2}=-\frac{1}{2} \to \frac{d^2x}{dz^2}=-\frac{1}{4}$$

$$-2\frac{d^2y}{dz^2}=-\frac{1}{2} \to \frac{d^2y}{dz^2}=\frac{1}{4}$$

Ответ: а) $\frac{dx}{dz}=0$; $\frac{dy}{dz}=-1$; $\frac{d^2x}{dz^2}=-\frac{d^2y}{dz^2}=-\frac{1}{4}$

3402.(б) Найти $\frac{\partial u}{\partial x}$; $\frac{\partial u}{\partial y}$, $\frac{\partial v}{\partial x}$ и $\frac{\partial v}{\partial y}$, если

$$\begin{cases} xu-yv=0 \\ yu+xv=1 \end{cases} \qquad \begin{matrix} u=u(x,y) \\ v=v(x,y) \end{matrix} \qquad \{\, d(uv)=v\,du+u\,dv \,\}$$

$$\begin{cases} u\,dx+x\,du-v\,dy-y\,dv=0 \\ y\,du+u\,dy+v\,dx+x\,dv=0 \end{cases}$$

$$\begin{cases} x\,du-y\,dv=v\,dy-u\,dx \\ y\,du+x\,dv=-v\,dx-u\,dy \end{cases}$$

$$\Delta = \begin{vmatrix} x & -y \\ y & x \end{vmatrix} = x^2+y^2.$$

$$\Delta_{du} = \begin{vmatrix} (v\,dy - u\,dx) & -y \\ (-v\,dx - u\,dy) & x \end{vmatrix} = x(v\,dy - u\,dx) - y(v\,dx - u\,dy) = dx(-xu - yv) + dy(xv - yu)$$

$$du = \frac{\Delta du}{\Delta} = \frac{dx(-xu - yv) + dy(xv - yu)}{x^2+y^2} = \underbrace{\frac{xu+yv}{x^2+y^2}}_{\frac{\partial u}{\partial x}}dx + \underbrace{\frac{xv-yu}{x^2+y^2}}_{\frac{\partial u}{\partial y}}dy, \quad x^2+y^2 \neq 0$$

$$\frac{\partial u}{\partial x} = \frac{xu+yv}{x^2+y^2}; \quad \frac{\partial u}{\partial y} = \frac{xv-yu}{x^2+y^2}$$

$$\Delta dv = \begin{vmatrix} x & (v\,dy - u\,dx) \\ y & (-v\,dx - u\,dy) \end{vmatrix} = -x(v\,dx + u\,dy) - y(v\,dy - u\,dx) = dx(-xv+yu) +$$
$$+ dy(-xu - yv)$$

$$dv = \underbrace{\frac{yu - xv}{x^2+y^2}}_{\frac{dv}{dx}}dx - \underbrace{\frac{xu+yv}{x^2+y^2}}_{\frac{dv}{dy}}dy.$$

$$\frac{\partial v}{\partial x} = \frac{yu - xv}{x^2+y^2}, \quad \frac{\partial v}{\partial y} = -\frac{xu+yv}{x^2+y^2}$$

3407. В какой области плоскости xOy система уравнений

$$\begin{cases} x = u+v \\ y = u^2+v^2 \\ z = u^3+v^3 \end{cases}$$

где u и v принимают всевозможные вещественные значения, опр. z как функцию от переменных x и y.

$$\frac{\partial z}{\partial x}; \frac{\partial z}{\partial y} - ?$$

$$u = u(x, y)$$
$$v = v(x, y)$$

$$\frac{\partial z}{\partial x} = 3u^2 \cdot \frac{\partial u}{\partial x} + 3v^2 \frac{\partial v}{\partial x}$$

$$\frac{\partial z}{\partial y} = 3u^2 \frac{\partial u}{\partial y} + 3v^2 \frac{\partial v}{\partial y}$$

Дифференцируем по x первое и второе уравнения системы.

$$\begin{cases} 1 = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial x} \\ 0 = 2u\frac{\partial u}{\partial x} + 2v\frac{\partial v}{\partial x} \end{cases}$$

$\Delta = \begin{vmatrix} 1 & 1 \\ u & v \end{vmatrix} = v - u \quad ; \quad v \neq 0$

$\frac{\partial u}{\partial x} = \frac{1}{v-u} \begin{vmatrix} 1 & 1 \\ 0 & v \end{vmatrix} = \frac{v}{v-u}$

$\frac{\partial v}{\partial x} = \frac{1}{v-u} \begin{vmatrix} 1 & 1 \\ u & 0 \end{vmatrix} = -\frac{u}{v-u}$

$\begin{cases} 0 = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} \\ 1 = 2u \frac{\partial u}{\partial y} + 2v \frac{\partial v}{\partial y} \end{cases}$

$\Delta = \begin{vmatrix} 1 & 1 \\ 2u & 2v \end{vmatrix} = 2(v-u)$

$\frac{\partial u}{\partial y} = \frac{1}{2(v-u)} \cdot \begin{vmatrix} 0 & 1 \\ 1 & 2v \end{vmatrix} = \frac{-1}{2(v-u)} = \frac{1}{2(u-v)}$

$\frac{\partial v}{\partial y} = \frac{1}{2(v-u)} \cdot \begin{vmatrix} 1 & 0 \\ 2u & 1 \end{vmatrix} = \frac{1}{2(v-u)}$

$\frac{\partial z}{\partial x} = 3u^2 \cdot \frac{v}{v-u} + 3v^2 \cdot \left(-\frac{u}{v-u}\right) = \frac{3u^2v - 3v^2u}{v-u} = -\frac{3uv(u-v)}{\cancel{v-u}} =$

$= -3uv$

$\frac{\partial z}{\partial y} = 3u^2 \cdot \frac{1}{2(u-v)} + 3v^2 \cdot \frac{1}{2(v-u)} = \frac{3}{2}(u+v)$

$u \neq v$

$x \neq 2u$

$y \neq 2u^2$

$y = 2\frac{x^2}{4} = \frac{x^2}{2}$

$\begin{cases} x = u + v \\ y = u^2 + v^2 \end{cases} \qquad v = u - x$

$y = u^2 + (u-x)^2 = u^2 + u^2 - 2ux + x^2 = 2u^2 + 2xu + x^2$

$y = 2u^2 - 2xu + x^2$

$2u^2 - 2x \quad u + x^2 - y = 0$

$\frac{D}{4} = \left(\frac{b}{2}\right)^2 - ac = x^2 - 2(x^2 - y) = x^2 - 2x^2 + 2y = 2y - x^2 \geq 0$

$y \geq \frac{x^2}{2}$

Чтобы $z = z(x, y)$ при $y > \frac{x^2}{2}$

2 способ с помощью дифференциалов

3408. (в)

$$\frac{\partial^2 z}{\partial x^2}, \text{ если } \begin{cases} x = \cos\varphi \cos\psi \\ y = \cos\varphi \sin\psi \\ z = \sin\varphi \end{cases} \longrightarrow \begin{matrix} \varphi = \varphi(x, y) \\ \psi = \psi(x, y) \end{matrix}$$

$$\frac{\partial z}{\partial x} = \cos\varphi \cdot \frac{\partial \varphi}{\partial x}$$

$$\begin{cases} 1 = -\sin\varphi \cdot \frac{\partial\varphi}{\partial x} \cdot \cos\psi + \sin\psi \cdot \cos\varphi \cdot \frac{\partial\psi}{\partial x} \\ 0 = -\sin\varphi \cdot \frac{\partial\varphi}{\partial x} \sin\psi + \cos\varphi \cdot \cos\varphi \cdot \frac{\partial\psi}{\partial x} \end{cases}$$

Метод Крамера:

$$\Delta = \begin{vmatrix} -\sin\varphi \cos\psi & -\sin\psi\cos\varphi \\ -\sin\varphi \sin\psi & \cos\varphi\cos\psi \end{vmatrix} = -\sin\varphi \overset{\cos\varphi}{\vee} \cos^2\psi - \sin\varphi \sin^2\psi \cos\varphi =$$

$$= -\sin\varphi\cos\varphi(\cos^2\psi + \sin^2\psi) = -\sin\varphi\cos\varphi$$

$$\frac{\partial\varphi}{\partial x} = -\frac{1}{\sin\varphi\cos\varphi} \begin{vmatrix} 1 & -\sin\psi\cos\varphi \\ 0 & \cos\varphi\cos\psi \end{vmatrix} = -\frac{\cos\psi}{\sin\varphi}$$

$$\frac{\partial\psi}{\partial x} = -\frac{1}{\sin\varphi\cos\varphi} \begin{vmatrix} -\sin\varphi\cos\psi & 1 \\ -\sin\varphi\sin\psi & 0 \end{vmatrix} = -\frac{\sin\varphi \sin\psi}{\sin\varphi \cos\varphi} = -\frac{\sin\psi}{\cos\varphi}$$

$$\frac{\partial z}{\partial x} = \cos\varphi\left(-\frac{\cos\psi}{\sin\varphi}\right) = -\operatorname{ctg}\varphi\cos\psi$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{\cos\psi \cdot \frac{\partial\varphi}{\partial x}}{\sin^2\varphi} + \sin\psi \cdot \operatorname{ctg}\varphi \cdot \frac{\partial\psi}{\partial x} = \frac{\cos\psi}{\sin^2\varphi}\left(-\frac{\cos\psi}{\sin\varphi}\right) - \frac{\cos\varphi\sin^2\psi}{\sin\varphi\cos\varphi} =$$

$$= \frac{-\cos^2\psi - \sin^2\psi\sin^2\varphi}{\sin^3\varphi} = -\frac{\cos^2\psi + \sin^2\psi \cdot \sin^2\varphi}{\sin^3\varphi}$$

34.10.

$$z = z(x, y)$$

$$\begin{cases} x = e^{u+v} \\ y = e^{u-v} \\ z = uv \end{cases} \quad \text{Найти } dz,\ d^2z \quad u = v = 0$$

$$\begin{cases} dz = v\,du + u\,dv \\ dx = e^{u+v}(du + dv) \\ dy = e^{u-v}(du - dv) \end{cases}$$

$$+\begin{cases} du + dv = e^{-u-v} \cdot dx \\ du - dv = e^{-u+v} \cdot dy \end{cases}$$

$$\to du = \frac{1}{2}e^{-u}\left(e^{-v}dx + e^{v}dy\right)$$

$$dv = -\frac{1}{2}e^{-u}\left(e^{-v}dx + e^{v}dy\right)$$

При $u = v = 0$

$$du = \frac{1}{2}(dx + dy)$$

$$dv = \frac{1}{2}(dx - dy)$$

$$d^2z = dv\,du + v\,d^2u^{\to 0} + du\,dv + u\,d^2v^{\to 0} = 2du\,dv =$$

$$= 2\left(\frac{dx+dy}{2}\right)\left(\frac{dx-dy}{2}\right) = dx^2 - dy^2$$

Д/З: ~~3401~~, ~~3404~~, ~~3408 (а)~~, ~~3409~~, ~~3413~~, 3415 (а, б)

Домашняя работа

3401.

$$\begin{cases} x+y+z=0 \\ x^2+y^2+z^2=1 \end{cases} \qquad \frac{dx}{dz} - ? \quad \frac{dy}{dz} - ?$$

$x = x(z)$
$y = y(z)$

$$\begin{cases} \frac{dx}{dz} + \frac{dy}{dz} = -1 \quad \to \quad \frac{dx}{dz} = -1 - \frac{dy}{dz} \\ 2x\frac{dx}{dz} + 2y\frac{dy}{dz} + 2z = 0 \quad |:2 \end{cases}$$

$$x\left(-1 - \frac{dy}{dz}\right) + \frac{y\,dy}{dz} = -z$$

$$-x - \frac{x\,dy}{dz} + \frac{y\,dy}{dz} = -z$$

$$\frac{dy}{dz}(y-x) = x - z \quad \Rightarrow \quad \frac{dy}{dz} = \frac{x-z}{y-x} = \frac{z-x}{x-y}$$

$$\frac{dx}{dz} = -1 - \frac{x-z}{y-x} = \frac{-y+x-x+z}{y-x} = \frac{z-y}{y-x} = \frac{y-z}{x-y}$$

Ответ: $\frac{dx}{dz} = \frac{y-z}{x-y}$; $\frac{dy}{dz} = \frac{z-x}{x-y}$

3408 (а)

$$\begin{cases} x = u + \ln v \\ y = v - \ln u \\ z = 2u + v \end{cases} \qquad \frac{\partial z}{\partial x} - ? \text{ и } \frac{\partial z}{\partial y} \text{ в точке } u=1,\ v=1$$

$u = u(x, y)$
$v = v(x, y)$

$$\frac{\partial z}{\partial x} = 2\frac{\partial u}{\partial x} + \frac{\partial v}{\partial x}$$

$$\begin{cases} 1 = \dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial x}\cdot\dfrac{1}{v} \\ 0 = \dfrac{\partial v}{\partial x} - \dfrac{1}{u}\dfrac{\partial u}{\partial x} \end{cases} \iff \begin{cases} \dfrac{1}{v}\dfrac{\partial v}{\partial x} + \dfrac{\partial u}{\partial x} = 1 \\ \dfrac{\partial v}{\partial x} - \dfrac{1}{u}\cdot\dfrac{\partial u}{\partial x} = 0 \end{cases}$$

Метод Крамера:

$$\Delta = \begin{vmatrix} \frac{1}{v} & 1 \\ 1 & -\frac{1}{u} \end{vmatrix} = -\frac{1}{uv} - 1 = \frac{-1-uv}{uv} = -\frac{(1+uv)}{uv}$$

$(\neq 0)$

$$\frac{\partial v}{\partial x} = -\frac{uv}{(1+uv)}\begin{vmatrix} 1 & 1 \\ 0 & -\frac{1}{u} \end{vmatrix} = \frac{v}{1+uv}$$

$u=1$; $v=1$: $\dfrac{\partial v}{\partial x} = \dfrac{1}{2}$

$$\frac{\partial u}{\partial x} = -\frac{uv}{(1+uv)}\cdot\begin{vmatrix} \frac{1}{v} & 1 \\ 1 & 0 \end{vmatrix} = -\frac{uv}{(1+uv)}\cdot(-1) = \frac{uv}{1+uv}$$

$u=1$; $v=1$; $\dfrac{\partial u}{\partial x} = \dfrac{1}{2}$

$$\Rightarrow \frac{\partial z}{\partial x} = 2\frac{\partial u}{\partial x} + \frac{\partial v}{\partial x} = 2\cdot\frac{1}{2} + \frac{1}{2} = \frac{3}{2}$$

$$\frac{\partial z}{\partial y} = 2\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

$$\begin{cases} 0 = \dfrac{\partial u}{\partial y} + \dfrac{1}{v}\cdot\dfrac{\partial v}{\partial y} \\ 1 = \dfrac{\partial v}{\partial y} - \dfrac{1}{u}\cdot\dfrac{\partial u}{\partial y} \end{cases} \iff \begin{cases} \dfrac{1}{v}\cdot\dfrac{\partial v}{\partial y} + \dfrac{\partial u}{\partial y} = 0 \\ \dfrac{\partial v}{\partial y} - \dfrac{1}{u}\cdot\dfrac{\partial u}{\partial y} = 1 \end{cases}$$

$$\Delta = \begin{vmatrix} \frac{1}{v} & 1 \\ 1 & -\frac{1}{u} \end{vmatrix} = -\frac{1}{uv} - 1 = \frac{-1-uv}{uv} = \frac{-(1+uv)}{uv}$$

$(\neq 0)$

$$\frac{\partial v}{\partial y} = -\frac{uv}{1+uv}\cdot\begin{vmatrix} 0 & 1 \\ 1 & -\frac{1}{u} \end{vmatrix} = -\frac{uv}{1+uv}\cdot(-1) = \frac{uv}{1+uv}$$

$u=1$, $v=1$: $\dfrac{\partial v}{\partial y} = \dfrac{1}{2}$

$$\frac{\partial u}{\partial y} = \frac{-uv}{1+uv}\cdot\begin{vmatrix} \frac{1}{v} & 0 \\ 1 & 1 \end{vmatrix} = -\frac{u}{1+uv}$$

$u=1$; $v=1$: $\dfrac{\partial u}{\partial y} = -\dfrac{1}{2}$

$$\frac{\partial z}{\partial y} = -\frac{1}{2}\cdot 2 + \frac{1}{2} = -1 + \frac{1}{2} = -\frac{1}{2}$$

Ответ: $\dfrac{\partial z}{\partial x} = \dfrac{3}{2}$

$\dfrac{\partial z}{\partial y} = -\dfrac{1}{2}$

3409. $\frac{\partial^2 z}{\partial x^2}$, $\frac{\partial^2 z}{\partial x \partial y}$ и $\frac{\partial^2 z}{\partial y^2}$, если:

$$\begin{cases} x = u\cos v \\ y = u\sin v \\ z = v \end{cases} \Rightarrow \begin{cases} dx = du\cdot\cos v - \sin v\, dv\cdot u \\ dy = \sin v\, du + \cos v\cdot dv\cdot u \end{cases}$$

$$\begin{cases} \cos v\cdot du - u\cdot\sin v\, dv = dx \\ \sin v\cdot du + \cos v\cdot u\, dv = dy \end{cases}$$

$$\Delta = \begin{vmatrix} \cos v & -u\sin v \\ \sin v & u\cos v \end{vmatrix} = u\cos^2 v + u\sin^2 v =$$

$$= u\underbrace{(\cos^2 v + \sin^2 v)}_{=1} = u$$

$$du = \frac{1}{u}\cdot\begin{vmatrix} dx & -u\sin v \\ dy & u\cos v \end{vmatrix} = \frac{dx\, u\cos v + u\sin v\, dy}{u} = \cos v\, dx + \sin v\, dy$$

$$dv = \frac{1}{u}\cdot\begin{vmatrix} \cos v & dx \\ \sin v & dy \end{vmatrix} = \frac{1}{u}\cdot(\cos v\, dy - \sin v\, dx)$$

$$u\, dv = \cos v\, dy - \sin v\, dx$$

$$u\, d^2 v + du\, dv = -\sin v\, dv\, dy - \cos v\, dv\, dx$$

$$z = v \to dz = dv \to d^2 z = d^2 v$$

$$u\, d^2 v + du\, dv = -\cos v\, dv\, dx - \sin v\, dv\, dy = -dv\underbrace{(\cos v\, dx + \sin v\, dy)}_{=du} = -dv\, du$$

$$\Rightarrow d^2 v = \frac{-2\, dv\, du}{u}$$

$$\Rightarrow d^2 z = \frac{-2\, dv\, du}{u} = -\frac{2}{u^2}(\cos v\, dy - \sin v\, dx)(\cos v\, dx + \sin v\, dy) =$$

$$= \frac{2}{u^2}(\sin v\, dx - \cos v\, dy)(\cos v\, dx + \sin v\, dy) =$$

$$= \frac{2}{u^2}(\sin v\cos v\, dx^2 + \sin^2 v\, dy\, dx - \cos^2 v\, dx\, dy - \cos v\sin v\, dy^2) =$$

$$= \frac{2}{u^2}(\sin v\cos v\, dx^2 - \cos 2v\, dy\cdot dx - \sin v\cos v\, dy^2)$$

$$\Rightarrow \frac{\partial^2 z}{\partial x^2} = \frac{2\sin v\cos v}{u^2} = \frac{\sin 2v}{u^2}$$

$$\frac{\partial^2 z}{\partial y^2} = -\frac{2\cos v\sin v}{u^2} = \frac{-\sin 2v}{u^2}$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{-\cos 2v}{u^2}$$

3404. Найти du, dv, d^2u и d^2v

$$\begin{cases} u+v=x+y \\ \dfrac{\sin u}{\sin v}=\dfrac{x}{y} \end{cases}$$

$du+dv=dx+dy$

$\rightarrow \sin u \cdot y=\sin v \cdot x$

$$\cancel{\frac{\cos u\, du \sin v-\cos v\, dv \cdot \sin u}{\sin^2 v}=\frac{dx\, y-dy\cdot x}{y^2}}$$

$$\cos u\cdot y\, du+dy \sin u=\cos v\, dv\cdot x+\sin v\, dx$$

$$\begin{cases} du+dv=dx+dy \\ \cos u\cdot y\, du-\cos v\cdot x\, dv=\sin v\, dx-\sin u\, dy. \end{cases}$$

$$\Delta=\begin{vmatrix} 1 & 1 \\ \cos u\cdot y & -\cos v\cdot x \end{vmatrix}=-\cos v\cdot x-\cos u\cdot y=-(\cos v\cdot x+\cos u\cdot y)$$

$$du=\frac{-1}{\cos v\cdot x+\cos u\cdot y}\cdot\begin{vmatrix} (dx+dy) & 1 \\ (\sin v\cdot dx-\sin u\, dy) & (-\cos v x) \end{vmatrix}=$$

$$=\frac{-1}{(\cos v\cdot x+\cos u\cdot y)}\cdot\left(-\cos v\cdot x\,(dx+dy)-(\sin v\cdot dx-\sin u\, dy)\right)=$$

$$=\frac{1}{\cos v\cdot x+\cos u\cdot y}\left(\cos v\cdot x(dx+dy)+\sin v\, dx\ \ \sin u\, dy\right)=$$

$$=\frac{x\cos v\, dx+x\cos v\, dy+\sin v\, dx-\sin u\, dy}{x\cos v+y\cos u}=$$

$$=\frac{dx\,(x\cos v+\sin v)+dy\,(x\cos v-\sin u)}{x\cos v+y\cos u}$$

$$dv=\frac{-1}{\cos v\cdot x+\cos u\cdot y}\cdot\begin{vmatrix} 1 & (dx+dy) \\ y\cdot\cos u & \sin v\, dx-\sin u\, dy \end{vmatrix}=$$

$$=\frac{-1}{x\cdot\cos v+y\cos u}\cdot\left(\sin v\, dx-\sin u\, dy-y\cos u\,(dx+dy)\right)=$$

$$=\frac{(y\cos u\,(dx+dy)+\sin u\, dy-\sin v\, dx)}{x\cos v+y\cos u}=$$

$$=\frac{y\cos u\, dx+y\cos u\, dy+\sin u\, dy-\sin v\, dx}{x\cos v+y\cos u}=$$

$$=\frac{dx\,(y\cos u-\sin v)+dy\,(y\cos u+\sin u)}{x\cos v+y\cos u}.$$

d^2u и d^2v - ?

Аналогично:

$$d^2u + d^2v = \underbrace{d^2x}_{=0} + \underbrace{d^2y}_{=0}$$

$$d^2u + d^2v = 0$$

$$y\cdot\cos u\, du + dy \sin u = x\cos v\, dv + \sin v\, dx$$

$$y\cos u\, d^2u + \cos u\, dy\, du - y\sin u\, du^2 + \underbrace{d^2y \sin u}_{=0} + \cos u\, dy\, du =$$

$$= x\cos v\, d^2v + \cos v\, dx\, dv - x\sin v\, dv^2 + \underbrace{d^2x\sin v}_{=0} + \cos v\, dx\, dv$$

$$y\cos u\, d^2u + 2\cos u\, dy\, du - y\sin u\, du^2 = x\cos v\, d^2v + 2\cos v\, dx\, dv - x\sin v\, dv^2$$

$$\begin{cases} d^2u + d^2v = 0 \\ y\cos u\, d^2u + 2\cos u\, dy\, du - y\sin u\, du^2 = x\cos v\, d^2v + 2\cos v\, dx\, dv - x\sin v\, dv^2 \end{cases}$$

$$d^2u = -d^2v$$

$$y\cos u\, d^2u + 2\cos u\, dy\, du + y\sin u\, du^2 = -x\cos v\, d^2u + 2\cos v\, dx\, dv - x\sin v\, dv^2$$

$$d^2u(y\cos u + x\cos v) = 2\cos v\, dx\, dv - x\sin v\, dv^2 - 2\cos u\, dy\, du - y\sin u\, du^2$$

$$d^2u = \frac{2\cos v\, dx\, dv - x\sin v\, dv^2 - 2\cos u\, dy\, du - y\sin u\, du^2}{y\cos u + x\cos v}, \text{ где}$$

$$dv = \frac{dx(y\cos u - \sin v) + dy(y\cos u + \sin u)}{x\cos v + y\cos u}$$

$$-y\cos u\, d^2v + 2\cos u\, dy\, du - y\sin u\, du^2 = x\cos v\, d^2v + 2\cos v\, dx\, dv - x\sin v\, dv^2$$

$$d^2v(x\cos v + y\cos u) = 2\cos u\, dy\, du - y\sin u\, du^2 + x\sin v\, dv^2 - 2\cos v\, dx\, dv$$

$$d^2v = \frac{2\cos u\, dy\, du - y\sin u\, du^2 + x\sin v\, dv^2 - 2\cos v\, dx\, dv}{x\cos v + y\cos u}$$

3413.

$x = \varphi(u,v)$, $y = \psi(u,v)$, $z = \chi(u,v)$.

Найти $\frac{\partial z}{\partial x}$ и $\frac{\partial z}{\partial y}$.

1) $\frac{\partial}{\partial x}$: $1 = (\varphi'_1)\cdot\frac{\partial u}{\partial x} + (\varphi'_2)\frac{\partial v}{\partial x}$ $\Rightarrow$ $1 = \frac{\partial\varphi}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial\varphi}{\partial v}\cdot\frac{\partial v}{\partial x}$

2) $\frac{\partial}{\partial x}$: $0 = (\psi'_1)\cdot\frac{\partial u}{\partial x} + (\psi'_2)\cdot\frac{\partial v}{\partial x}$ $\Rightarrow$ $0 = \frac{\partial\psi}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial\psi}{\partial v}\cdot\frac{\partial v}{\partial x}$

3) $\frac{\partial}{\partial x}$: $\frac{\partial z}{\partial x} = (\chi'_1)\frac{\partial u}{\partial x} + (\chi'_2)\frac{\partial v}{\partial x}$ $\Rightarrow$ $\frac{\partial z}{\partial x} = \frac{\partial\chi}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial\chi}{\partial v}\cdot\frac{\partial v}{\partial x}$

$$\begin{cases} 1 = \frac{\partial\varphi}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial\varphi}{\partial v}\cdot\frac{\partial v}{\partial x} \\ 0 = \frac{\partial\psi}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial\psi}{\partial v}\cdot\frac{\partial v}{\partial x} \end{cases}$$

$$\Delta = \begin{vmatrix} \frac{\partial\varphi}{\partial u} & \frac{\partial\varphi}{\partial v} \\ \frac{\partial\psi}{\partial u} & \frac{\partial\psi}{\partial v} \end{vmatrix} = \frac{\partial\varphi\cdot\partial\psi}{\partial u\,\partial v} - \frac{\partial\varphi\cdot\partial\psi}{\partial u\cdot\partial v} \neq 0$$

Метод Крамера не применим

$$\Delta = \begin{vmatrix} \frac{\partial \varphi}{\partial u} & \frac{\partial \varphi}{\partial v} \\ \frac{\partial \psi}{\partial u} & \frac{\partial \psi}{\partial v} \end{vmatrix} = \frac{\partial \varphi}{\partial u}\frac{\partial \psi}{\partial v} - \frac{\partial \varphi}{\partial u}\frac{\partial \psi}{\partial v} \neq 0$$ почему?

$$\Rightarrow \frac{\partial u}{\partial x} = \frac{1}{\Delta} \begin{vmatrix} 1 & \frac{\partial \varphi}{\partial v} \\ 0 & \frac{\partial \psi}{\partial v} \end{vmatrix} = \frac{1}{\Delta} \cdot \frac{\partial \psi}{\partial v}$$

$$\frac{\partial v}{\partial x} = \frac{1}{\Delta} \begin{vmatrix} \frac{\partial \varphi}{\partial u} & 1 \\ \frac{\partial \psi}{\partial u} & 0 \end{vmatrix} = \frac{-1}{\Delta} \cdot \frac{\partial \psi}{\partial u}$$

$$\Rightarrow \frac{\partial z}{\partial x} = \frac{\partial \chi}{\partial u} \cdot \frac{1}{\Delta} \cdot \frac{\partial \psi}{\partial v} + \frac{\partial \chi}{\partial v} \cdot \left(-\frac{1}{\Delta}\frac{\partial \psi}{\partial u}\right) =$$

$$= \frac{1}{\Delta}\left(\frac{\partial \chi}{\partial u} \cdot \frac{\partial \psi}{\partial v} - \frac{\partial \chi}{\partial v} \cdot \frac{\partial \psi}{\partial u}\right)$$

Если искать $\frac{\partial z}{\partial y}$, то во втором ур-ии системы будет "1", а в первом "0", тогда ответ будет просто противоположного знака и вместо $\partial \psi \to \partial \varphi$

$$\frac{\partial z}{\partial y} = \frac{1}{\Delta}\left(\frac{\partial \chi}{\partial v} \cdot \frac{\partial \varphi}{\partial u} - \frac{\partial \chi}{\partial u} \cdot \frac{\partial \varphi}{\partial v}\right)$$

т.е. $$\begin{cases} \frac{\partial \varphi}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial \varphi}{\partial v} \cdot \frac{\partial v}{\partial y} = 0 \\ \frac{\partial \psi}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial \psi}{\partial v} \cdot \frac{\partial v}{\partial y} = 1. \end{cases}$$

$$\Delta = \frac{\partial \varphi}{\partial u}\frac{\partial \psi}{\partial v} - \frac{\partial \varphi}{\partial u}\frac{\partial \psi}{\partial v}$$

$$\frac{\partial u}{\partial y} = \frac{1}{\Delta} \begin{vmatrix} 0 & \frac{\partial \varphi}{\partial v} \\ 1 & \frac{\partial \psi}{\partial v} \end{vmatrix} = -\frac{1}{\Delta}\frac{\partial \varphi}{\partial v}$$

$$\frac{\partial v}{\partial y} = \frac{1}{\Delta} \begin{vmatrix} \frac{\partial \varphi}{\partial u} & 0 \\ \frac{\partial \psi}{\partial u} & 1 \end{vmatrix} = \frac{1}{\Delta}\frac{\partial \varphi}{\partial u}$$

3475. (а, б) $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$; $\frac{\partial v}{\partial x}$; $\frac{\partial v}{\partial y}$

а) $$\begin{cases} x = u\cos\frac{v}{u} \\ y = u\sin\frac{v}{u} \end{cases} \qquad \begin{cases} dx = \cos\frac{v}{u}\,du - u \cdot \sin\frac{v}{u} \cdot \frac{u\,dv - du\,v}{u^2} \\ dy = \sin\frac{v}{u}\,du + u\cos\frac{v}{u} \cdot \frac{u\,dv - du\,v}{u^2} \end{cases}$$

$$\begin{cases} dx = \cos\frac{v}{u}\,du - \sin\frac{v}{u}\,dv + \frac{v\sin\frac{v}{u}}{u} \cdot du \\ dy = \sin\frac{v}{u}\,du + \cos\frac{v}{u} \cdot dv - \frac{v\cos\frac{v}{u}}{u}\,du \end{cases}$$

$$\begin{cases} dx = du\left(\cos\frac{v}{u} + \frac{v\sin\frac{v}{u}}{u}\right) - \sin\frac{v}{u}\, dv \\ dy = du\left(\sin\frac{v}{u} - \frac{v\cos\frac{v}{u}}{u}\right) + \cos\frac{v}{u}\, dv \end{cases}$$

4.03.2021.

3459. $y\frac{\partial z}{\partial x} - x\frac{\partial z}{\partial y} = 0$, если $\xi = x$ и $\eta = x^2 + y^2$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial \xi}\cdot\overset{=1}{\frac{\partial \xi}{\partial x}} + \frac{\partial z}{\partial \eta}\cdot\overset{=2x}{\frac{\partial \eta}{\partial x}}$$

$$z = z(x, y) = z\left(x(\xi,\eta),\ y(\xi,\eta)\right) = z(\xi, \eta)$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial \xi}\cdot\overset{=0}{\frac{\partial \xi}{\partial y}} + \frac{\partial z}{\partial \eta}\cdot\overset{=2y.}{\frac{\partial \eta}{\partial y}}$$

$$\frac{\partial \xi}{\partial x} = 1\ ;\ \frac{\partial \xi}{\partial y} = 0\ ;\ \frac{\partial \eta}{\partial x} = 2x\ ;\ \frac{\partial \eta}{\partial y} = 2y$$

$$y\cdot\left(\frac{\partial z}{\partial \xi} + 2x\cdot\frac{\partial z}{\partial \eta}\right) - 2xy\frac{\partial z}{\partial \eta} = 0$$

$$y\cdot\frac{\partial z}{\partial \xi} = 0$$

$$\frac{\partial z}{\partial \xi} = 0 \longrightarrow z = \varphi(\eta) = \varphi(x^2 + y^2)$$

3461. $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = z$, если $\xi = x$ и $\eta = \frac{y}{x}$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial \xi}\cdot\frac{\partial \xi}{\partial x} + \frac{\partial z}{\partial \eta}\cdot\frac{\partial \eta}{\partial x}.$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial \xi}\cdot\frac{\partial \xi}{\partial y} + \frac{\partial z}{\partial \eta}\cdot\frac{\partial \eta}{\partial y}.$$

$$\frac{\partial \xi}{\partial x} = 1\ ;\ \frac{\partial \xi}{\partial y} = 0$$

$$\frac{\partial \eta}{\partial x} = -\frac{y}{x^2}\ ;\ \frac{\partial \eta}{\partial y} = \frac{1}{x}$$

$$\Rightarrow \begin{cases} \frac{\partial z}{\partial x} = \frac{\partial z}{\partial \xi}\cdot 1 + \frac{\partial z}{\partial \eta}\cdot\left(-\frac{y}{x^2}\right) \\ \frac{\partial z}{\partial y} = \frac{\partial z}{\partial \xi}\cdot 0 + \frac{\partial z}{\partial \eta}\cdot\frac{1}{x} \end{cases}$$

$$x\left(\frac{\partial z}{\partial \xi} + -\frac{y}{x^2}\frac{\partial z}{\partial \eta}\right) + \frac{y}{x}\frac{\partial z}{\partial \eta} = z$$

$$x\frac{\partial z}{\partial \xi} = z\ ;\ \xi\frac{\partial z}{\partial \xi} = z \longrightarrow \frac{\partial z}{z} = \frac{\partial \xi}{\xi}$$

$$\ln|z| = \ln|\xi| + \ln|\varphi(\eta)|$$

$$z=\varphi(\eta)\cdot\xi$$

$$z=x\cdot\varphi\left(\frac{y}{x}\right)$$

Ответ: $z=x\cdot\varphi\left(\frac{y}{x}\right)$

3463. $(x+y)\frac{\partial z}{\partial x}-(x-y)\frac{\partial z}{\partial y}=0$, если

$u=\ln\sqrt{x^2+y^2}$ и $v=\operatorname{arctg}\frac{y}{x}$

$$z(x,y)=z(x(u,v),y(v,u))=z(u,v)$$

$$\frac{\partial u}{\partial x}=\frac{1}{\sqrt{x^2+y^2}}\cdot\frac{1}{2}\cdot\frac{1}{\sqrt{x^2+y^2}}\cdot 2x=\frac{x}{x^2+y^2}$$

$$\frac{\partial u}{\partial y}=\frac{y}{x^2+y^2}$$

$$\frac{\partial v}{\partial x}=\frac{1}{1+\frac{y^2}{x^2}}\cdot\left(-\frac{y}{x^2}\right)=-\frac{x^2}{x^2+y^2}\cdot\frac{y}{x^2}=-\frac{y}{x^2+y^2}$$

$$\frac{\partial v}{\partial y}=\frac{x^2}{x^2+y^2}\cdot\frac{1}{x}=\frac{x}{x^2+y^2}$$

$$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x}\quad;\quad\frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y}$$

$$(x+y)\left(\frac{\partial z}{\partial u}\cdot\frac{x}{(x^2+y^2)}+\frac{\partial z}{\partial v}\cdot\left(\frac{-y}{x^2+y^2}\right)\right)-(x-y)\left(\frac{\partial z}{\partial u}\cdot\frac{y}{(x^2+y^2)}+\frac{\partial z}{\partial v}\cdot\frac{x}{x^2+y^2}\right)=0$$

$$(x+y)\left(x\cdot\frac{\partial z}{\partial u}-y\cdot\frac{\partial z}{\partial v}\right)-(x-y)\left(y\cdot\frac{\partial z}{\partial u}+x\frac{\partial z}{\partial v}\right)=0$$

$$\frac{\partial z}{\partial u}\left((x+y)x-(x-y)y\right)+\frac{\partial z}{\partial v}\left(-y(x+y)+x(y-x)\right)=0$$

$$\frac{\partial z}{\partial u}=\frac{\partial z}{\partial v}$$

3468. $\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2$

$x=uv$

$y=\frac{1}{2}(u^2-v^2)$

$$z(x,y)=z(x(u,v);y(u,v))=z(u,v)$$

$$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x}$$

$$\frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y}$$

$$\begin{cases} 1 = v\cdot\dfrac{\partial u}{\partial x} + u\dfrac{\partial v}{\partial x} \\[2ex] 0 = u\cdot\dfrac{\partial u}{\partial x} - v\dfrac{\partial v}{\partial x} \end{cases} \qquad \Delta = \begin{vmatrix} v & u \\ u & -v \end{vmatrix} = -u^2 - v^2 = -(u^2+v^2)$$

$$\frac{\partial u}{\partial x} = -\frac{1}{(u^2+v^2)}\begin{vmatrix} 1 & u \\ 0 & -v \end{vmatrix} = \frac{v}{u^2+v^2}$$

$$\frac{\partial v}{\partial x} = -\frac{1}{u^2+v^2}\begin{vmatrix} v & 1 \\ u & 0 \end{vmatrix} = \frac{u}{u^2+v^2}$$

$$\begin{cases} 0 = v\dfrac{\partial u}{\partial y} + u\dfrac{\partial v}{\partial y} \\[2ex] 1 = u\dfrac{\partial u}{\partial y} - v\dfrac{\partial v}{\partial y} \end{cases} \qquad \Delta = \begin{vmatrix} v & u \\ u & -v \end{vmatrix} = -(u^2+v^2)$$

$$\frac{\partial u}{\partial y} = -\frac{1}{u^2+v^2}\begin{vmatrix} 0 & u \\ 1 & -v \end{vmatrix} = \frac{u}{u^2+v^2}$$

$$\frac{\partial v}{\partial y} = -\frac{1}{u^2+v^2}\begin{vmatrix} v & 0 \\ u & 1 \end{vmatrix} = -\frac{v}{u^2+v^2}$$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\cdot\frac{v}{u^2+v^2} + \frac{\partial z}{\partial v}\cdot\frac{u}{u^2+v^2}$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\cdot\frac{u}{u^2+v^2} - \frac{\partial z}{\partial v}\cdot\frac{v}{u^2+v^2}$$

$$\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = \left(\frac{\partial z}{\partial u}\cdot\frac{v}{u^2+v^2}\right)^2 + \cancel{2\frac{vu}{(u^2+v^2)^2}\cdot\frac{\partial z}{\partial u}\cdot\frac{\partial z}{\partial v}} + \left(\frac{\partial z}{\partial v}\cdot\frac{u}{u^2+v^2}\right)^2 +$$

$$+\left(\frac{\partial z}{\partial u}\,\frac{u}{u^2+v^2}\right)^2 - \cancel{2\frac{uv}{(u^2+v^2)^2}\cdot\frac{\partial z}{\partial u}\cdot\frac{\partial z}{\partial v}} + \left(\frac{\partial z}{\partial v}\cdot\frac{v}{u^2+v^2}\right)^2 =$$

$$= \left(\frac{\partial z}{\partial u}\right)^2\frac{v^2}{(u^2+v^2)^2} + \left(\frac{\partial z}{\partial u}\right)^2\frac{u^2}{(u^2+v^2)^2} + \left(\frac{\partial z}{\partial v}\right)^2\cdot\frac{u^2}{(u^2+v^2)^2} + \left(\frac{\partial z}{\partial v}\right)^2\frac{v^2}{(u^2+v^2)^2}$$

$$= \left(\frac{\partial z}{\partial u}\right)^2\cdot\frac{1}{(u^2+v^2)^2}(v^2+u^2) + \left(\frac{\partial z}{\partial v}\right)^2\cdot\frac{1}{(u^2+v^2)^2}(v^2+u^2) =$$

$$= \frac{v^2+u^2}{(u^2+v^2)^2}\left(\left(\frac{\partial z}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2\right) = \frac{\left(\frac{\partial z}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2}{u^2+v^2}$$

Домашняя работа.

3458 $\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y} = 0$, если $\xi = x+y$ и $\eta = x-y$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial \xi}\cdot\frac{\partial \xi}{\partial x} + \frac{\partial z}{\partial \eta}\cdot\frac{\partial \eta}{\partial x} = \frac{\partial z}{\partial \xi}\cdot 1 + \frac{\partial z}{\partial \eta}\cdot 1$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial \xi}\cdot\frac{\partial \xi}{\partial y} + \frac{\partial z}{\partial \eta}\cdot\frac{\partial \eta}{\partial y} = \frac{\partial z}{\partial \xi}\cdot 1 + \frac{\partial z}{\partial \eta}(-1)$$

$$\left(\frac{\partial z}{\partial \xi} + \frac{\partial z}{\partial \eta}\right) - \frac{\partial z}{\partial \xi} + \frac{\partial z}{\partial \eta} = 0$$

$$\frac{2\partial z}{\partial \eta} = 0 \quad \Rightarrow \frac{\partial z}{\partial \eta} = 0 \quad \Rightarrow z = \varphi(\xi) = \varphi(x+y)$$

Ответ: $z = \varphi(x+y)$

3462. $x\frac{\partial z}{\partial x} + \sqrt{1+y^2}\frac{\partial z}{\partial y} = xy$, если $u = \ln x$ и $v = \ln(y+\sqrt{1+y^2})$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\cdot\frac{1}{x} + \frac{\partial z}{\partial v}\cdot 0$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial y}\cdot 0 + \frac{\partial z}{\partial v}\cdot\frac{1}{y+\sqrt{1+y^2}}\cdot\left(1+\frac{2y}{2\sqrt{1+y^2}}\right)$$

$$x\left(\frac{1}{x}\cdot\frac{\partial z}{\partial u}\right) + \sqrt{1+y^2}\cdot\left(\frac{\sqrt{1+y^2}+y}{(y+\sqrt{1+y^2})\cdot\sqrt{1+y^2}}\right)\frac{\partial z}{\partial v} = xy$$

$$\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v} = xy$$

$u = \ln x \quad \rightarrow x = e^u$

$v = \ln(y+\sqrt{1+y^2})$

$y+\sqrt{1+y^2} = e^v$

$\sqrt{1+y^2} = e^v - y \quad {}^2 \Rightarrow 1+y^2 = e^{2v} - 2e^v y + y^2$

$2e^v y = e^{2v} - 1$

$$y = \frac{e^{2v}-1}{2e^v} = \frac{e^v - e^{-v}}{2} = \operatorname{sh} v$$

$$\Rightarrow \frac{\partial z}{\partial u} + \frac{\partial z}{\partial v} = e^u \operatorname{sh} v$$

Ответ: $\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v} = e^u \operatorname{sh} v$

3465. $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = \frac{x}{z}$, если $u = 2x - z^2 \rightarrow x = \frac{z^2+u}{2}$

$v = \frac{y}{z} \rightarrow y = zv \qquad 2x = z^2+u$

$$\begin{cases} du = 2dx - 2z\,dz \\ dv = \dfrac{z\,dy - y\,dz}{z^2} \end{cases} \iff \begin{cases} du = 2dx - 2z\,dz \\ dv = \dfrac{dy}{z} - \dfrac{y\,dz}{z^2} \end{cases}$$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x}$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y}$$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\cdot\left(2 - 2z\frac{\partial z}{\partial x}\right) + \frac{\partial z}{\partial v}\left(-\frac{y}{z^2}\cdot\frac{\partial z}{\partial x}\right) = 2\frac{\partial z}{\partial u} - 2z\frac{\partial z}{\partial u}\frac{\partial z}{\partial x} - \frac{y}{z^2}\frac{\partial z}{\partial v}\cdot\frac{\partial z}{\partial x}$$

$$\frac{\partial z}{\partial x}\left(1 + 2z\frac{\partial z}{\partial u} + \frac{y}{z^2}\cdot\frac{\partial z}{\partial v}\right) = 2\frac{\partial z}{\partial u} \rightarrow \frac{\partial z}{\partial x} = \frac{2\frac{\partial z}{\partial u}}{\left(1 + 2z\frac{\partial z}{\partial u} + \frac{y}{z^2}\cdot\frac{\partial z}{\partial v}\right)}$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\left(-2z\frac{\partial z}{\partial y}\right) + \frac{\partial z}{\partial v}\left(\frac{z - y\frac{\partial z}{\partial y}}{z^2}\right)$$

$$\frac{\partial z}{\partial y} = -2z\frac{\partial z}{\partial u}\frac{\partial z}{\partial y} + \frac{1}{z}\cdot\frac{\partial z}{\partial v} - \frac{y}{z^2}\frac{\partial z}{\partial v}\cdot\frac{\partial z}{\partial y}$$

$$\frac{\partial z}{\partial y}\left(1 + 2z\frac{\partial z}{\partial u} + \frac{y}{z^2}\cdot\frac{\partial z}{\partial v}\right) = \frac{1}{z}\cdot\frac{\partial z}{\partial v}$$

$$\Rightarrow \frac{\partial z}{\partial y} = \frac{\frac{1}{z}\cdot\frac{\partial z}{\partial v}}{\left(1 + 2z\frac{\partial z}{\partial u} + \frac{y}{z^2}\cdot\frac{\partial z}{\partial v}\right)}$$

$$x\cdot\frac{2\frac{\partial z}{\partial u}}{\left(1 + 2z\frac{\partial z}{\partial u} + \frac{y}{z^2}\cdot\frac{\partial z}{\partial v}\right)} + y\cdot\frac{\frac{1}{z}\cdot\frac{\partial z}{\partial v}}{\left(1 + 2z\frac{\partial z}{\partial u} + \frac{y}{z^2}\cdot\frac{\partial z}{\partial v}\right)} = \frac{x}{z}$$

$$2x\frac{\partial z}{\partial u} + \frac{y}{z}\cdot\frac{\partial z}{\partial v} = \frac{x}{z}\left(1 + 2z\cdot\frac{\partial z}{\partial u} + \frac{y}{z^2}\cdot\frac{\partial z}{\partial v}\right)$$

$$\cancel{2x\frac{\partial z}{\partial u}} + \frac{y}{z}\cdot\frac{\partial z}{\partial v} = \frac{x}{z} + \cancel{2x\frac{\partial z}{\partial u}} + \frac{xy}{z^3}\frac{\partial z}{\partial v} \quad \Big|\cdot z$$

$$y\cdot\frac{\partial z}{\partial v} = x + \frac{xy}{z^2}\cdot\frac{\partial z}{\partial v}$$

$$\frac{\partial z}{\partial v}\left(y - \frac{xy}{z^2}\right) = x \rightarrow \frac{\partial z}{\partial v} = \frac{x}{\left(y - \frac{xy}{z^2}\right)} = \frac{z^2 x}{yz^2 - xy}$$

$$\frac{\partial z}{\partial v} = \frac{z^2\cdot(z^2+u)}{2\left(zv\cdot z^2 - \frac{z^2+u}{2}\cdot zv\right)} = \frac{z(z^2+u)\cdot\cancel{2}}{\cancel{2}(2z^2v - z^2v - uv)} = \frac{z}{v}\cdot\frac{z^2+u}{z^2-u}$$

Ответ: $\dfrac{\partial z}{\partial v} = \dfrac{z}{v}\cdot\dfrac{z^2+u}{z^2-u}$

3489. $\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}=0$

$\xi=x\,;\quad \eta=y-x\,;\quad \zeta=z-x$

$\frac{\partial u}{\partial x}=\frac{\partial u}{\partial \xi}\cdot 1+\frac{\partial u}{\partial \eta}(-1)+\frac{\partial u}{\partial \zeta}(-1)=\frac{\partial u}{\partial \xi}-\frac{\partial u}{\partial \eta}-\frac{\partial u}{\partial \zeta}$

$\frac{\partial u}{\partial y}=\frac{\partial u}{\partial \xi}\cdot 0+\frac{\partial u}{\partial \eta}\cdot 1+\frac{\partial u}{\partial \zeta}\cdot 0=\frac{\partial u}{\partial \eta}$

$\frac{\partial u}{\partial z}=\frac{\partial u}{\partial \xi}\cdot 0+\frac{\partial u}{\partial \eta}\cdot 0+\frac{\partial u}{\partial \zeta}\cdot 1=\frac{\partial u}{\partial \zeta}$

$\Rightarrow \frac{\partial u}{\partial \xi}-\cancel{\frac{\partial u}{\partial \eta}}-\cancel{\frac{\partial u}{\partial \zeta}}+\cancel{\frac{\partial u}{\partial \eta}}+\cancel{\frac{\partial u}{\partial \zeta}}=0$

$\frac{\partial u}{\partial \xi}=0$

Ответ: $\frac{\partial u}{\partial \xi}=0$

Д/З: ~~3621~~, ~~3622~~, ~~3625~~, ~~3628~~, 3643, 3644, ? ~~дрез. 3677~~

Экстремум функции многих переменных.

Пусть $u = f(M)$ опр-на в некоторой окрестности (.) $M_0 \in R^m$

Опр. Говорят, что функция $u = f(M)$ имеет в точке M_0 локальный максимум (минимум), если $\exists$ ε-окрестность (.) M_0, в которой $f(M_0) > f(M)$ $\left(f(M_0) < f(M)\right)$

Необх. условие экстремума: Если в точке $M_0(x_1^0, \dots, x_m^0)$ функция $u = f(x_1, \dots, x_m)$ имеет локальный экстремум и если в (.) M_0 $\exists$ частная производная $\frac{\partial u}{\partial x_k}$, $k = \overline{1, m}$, то $\frac{\partial u}{\partial x_k}(M_0) = 0$

Следствие: Если ф-я $u = F(M)$ имеет в (.) M_0 локальный экстремум и дифференцируема в (.) M_0, то

$$du\Big|_{M_0} = \frac{\partial u}{\partial x_1}(M_0)\,dx_1 + \dots + \frac{\partial u}{\partial x_m}(M_0)\,dx_m = 0$$

Достаточное условие экстремума:

~ Функция $u(M)$ имеет в точке M_0 локальный максимум (минимум) если

1) $du\big|_{M_0} = 0$

2) $d^2u\big|_{M_0} < 0$, то лок. максимум ($d^2u\big|_{M_0} > 0$ - минимум)

3624.

$$z = x^2 - xy + y^2 - 2x + y$$

$$dz = (2x - y - 2)\,dx + (2y - x + 1)\,dy = 0$$

$$\begin{cases} 2x - y - 2 = 0 \\ 2y - x + 1 = 0 \end{cases} \Longleftrightarrow \begin{cases} 2x - y = 2 \\ -x + 2y = -1 \quad | \cdot 2 \end{cases}$$

$$3y = 0 \qquad \begin{cases} y = 0 \\ x = 1 \end{cases} \qquad M_0(1; 0)$$

$$d^2z = d(dz) = (2dx - dy)\,dx + (2dy - dx)\,dy = 2dx^2 - 2dx\,dy + 2dy^2 =$$

$$= 2(dx^2 - dx\,dy + dy^2) = \cancel{2(dx - dy)^2} = 2\Big(\underbrace{dx^2 - 2\cdot\tfrac{1}{2}\,dx\,dy + \tfrac{1}{4}dy^2}_{(dx - \frac{1}{2}dy)^2} + \tfrac{3}{4}dy^2\Big) > 0$$

Пусть $dy \neq 0$

$$= 2dy^2\Big(\underbrace{\frac{dx^2}{dy^2} - \frac{dx}{dy} + 1}_{t^2 - t + 1}\Big)$$

$$\frac{dx}{dy} = t$$

$\Rightarrow M_0(1; 0)$ - точка локального минимума.

$$z_{min}(1, 0) = 1 - 2 = -1.$$

3626.

$z = x^3 + y^3 - 3xy$

$dz = (3x^2 - 3y)dx + (3y^2 - 3x)dy$

$$\begin{cases} 3x^2 - 3y = 0 \\ 3y^2 - 3x = 0 \end{cases} \iff \begin{cases} y = x^2 \\ y^2 = x \end{cases}$$

$x = y^2 \rightarrow 3y^4 - 3y = 0 \quad y(y^3 - 1) = 0$

$y = 0$ и $y = 1$.
$x = 0$ и $x = 1$.

$\Rightarrow M_1(0;0)$ и $M_2(1;1)$

$d^2z = (6x\,dx - 3dy)dx + (6y\,dy - 3dx)dy = 6(x\,dx^2 - dx\,dy + y\,dy^2)$

$d^2z\big|_{M_2} = 6(dx^2 - dx\,dy + dy^2) > 0$

M_2 – точка лок. минимума.

$d^2z\big|_{M_1} = -6\,dx\,dy \quad \begin{matrix} <0 \\ >0 \end{matrix}$ экстремума нет.

$z_{min}(1;1) = -1$

3627.

$z = x^4 + y^4 - x^2 - 2xy - y^2$

$dz = (4x^3 - 2x - 2y)dx + (4y^3 - 2y - 2x)dy$

$$\begin{cases} 4x^3 - 2x - 2y = 0 \\ 4y^3 - 2y - 2x = 0 \end{cases} \iff \begin{cases} 4x^3 = 2x + 2y \\ 4y^3 = 2x + 2y \end{cases} \Rightarrow x = y.$$

$4x^3 - 2x - 2x = 0$
$x = 0 \ ; \ x = \pm 1$

$M_1(0;0)$; $M_2(1,1)$; $M_3(-1,-1)$

$d^2z = (12x^2dx - 2dx - 2dy)dx + (12y^2dy - 2dy - 2dx)dy =$
$= (12x^2 - 2)dx^2 - 4\,dx\,dy + (12y^2 - 2)dy^2$

$M_1(0,0) \Rightarrow d^2z(0,0) = -2dx^2 - 4dx\,dy - 2dy^2 = -2(dx + dy)^2 \leq 0$

Сравнивая $z(x,y)$ с $z(0,0)$

$z(x,y) - z(0,0) = x^4 + y^4 - x^2 - 2xy - y^2 = x^4 + y^4 - (x+y)^2$

при $y = -x \quad z(x,y) - z(0,0) = 2x^4 > 0$

$z(x,y) > z(0,0)$

$x = 0,\ y \neq 0 \quad z(x,y) - z(0,0) = y^4 - y^2 = y^2(y^2 - 1) < 0$

$0<|y|<1.$

– экстремума нет.

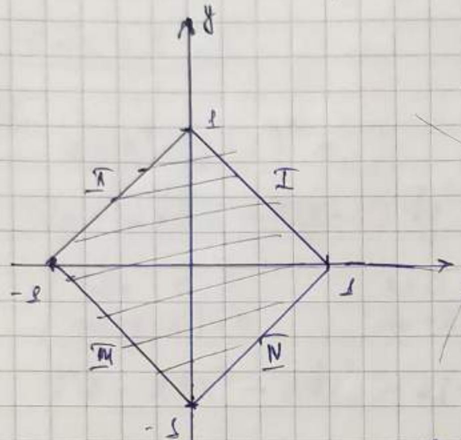
$d^2z(1,1) = d^2z(-1,-1) = 10dx^2 - 4dx\,dy + 10dy^2 = 2(5dx^2 - 2dx\,dy + 5dy^2)$ –

$5t^2-2t+5;\ D<0$

$z_{min} = z(1,1) = z(-1,-1) = -2$

(3677.) Определить наибольшее и наименьшее значение ф-ии в указанной области.

$z = x^2 - xy + y^2$, если $|x|+|y| \le 1$.



$dz = (2x-y)\,dx + (2y-x)\,dy.$

$\begin{cases} 2x-y=0 \\ 2y-x=0 \end{cases} \quad y=2x \quad \Rightarrow M_1(0,0)$

$d^2z = (2dx-dy)dx + (2dy-dx)dy =$

$= 2dx^2 - dx\,dy - dx\,dy + 2dy^2 =$

$= 2(dx^2 - dx\,dy + dy^2) > 0$

$M_1(0,0)$ – точка минимума.

I граница

$y = 1-x, \quad 0 \le x \le 1.$

$z = x^2 - x(1-x) + (1-x)^2 = x^2 - x + x^2 + x^2 - 2x + 1 = 3x^2 - 3x + 1.$

$z' = 6x - 3 = 0 \Rightarrow x = \frac{1}{2} \quad y = \frac{1}{2} \quad M_2(\frac{1}{2}; \frac{1}{2})$

$M_3(1,0), \quad M_4(0;1)$

II область

$y = x+1$

$-1 \le x \le 0$

Доделать для всех областей и найти наиб. и наим. знач. из всех.

$z = x^2 - x(1+x) + (1+x)^2 = x^2 - x - x^2 + 1 + 2x + x^2 =$

$= x^2 + x + 1.$ — корней нет.

$D = b^2 - 4ac = 1-4 < 0$

III область.

$y = -x-1.$

$z = x^2 - x(-x-1) + (x+1)^2 = x^2 + x^2 + x + x^2 + 2x + 1 = 3x^2 + 3x + 1.$

$D = 9 - 4\cdot 3\cdot 1 < 0$

– корней нет.

IV область.

$y = x - 1.$

$z = x^2 - x(x-1) + (x-1)^2 = \cancel{x^2} - \cancel{x^2} + x + x^2 - 2x + 1 = x^2 - x + 1.$

корней нет.

$M_1(0;0)$; $M_2\left(\frac{1}{2}; \frac{1}{2}\right)$; $M_3(1;0)$; $M_4(0;1)$

$z = x^2 - xy + y^2$

$z_1 = 0$

$z_2 = \frac{1}{4} - \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{4} = \frac{1}{4}$

$z_3 = 1.$

$z_4 = 1.$

Ответ: $z_{min} = 0$
$z_{max} = 1.$

Домашняя работа.

3621.

$z = x^2 + (y-1)^2$

$dz = 2x\,dx + 2(y-1)\,dy.$

$\begin{cases} 2x = 0 \\ 2(y-1) = 0 \end{cases} \rightarrow \begin{matrix} x = 0 \\ y = 1. \end{matrix}$ $M_0(0;1)$

$d^2z = 2dx \cdot dx + 2dy \cdot dy = 2dx^2 + 2dy^2$

M_0 – точка локального минимума.

$z_{min.} = 0$

3622.

$z = x^2 - (y-1)^2 =$

$dz = 2x\,dx - 2(y-1)\,dy.$

$\begin{cases} 2x = 0 \\ -2(y-1) = 0 \end{cases}$ $\begin{matrix} x = 0 \\ -2y + 2 = 0 \rightarrow y = 1 \end{matrix}$ $M_0(0;1)$

$d^2z = 2dx^2 - 2dy^2 = 2(dx^2 - dy^2) = 2(dx-dy)(dx+dy)$

$\sim$ экстремумов нет.

3623.

$z = (x-y+1)^2$

$dz = 2(x-y+1)dx - 2(x-y+1)\,dy$

$\begin{cases} x - y + 1 = 0 \\ x - y + 1 = 0 \end{cases}$ $x = y - 1.$ $M_0(x; y-1)$

$d^2z = 2(dx-dy)dx - 2(dx-dy)dy =$

$= 2dx^2 - 2dx\,dy - 2dx\,dy + dy^2 = 2(dx^2 - 2dx\,dy + dy^2) =$

$= 2(dx-dy)^2$

Если $dy \neq 0$ $2\left(\left(\frac{dx}{dy}\right)^2 - 2\left(\frac{dx}{dy}\right) + 1\right) \geq 0$ – нестрогий минимум для точек $x - y + 1 = 0$

3628.

$z = xy + \frac{50}{x} + \frac{20}{y}$ $(x>0, y>0)$

$dz = \left(y - \frac{50}{x^2}\right)dx + \left(x - \frac{20}{y^2}\right)dy.$

$\begin{cases} y - \frac{50}{x^2} = 0 \\ x - \frac{20}{y^2} = 0 \end{cases}$; $y = \frac{50}{x^2} \rightarrow y^2 = \frac{50^2}{x^4}$

$\rightarrow x - \frac{20}{50^2}x^4 = 0$; $x\left(1 - \frac{20x^3}{50^2}\right) = 0$

$$1-\frac{20x^3}{50^2}=0 \qquad 20x^3=50^2 \quad \to \quad x=5$$

$$y=\frac{50}{25}=2$$

$$M_0(5;2)$$

$$d^2z=\left(dy+\frac{100}{x^3}dx\right)dx+\left(dx+\frac{40}{y^3}dy\right)dy=$$

$$=dx\,dy+\frac{100}{x^3}dx^2+dx\,dy+\frac{40}{y^3}dy^2=\frac{100}{x^3}dx^2+2dx\,dy+\frac{40}{y^3}dy^2.$$

В т. $M_0(5;2)$

$$d^2z=\frac{100}{125}dx^2+2dx\,dy+\frac{40}{8}dy^2=\frac{4}{5}dx^2+2dx\,dy+5dy^2$$

Т.к. $x>0$ и $y>0$ по усл.

$$\Rightarrow \underline{z_{min}}=z(5;2)=10+\frac{50}{5}+\frac{20}{2}=\underline{30}$$

3643.

$$u=x^2+y^2+z^2+12xy+2z$$

$$du=(2x+12y)dx+(2y+12x)dy+(2z+2)dz$$

$$\begin{cases}2x+12y=0\\2y+12x=0\\2z+2=0\end{cases} \Leftrightarrow \begin{cases}x+6y=0 \quad |\cdot 6\\6x+y=0\\z=-1\end{cases} \quad -\begin{cases}6x+36y=0\\6x+y=0\end{cases}$$

$$35y=0 \quad \to y=0\;;\; x=0$$

$$M_0(0;0;-1)$$

3644.

$$u = x + \frac{y^2}{4x} + \frac{z^2}{y} + \frac{2}{z} \quad (x>0,\ y>0,\ z>0)$$

$$du = \left(1 - \frac{y^2}{4x^2}\right)dx + \left(\frac{y}{2x} - \frac{z^2}{y^2}\right)dy + \left(\frac{2z}{y} - \frac{2}{z^2}\right)dz$$

т.к. $x>0,\ y>0,\ z>0$

$$\begin{cases} 1 - \frac{y^2}{4x^2} = 0 \\ \frac{y}{2x} - \frac{z^2}{y^2} = 0 \\ \frac{2z}{y} - \frac{2}{z^2} = 0 \end{cases}$$

$1 = \frac{y^2}{4x^2} \rightarrow y^2 = 4x^2 \quad y = 2x^2 \quad x = \sqrt{\frac{2}{y}} = \sqrt{\frac{2}{z^3}}$

$\frac{y^2}{x^2} = 4 \ ; \ \frac{y}{x} = 2.$

$1 - \frac{z^2}{y^2} = 0 \Rightarrow \frac{z}{y} = 1.$

$2 - \frac{2}{z^2} = 0 \rightarrow z^2 = 1 \rightarrow z = 1. \Rightarrow y = 1. \rightarrow x = \frac{1}{2}.$

$M_0\left(\frac{1}{2}; 1; 1\right)$

$d^2u - ?$

$$d\left(\frac{y}{x}\right)^2 = 2\frac{y}{x}\cdot\frac{x\,dy - y\,dx}{x^2}$$

$$d\left(\frac{y}{x}\right) = \frac{x\,dy - y\,dx}{x^2}$$

$$d\left(\frac{z}{y}\right)^2 = 2\frac{z}{y}\cdot\frac{y\,dz - z\,dy}{y^2}$$

$$d\left(-\frac{2}{z^2}\right) = +\frac{4}{z^3}dz$$

$$d^2u = \frac{1}{4}\left(-2\frac{y}{x}\cdot\frac{x\,dy - y\,dx}{x^2}\right)dx + \left(\frac{1}{2}\frac{x\,dy - y\,dx}{x^2} - 2\frac{z}{y}\cdot\frac{y\,dz - z\,dy}{y^2}\right)dy +$$

$$+ \left(2\frac{y\,dz - z\,dy}{y^2} + \frac{4\,dz}{z^3}\right)dz =$$

$$= \frac{1}{4}\left(+\frac{2y^2dx - 2xy\,dy}{x^3}\right)dx + \left(\frac{x\,dy - y\,dx}{2x^2} + \frac{2z^2dy - 2zy\,dz}{y^3}\right)dy +$$

$$+ \left(\frac{2y\,dz - 2z\,dy}{y^2} + \frac{4\,dz}{z^3}\right)dz =$$

$$= \frac{y^2dx^2}{2x^3} - \frac{xy\,dx\,dy}{2x^3} + \frac{x\,dy^2}{2x} - \frac{y\,dx\,dy}{2x^2} + \frac{2z^2dy^2}{y^3} - \frac{2zy\,dz\,dy}{y^3} +$$

$$+ \frac{2\,dz^2}{y} - \frac{2z\,dy\,dz}{y^2} + \frac{4\,dz^2}{z^3} =$$

$$= \frac{y^2}{2x^3}dx^2 - \frac{y}{x^2}dx\,dy + \left(\frac{1}{2x} + \frac{2z^2}{y^3}\right)dy^2 - \frac{4z}{y^2}dy\,dz + \left(\frac{2}{y} + \frac{4}{z^3}\right)dz^2$$

Подставим $M_0\left(\frac{1}{2}; 1; 1\right)$

$$d^2u = \frac{1}{2\cdot\frac{1}{8}}dx^2 - \frac{1}{\frac{1}{4}}dx\,dy + (1+2)dy^2 - 4dy\,dz + (2+4)dz^2 =$$

$$= 4dx^2 - 4dx\,dy + 3dy^2 - 4dy\,dz + 6dz^2 =$$

$$= (2dx - dy)^2 + 2dy^2 - 4dy\,dz + 2dz^2 + 4dz^2 =$$

$$= (2dx - dy)^2 + dy^2 + (dy - 2dz)^2 + 2dz^2 \quad > 0$$

при $x > 0,\ y > 0,\ z > 0$

$\Rightarrow M_0\left(\frac{1}{2}; \frac{1}{1}; 1\right)$ - точка минимума.

$$u = x + \frac{y^2}{4x} + \frac{z^2}{y} + \frac{2}{z}$$

$$u = \frac{1}{2} + \frac{1}{2} + \frac{1}{1} + 2 = 4$$

Ответ: $u = 4$.

11.03.2021г. (практика - пропуск.)

3481.

$$(y - z)\frac{\partial z}{\partial x} + (y + z)\frac{\partial z}{\partial y} = 0,$$ приняв x за функцию, а $u = y - z$, $v = y + z$ - за незав.

$$x = x(u, v) = x(y - z; y + z)$$

$$1 = \frac{\partial x}{\partial u} \cdot \left(-\frac{\partial z}{\partial x}\right) + \frac{\partial x}{\partial v} \cdot \frac{\partial z}{\partial x}$$

$$\frac{\partial z}{\partial x} = \frac{1}{\frac{\partial x}{\partial v} - \frac{\partial x}{\partial u}} \rightarrow \frac{\partial z}{\partial x} = \frac{-1}{\frac{\partial x}{\partial u} - \frac{\partial x}{\partial v}}$$

$$0 = \frac{\partial x}{\partial u}\left(1 - \frac{\partial z}{\partial y}\right) + \frac{\partial x}{\partial v} \cdot \left(1 + \frac{\partial z}{\partial y}\right)$$

$$0 = \frac{\partial x}{\partial u} - \frac{\partial x}{\partial u} \cdot \frac{\partial z}{\partial y} + \frac{\partial x}{\partial v} + \frac{\partial x}{\partial v} \cdot \frac{\partial z}{\partial y}$$

$$\frac{\partial x}{\partial u} + \frac{\partial x}{\partial v} = \frac{\partial z}{\partial y} \cdot \left(\frac{\partial x}{\partial u} - \frac{\partial x}{\partial v}\right)$$

$$\frac{\partial z}{\partial y} = \frac{\frac{\partial x}{\partial u} + \frac{\partial x}{\partial v}}{\frac{\partial x}{\partial u} - \frac{\partial x}{\partial v}} \quad ; \quad -u + v \cdot \left(\frac{\partial x}{\partial u} + \frac{\partial x}{\partial v}\right) = 0$$

$$\frac{\partial x}{\partial u} + \frac{\partial x}{\partial v} = \frac{u}{v}$$

Ответ: $\frac{\partial x}{\partial u} + \frac{\partial x}{\partial v} = \frac{u}{v}$

3484.

$$\omega = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}, \quad x = r\cos\varphi,\ y = r\sin\varphi$$

$$u = u(x, y) = u(r\cos\varphi; r\sin\varphi) = u(r; \varphi)$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r} \cdot \frac{\partial r}{\partial x} + \frac{\partial u}{\partial \varphi} \cdot \frac{\partial \varphi}{\partial x} \qquad 0 = \frac{\partial r}{\partial x}\sin\varphi + r\cos\varphi\frac{\partial \varphi}{\partial x}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r} \cdot \frac{\partial r}{\partial y} + \frac{\partial u}{\partial \varphi} \cdot \frac{\partial \varphi}{\partial y} \qquad 0 = \frac{\partial r}{\partial y}\cos\varphi - r\sin\varphi\frac{\partial \varphi}{\partial y}$$

$$x = r\cos\varphi \Rightarrow 1 = \frac{\partial r}{\partial x} \cdot \cos\varphi + r \cdot \frac{\partial(\cos\varphi)}{\partial x} \Rightarrow 1 = \frac{\partial r}{\partial x} \cdot \cos\varphi - r\sin\varphi\frac{\partial \varphi}{\partial x}$$

$y = r\sin\varphi \Rightarrow 1 = \frac{\partial r}{\partial y}\cdot\sin\varphi + r\cos\varphi\frac{\partial\varphi}{\partial y}$.

① $\begin{cases} \frac{\partial r}{\partial x}\cos\varphi - r\sin\varphi\cdot\frac{\partial\varphi}{\partial x} = 1 \\ \frac{\partial r}{\partial x}\sin\varphi + r\cos\varphi\frac{\partial\varphi}{\partial x} = 0 \end{cases}$

$\Delta_1 = \begin{vmatrix} \cos\varphi & -r\sin\varphi \\ \sin\varphi & r\cos\varphi \end{vmatrix} = r\cos^2\varphi + r\sin^2\varphi = r$

$\Delta\left(\frac{\partial r}{\partial x}\right) = \begin{vmatrix} 1 & -r\sin\varphi \\ 0 & r\cos\varphi \end{vmatrix} = r\cos\varphi$

② $\begin{cases} \frac{\partial r}{\partial y}\sin\varphi + r\cos\varphi\frac{\partial\varphi}{\partial y} = 1 \\ \frac{\partial r}{\partial y}\cos\varphi + r\sin\varphi\frac{\partial\varphi}{\partial y} = 0 \end{cases}$

$\Delta\left(\frac{\partial\varphi}{\partial x}\right) = \begin{vmatrix} \cos\varphi & 1 \\ \sin\varphi & 0 \end{vmatrix} = -\sin\varphi$.

$\Delta_2 = -r$

$\frac{\partial\varphi}{\partial x} = -\frac{\sin\varphi}{r}$; $\frac{\partial r}{\partial x} = \cos\varphi$; $\frac{\partial r}{\partial y} = \sin\varphi$; $\frac{\partial\varphi}{\partial y} = \frac{\cos\varphi}{r}$.

$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\cos\varphi - \frac{\partial u}{\partial\varphi}\cdot\frac{\sin\varphi}{r}$

$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial r}\cos\varphi - \frac{\partial u}{\partial\varphi}\cdot\frac{\sin\varphi}{r}\right) = \left(\frac{\partial^2 u}{\partial r^2}\cdot\frac{\partial r}{\partial x} + \frac{\partial^2 u}{\partial r\partial\varphi}\cdot\frac{\partial \varphi}{\partial x}\right)\cdot\cos\varphi + \frac{\partial u}{\partial r}\cdot(-\sin\varphi)\frac{\partial\varphi}{\partial x} - \left(\frac{\partial^2 u}{\partial\varphi\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial^2 u}{\partial\varphi^2}\cdot\frac{\partial\varphi}{\partial x}\right)\cdot\frac{\sin\varphi}{r} - \frac{\partial u}{\partial\varphi}\cdot\frac{\cos\varphi\frac{\partial\varphi}{\partial x} - \frac{\partial r}{\partial x}}{r^2} = \frac{\partial^2 u}{\partial r^2}\cos^2\varphi + \ldots$

3513. $y\frac{\partial^2 z}{\partial y^2} + \frac{2\partial z}{\partial y} = \frac{2}{x}$, если $u = \frac{x}{y}$, $v = x$, $\omega = xz - y$; $\omega = \omega(u, v)$

$\omega = \omega(u, v) = xz - y$.

$\frac{\partial\omega}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial\omega}{\partial v}\cdot\frac{\partial v}{\partial y} = x\cdot\frac{\partial z}{\partial y} - 1$

$\frac{\partial\omega}{\partial u}\cdot\left(-\frac{x}{y^2}\right) = x\frac{\partial z}{\partial y} - 1$; $\frac{\partial z}{\partial y} = \frac{1}{x} - \frac{\partial\omega}{\partial u}\cdot\frac{1}{y^2}$

$-\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial\omega}{\partial u}\right)\cdot\frac{1}{y^2} - \frac{2}{y^3}\frac{\partial\omega}{\partial u} \rightarrow + \frac{\partial^2 z}{\partial y^2} = \left(\frac{\partial^2\omega}{\partial u^2}\cdot\frac{1}{y^2}\right)\cdot\frac{1}{y^2} + \frac{2}{y^3}\cdot\frac{\partial\omega}{\partial u}$

$\frac{1}{y^3}\cdot\frac{\partial^2\omega}{\partial u^2} + \frac{2\cdot\partial\omega}{y^2\cdot\partial u} + 2\cdot\left(\frac{1}{x} - \frac{\partial\omega}{\partial u}\cdot\frac{1}{y^2}\right) = \frac{2}{x}$

$\frac{x}{y^3}\cdot\frac{\partial^2\omega}{\partial u^2} = 0$; $\frac{\partial^2\omega}{\partial u^2} = 0$

Ответ: $\frac{\partial^2\omega}{\partial u^2} = 0$

Д/З: 3474, 3445, 3476, 3487, 89, 95, 96, 3514, 3517.

18.03.2021.

Условный экстремум.

3657.

б) $z = x^2 + 12xy + 2y^2$, если $4x^2 + y^2 = 25$

$4x^2 + y^2 - 25 = 0$

Функция Лагранжа $L = x^2 + 12xy + 2y^2 + \lambda(4x^2 + y^2 - 25)$

из условия экстремума

$$\begin{cases} \dfrac{\partial L}{\partial x} = 2x + 12y + 8\lambda x = 0 \\ \dfrac{\partial L}{\partial y} = 12x + 4y + 2\lambda y = 0 \\ \dfrac{\partial L}{\partial \lambda} = 4x^2 + y^2 - 25 = 0 \end{cases}$$

$$\begin{cases} x + 6y + 4\lambda x = 0 \\ 6x + 2y + \lambda y = 0 \\ 4x^2 + y^2 - 25 = 0 \end{cases}$$

$$\begin{cases} x(1+4\lambda) + 6y = 0 \\ 6x + y(2+\lambda) = 0 \end{cases}$$ – бесконечно много решений $(x \neq 0, y \neq 0)$

$$\Rightarrow \Delta = 0 = \begin{vmatrix} (1+4\lambda) & 6 \\ 6 & (2+\lambda) \end{vmatrix}$$

$(1+4\lambda)(2+\lambda) - 36 = 0$

$2 + \lambda + 8\lambda + 4\lambda^2 - 36 = 0$

$4\lambda^2 + 9\lambda - 34 = 0$

$D = 81 + 16 \cdot 34 = 625$; $\sqrt{D} = 25$

$\lambda_2 = \dfrac{-9+25}{8} = \dfrac{16}{8} = 2$

$\lambda_1 = \dfrac{-9-25}{8} = \dfrac{-34}{8} = -\dfrac{17}{4}$

$\underline{\lambda = 2}$: $\begin{cases} x + 6y + 8x = 0 \\ 6x + 2y + 2y = 0 \end{cases} \Leftrightarrow 8x + 2y = 0 \Rightarrow y = -\dfrac{3}{2}x$

$4x^2 + \dfrac{9}{4}x^2 - 25 = 0 \quad | \cdot 4$

$16x^2 + 8x^2 = 100$

$25x^2 = 100 \Rightarrow x = \pm 2$

$\Rightarrow (2; -3)$; $(-2; 3)$ – стационарные точки

$L = x^2 + 12xy + 2y^2 + 8x^2 + 2y^2 - 50 = 9x^2 + 12xy + 4y^2 - 50$

$dL = (18x + 12y)dx + (8y + 12x)dy$

$d^2L = (18dx + 12dy)dx + (8dy + 12dx)dy = 18dx^2 + 12dy\,dx + 8dy^2 + 12dx\,dy =$

$= 18dx^2 + 24dx\,dy + 8dy^2 = 2(3dx + 2dy)^2 \geq 0$

Из $4x^2 + y^2 - 25 = 0$

$8x\,dx + 2y\,dy = 0 \rightarrow 4x\,dx + y\,dy = 0$

$(2; -3) \rightarrow 8dx - 3dy = 0 \rightarrow dx = \dfrac{3}{8}dy$

$dy \neq 0 \Rightarrow 2\left(\underbrace{3 \cdot \dfrac{3}{8}dy + 2dy}_{2 \cdot 9}\right)^2 > 0$

$z_{min} = z(2; -3) = z(-2; 3) = 4 - 72 + 18 = 22 - 72 = -50$

$\lambda = -\frac{17}{4}$: $\quad x + 8y - 17x = 0 \quad \to 16x = 6y. \quad y = \frac{16}{6}x = \frac{8}{3}x$

$$4x^2 + \frac{64}{9}x^2 - 25 = 0 \;/\cdot 9$$

$$36x^2 + 64x^2 = 225$$

$$x^2 = \frac{225}{100} \quad \to x = \pm\frac{15}{10} = \pm\frac{3}{2}$$

$\left(\frac{3}{2}; 4\right)$; $\left(-\frac{3}{2}; -4\right)$ – стационарные точки.

$$L = x^2 + 12xy + 2y^2 - \frac{17}{4}(4x^2 + y^2 - 25) = x^2 + 12xy + 2y^2 - 17x^2 - \frac{17}{4}y^2 + \frac{425}{4} =$$

$$= -16x^2 + 12xy - \frac{9}{4}y^2 + \frac{425}{4}$$

$$dL = (-32x + 12y)dx + \left(-\frac{9}{2}y + 12x\right)dy.$$

$$d^2L = (-32dx + 12dy)dx + \left(12dx - \frac{9}{2}dy\right)dy = -32dx^2 + 24dx\,dy - \frac{9}{2}dy^2 =$$

$$= -\frac{1}{2}(64dx^2 - 48dx\,dy + 9dy^2) = -\frac{1}{2}(8dx - 3dy)^2$$

Из $4x^2 + y^2 - 25 = 0$

$$8x\,dx + 2y\,dy = 0 \to 4x\,dx + y\,dy = 0$$

$\left(\frac{3}{2}; 4\right) \to 6dx + 4dy = 0 \quad \to dx = -\frac{2}{3}dy \quad ; dy \neq 0$

$$-\frac{1}{2}\left(8\cdot\left(-\frac{2}{3}\right)dy - 3dy\right)^2 < 0$$

$\left(-\frac{3}{2}; -4\right) \to -6dx - 4dy = 0 \quad \to dx = -\frac{2}{3}dy.$

– точки максимума.

$$z_{max} = z\left(\frac{3}{2}; 4\right) = z\left(-\frac{3}{2}; -4\right) = \frac{9}{4} + 12\cdot\frac{3}{2}\cdot 4 + 2\cdot 16 = \frac{9}{4} + 72 + 32 =$$

$$= 104 + \frac{9}{4} = 106\frac{1}{4} = 106{,}25$$

3662.

$u = xy^2z^3$, если $x + 2y + 3z = a \quad (x > 0,\ y > 0,\ z > 0,\ a > 0)$

$x + 2y + 3z - a = 0$

$$\ln u = \ln x + 2\ln y + 3\ln z = v$$

$$L = \ln x + 2\ln y + 3\ln z + \lambda(x + 2y + 3z - a)$$

$$\frac{\partial L}{\partial x} = \frac{1}{x} + \lambda = 0$$

$$\frac{\partial L}{\partial y} = \frac{2}{y} + 2\lambda = 0$$

$$\frac{\partial L}{\partial \lambda} = x + 2y + 3z - a = 0$$

$$\frac{\partial L}{\partial z} = \frac{3}{z} + 3\lambda = 0$$

$$\Leftrightarrow \begin{cases} \frac{1}{x} + \lambda = 0 \\ \frac{1}{y} + \lambda = 0 \\ \frac{1}{z} + \lambda = 0 \\ x + 2y + 3z - a = 0 \end{cases} \qquad x = -\frac{1}{\lambda},\quad y = -\frac{1}{\lambda},\quad z = -\frac{1}{\lambda}$$

$$-\frac{1}{\lambda} - \frac{2}{\lambda} - \frac{3}{\lambda} - a = 0$$

$$\frac{1}{\lambda}+\frac{2}{\lambda}+\frac{3}{\lambda}+a=0 \quad |\cdot\lambda$$

$$\cancel{1}\cdot 6+a\lambda=0 \quad \Rightarrow \lambda=-\frac{6}{a}$$

$\Rightarrow M\left(\frac{a}{6};\frac{a}{6};\frac{a}{6}\right)$ - стационарная точка

$$L=\ln x+2\ln y+3\ln z-\frac{6}{a}(x+2y+3z-a)=\ln x+2\ln y+3\ln z-\frac{6}{a}x-\frac{12}{a}y-$$
$$-\frac{18}{a}z+6.$$

$$dL=\left(\frac{1}{x}-\frac{6}{a}\right)dx+\left(\frac{2}{y}-\frac{12}{a}\right)dy+\left(\frac{3}{z}-\frac{18}{a}\right)dz$$

$$d^2L=\left(-\frac{1}{x^2}\right)dx^2-\left(\frac{2}{y^2}\right)dy^2-\frac{3}{z^2}dz^2<0 \quad \text{- max}$$

$$u=\frac{a^6}{6^6}$$

Ответ: $u=\frac{a^6}{6^6}$

Д/З: 3653(а)(через λ и μ), ~~3654~~, 3656, 3659.

Домашняя работа.

3654. $z=xy$, если $x+y=1 \rightarrow x+y-1=0$

$$L=xy+\lambda(x+y-1)=xy+\lambda x+\lambda y-\lambda$$

$$\begin{cases}\frac{\partial L}{\partial x}=y+\lambda\\ \frac{\partial L}{\partial y}=x+\lambda\\ \frac{\partial L}{\partial \lambda}=x+y-1\end{cases} \Rightarrow \begin{cases}y+\lambda=0\\ x+\lambda=0\\ x+y-1=0\end{cases} \Leftrightarrow \begin{cases}y=-\lambda\\ x=-\lambda\\ -\lambda-\lambda-1=0\end{cases}$$

$$2\lambda=-1 \rightarrow \lambda=-\frac{1}{2}$$

$$\Rightarrow y=\frac{1}{2};\ x=\frac{1}{2}$$

$M_0\left(\frac{1}{2};\frac{1}{2}\right)$ - стационарная точка

$$L=xy-\frac{1}{2}x-\frac{1}{2}y+\frac{1}{2} \qquad x+y=1.$$

$$dL=\left(y-\frac{1}{2}\right)dx+\left(x-\frac{1}{2}\right)dy.$$

$$d^2L=(dy)dx+(dx)dy=2dx\,dy \ \vee\ 0$$

$dy\neq 0$ $\qquad 2dx\,dy<0$ $\quad$ - $M_0\left(\frac{1}{2};\frac{1}{2}\right)$ точка максимума.

$$z_{max}=\frac{1}{2}\cdot\frac{1}{2}=\frac{1}{4}$$

Ответ: $z_{max}=\frac{1}{4}$

3656. $z=x^2+y^2$, если $\frac{x}{a}+\frac{y}{b}=1. \quad |\cdot ab$

$$bx+ay=ab$$

$$bx+ay-ab=0$$

$$L=x^2+y^2+\lambda(bx+ay-ab)=x^2+y^2+\lambda bx+\lambda ay-\lambda ab$$

$$\begin{cases}\dfrac{\partial L}{\partial x}=2x+\lambda b\\ \dfrac{\partial L}{\partial y}=2y+\lambda a\\ \dfrac{\partial L}{\partial \lambda}=bx+ay-ab\end{cases} \Leftrightarrow \begin{cases}2x+\lambda b=0\\ 2y+\lambda a=0\\ bx+ay-ab=0\end{cases} \quad x=-\frac{\lambda b}{2};\quad y=-\frac{\lambda a}{2}$$

$$-\frac{\lambda b^2}{2}-\frac{\lambda a^2}{2}=ab \quad /\cdot 2$$

$$-\lambda b^2-\lambda a^2=2ab$$

$$\lambda(b^2+a^2)=-2ab \rightarrow \lambda=-\frac{2ab}{b^2+a^2}$$

$$\Rightarrow x=\frac{ab^2}{b^2+a^2};\quad y=\frac{a^2b}{b^2+a^2}$$

$$L=x^2+y^2-\frac{2ab}{b^2+a^2}(bx+ay-ab)=x^2+y^2-\frac{2ab^2}{b^2+a^2}x-\frac{2a^2b}{b^2+a^2}y+\frac{2a^2b^2}{b^2+a^2}$$

$$dL=\left(2x-\frac{2ab^2}{b^2+a^2}\right)dx+\left(2y-\frac{2a^2b}{b^2+a^2}\right)dy.$$

$$d^2L=(2dx^2+2dy^2)=2(dx^2+dy^2)>0$$

$$\Rightarrow M\left(\frac{2ab^2}{b^2+a^2};\ \frac{2a^2b}{b^2+a^2}\right) - \text{т. минимума}$$

$$z_{min}=\left(\frac{2ab^2}{b^2+a^2}\right)^2+\left(\frac{2a^2b}{b^2+a^2}\right)^2=\frac{4a^2b^4+4a^4b^2}{(b^2+a^2)^2}$$

$$L=x^2+y^2-\frac{2ab}{b^2+a^2}(bx+ay-ab)$$

$$d^2L=2(dx^2+dy^2)>0$$

$$M\left(\frac{ab^2}{b^2+a^2};\ \frac{a^2b}{b^2+a^2}\right) - \text{т. минимума.}$$

$$z_{min}=\frac{a^2b^4+a^4b^2}{(b^2+a^2)^2}=\frac{a^2b^2(b^2+a^2)}{(b^2+a^2)^2}=\frac{a^2b^2}{b^2+a^2}$$

Ответ. $z_{min}=\dfrac{a^2b^2}{a^2+b^2}$

3659.

$u=x-2y+2z$, если $x^2+y^2+z^2=1.$

$$x^2+y^2+z^2-1=0$$

$$L=x-2y+2z+\lambda(x^2+y^2+z^2-1)$$

$$\begin{cases}\dfrac{\partial L}{\partial x}=1+2\lambda x\\ \dfrac{\partial L}{\partial y}=2\lambda y-2\\ \dfrac{\partial L}{\partial z}=2\lambda z+2\\ \dfrac{\partial L}{\partial \lambda}=x^2+y^2+z^2-1\end{cases} \Leftrightarrow \begin{cases}2\lambda x+1=0\\ 2\lambda y-2=0\\ 2\lambda z+2=0\\ x^2+y^2+z^2=1.\end{cases}$$

$$\begin{cases} 2\lambda x + 1 = 0 \\ 2\lambda y - 2 = 0 \\ 2\lambda z + 2 = 0 \\ x^2+y^2+z^2 = 1 \end{cases}$$

$x = -\frac{1}{2\lambda}$; $y = \frac{1}{\lambda}$; $z = -\frac{1}{\lambda}$

$\frac{1}{4\lambda^2} + \frac{1}{\lambda^2} + \frac{1}{\lambda^2} = 1. \quad | \cdot 4\lambda^2$

$1 + 4 + 4 = 4\lambda^2$

$\lambda^2 = \frac{9}{4} \rightarrow \lambda = \pm \frac{3}{2}$

1) $\lambda = \frac{3}{2} \rightarrow x = -\frac{1}{3}$; $y = \frac{2}{3}$; $z = -\frac{2}{3}$

$M_1\left(-\frac{1}{3}; \frac{2}{3}; -\frac{2}{3}\right)$ - стационарная точка.

$L = x - 2y + 2z + \frac{3}{2}(x^2+y^2+z^2-1)$

$dL = (1+3x)dx + (3y-2)dy + (2+3z)dz$

$d^2L = (3dx)dx + (3dy)dy + (3dz)dz = 3(dx^2+dy^2+dz^2) > 0$

$M_1\left(-\frac{1}{3}; \frac{2}{3}; -\frac{2}{3}\right)$ - т. минимума.

$u\left(-\frac{1}{3}; \frac{2}{3}; -\frac{2}{3}\right) = -\frac{1}{3} - \frac{4}{3} - \frac{4}{3} = -\frac{9}{3} = -3$

2) $\lambda = -\frac{3}{2} \rightarrow x = \frac{1}{3}$; $y = -\frac{2}{3}$; $z = \frac{2}{3}$

$M_2\left(\frac{1}{3}; -\frac{2}{3}; \frac{2}{3}\right)$ - стационарная точка.

$L = x - 2y + 2z - \frac{3}{2}(x^2+y^2+z^2-1)$

$dL = (1-3x)dx + (-3y-2)dy + (-3z+2)dz$

$d^2L = -3(dx^2+dy^2+dz^2) < 0$

$M_2\left(\frac{1}{3}; -\frac{2}{3}; \frac{2}{3}\right)$ - точка максимума

$u\left(\frac{1}{3}; -\frac{2}{3}; \frac{2}{3}\right) = \frac{1}{3} - \frac{4}{3} + \frac{4}{3} = \frac{1}{3}$

Ответ: $u_{max} = \frac{1}{3}$

$u_{min} = -3.$

3663. а) $u = xyz$, если $x^2+y^2+z^2 = 1$, $x+y+z = 0$

$x^2+y^2+z^2-1 = 0$, $x+y+z = 0$

$L = xyz + \lambda(x^2+y^2+z^2-1) + \mu(x+y+z)$

$\frac{\partial L}{\partial x} = yz + 2x\lambda + \mu$

$\frac{\partial L}{\partial y} = xz + 2y\lambda + \mu$

$\frac{\partial L}{\partial z} = xy + 2z\lambda + \mu$

$\frac{\partial L}{\partial \lambda} = x^2+y^2+z^2-1.$

$\frac{\partial L}{\partial \mu} = x+y+z$

$$\begin{cases} yz+2x\lambda+\mu=0 \\ xz+2y\lambda+\mu=0 \\ xy+2z\lambda+\mu=0 \\ x^2+y^2+z^2=1 \\ x+y+z=0 \end{cases}$$

$\cancel{(x^2+y^2+z^2)\cdot(x+y+z)=0}$

$\underline{yz}+2x\lambda+\cancel{\mu}-\underline{xz}-2y\lambda-\cancel{\mu}=0$

$\underline{xz}+2y\lambda+\cancel{\mu}-xy-2z\lambda-\cancel{\mu}=0$

$z(y-x)+2\lambda(x-y)=0$; $-z(x-y)+2\lambda(x-y)=0$

$(x-y)(2\lambda-z)=0$

$x(z-y)+2\lambda(y-z)=0 \Rightarrow (y-z)(2\lambda-x)=0$

$$\begin{cases} (x-y)(2\lambda-z)=0 \\ (y-z)(2\lambda-x)=0 \end{cases}$$

$x=y \rightarrow 2y+z=0 \rightarrow z=-2y.$

$y^2+y^2+4y^2=1 \rightarrow y^2=\frac{1}{6} \rightarrow y=\pm\frac{1}{\sqrt6}$

$x=\pm\frac{1}{\sqrt6}$

$z=\mp\frac{2}{\sqrt6}$

$M_1\left(\frac{1}{\sqrt6};\frac{1}{\sqrt6};-\frac{2}{\sqrt6}\right)$; $M_2\left(-\frac{1}{\sqrt6};-\frac{1}{\sqrt6};\frac{2}{\sqrt6}\right)$

1) $\begin{cases} \frac{1}{\sqrt6}\cdot\left(-\frac{2}{\sqrt6}\right)+\frac{2}{\sqrt6}\lambda+\mu=0 \\ -\frac{2}{6}+\frac{2}{\sqrt6}\lambda+\mu=0 \end{cases} \Longleftrightarrow -\frac{1}{3}+\frac{2}{\sqrt6}\lambda=-\frac{1}{3}+\frac{2}{\sqrt6}\lambda$

?

$dL=(yz+2x\lambda+\mu)dx+(xz+2y\lambda+\mu)dy+(xy+2z\lambda+\mu)dz$

$d^2L=((zdy+ydz)+2\lambda dx)dx+((zdx+xdz)+2\lambda dy)dy+$

$+((xdy+ydx)+2\lambda dz)dz=$

$=\underline{zdxdy}+\underline{ydzdx}+2\lambda dx^2+\underline{zdxdy}+\underline{xdzdy}+2\lambda dy^2+\underline{xdzdy}+\underline{ydxdz}+2\lambda dz^2=$

$=2\lambda(dx^2+dy^2+dz^2)+2zdxdy+2ydzdx+2xdzdy$

От μ - не зависит.

При $M_1\left(\frac{1}{\sqrt6};\frac{1}{\sqrt6};-\frac{2}{\sqrt6}\right)\rightarrow d^2L=2\lambda(dx^2+dy^2+dz^2)-\frac{4}{\sqrt6}dxdy+\frac{2}{\sqrt6}dzdx+\frac{2}{\sqrt6}dzdy$

$\neq 0$?

При $M_2\left(-\frac{1}{\sqrt6};-\frac{1}{\sqrt6};\frac{2}{\sqrt6}\right)\rightarrow d^2L=2\lambda(dx^2+dy^2+dz^2)+\frac{4}{\sqrt6}dxdy-\frac{2}{\sqrt6}dzdx-\frac{2}{\sqrt6}dzdy$

$$u(M_1) = \frac{1}{\sqrt6}\cdot\frac{1}{\sqrt6}\left(-\frac{2}{\sqrt6}\right) = -\frac{2}{6\sqrt6} = -\frac{1}{3\sqrt6}$$

$$u(M_0) = \frac{1}{3\sqrt6}$$

Ответ: $u_{max} = \frac{1}{3\sqrt6}$

$u_{min} = -\frac{1}{3\sqrt6}$.

22.03.2021 г.

Двойное интегралы.
(расстановка пределов интегрирования)

Если $f(M)$ непрерывна в некоторой замкнутой плоской области D и если разбить эту область произвольным образом на n частичных областей с площадями $\Delta S_1, \Delta S_2, \dots, \Delta S_n$; выбрать в каждой из них по одной произвольной точке $M_1 \dots M_n$; вычислить значение ф-ии в этих точках и составить сумму.

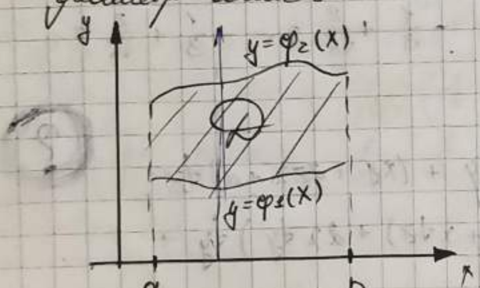
$$f(M_1)\Delta S_1 + f(M_2)\Delta S_2 + \dots + f(M_n)\Delta S_n =$$

$$= \sum_{i=1}^{n} f(M_i)\Delta S_i$$ — называется интегральной суммой ф-ии $f(M)$ по области D

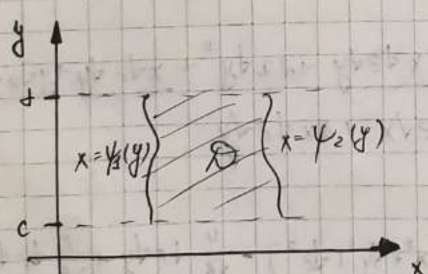
При $n \to \infty$; стремлении к нулю наибольшего из диаметров частичных областей все эти различные интегральные суммы имеют общий предел, который называется двойным интегралом от ф-и $f(M)$ по области D.

$$\iint\limits_D f(M)\,dS$$

диаметр области — наибольшая из её хорд.



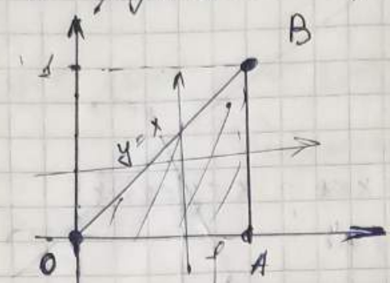
$$\iint\limits_D f(x,y)\,dx\,dy = \int_a^b dx \int_{\varphi_1(x)}^{\varphi_2(x)} f(x,y)\,dy$$



$$\iint\limits_D f(x,y)\,dx\,dy = \int_c^d dy \int_{\psi_1(y)}^{\psi_2(y)} f(x,y)\,dx$$

3916. $\iint\limits_D f(x,y)\,dx\,dy$.

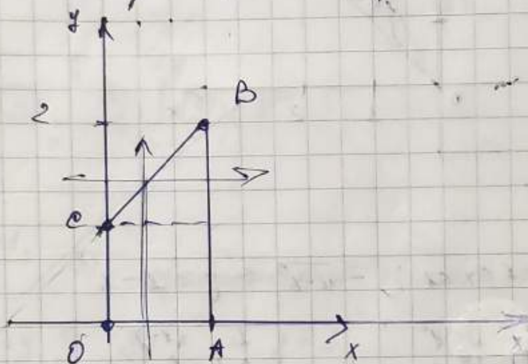
D – треугольник с вершинами $O(0,0)$, $A(1;0)$, $B(1;1)$



$$\iint\limits_D f(x,y)\,dx\,dy = \int_0^1 dx \int_0^x f(x,y)\,dy$$

$$\iint\limits_D f(x,y)\,dx\,dy = \int_0^1 dy \int_y^1 f(x,y)\,dx$$

3918. D – трапеция с вершинами $O(0,0)$; $A(1,0)$; $B(1;2)$; $C(0;1)$

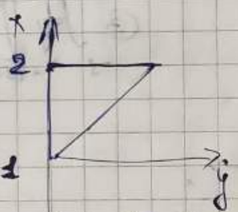


$$\iint\limits_D f(x,y)\,dx\,dy = \int_0^1 dx \int_0^{x+1} f(x,y)\,dy =$$

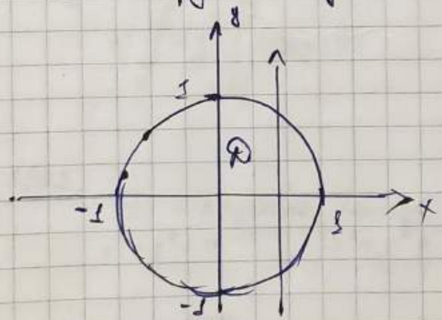
$$= \int dy \int f(x,y)\,dx$$

разбиваем на части

$$= \int_0^1 dy \int_0^1 f(x,y)\,dx + \int_1^2 dy \int_{y-1}^1 f(x,y)\,dx$$



3919. D – круг $x^2+y^2 \le 1$



$$\int_{-1}^1 dx \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} f(x,y)\,dy =$$

$$= \int_{-1}^1 dy \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} f(x,y)\,dx$$

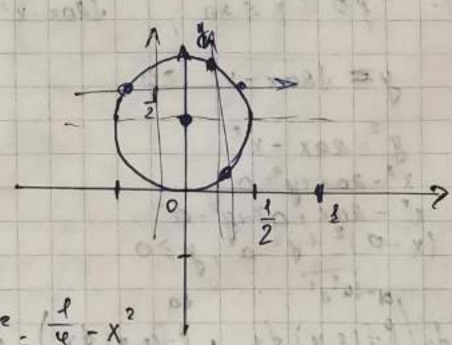
$y^2 = 1 - x^2$

$y = \pm\sqrt{1-x^2}$

3920. D – круг $x^2+y^2 \le y$

$x^2+y^2-y=0$

$x^2 + \left(y-\frac{1}{2}\right)^2 = \frac{1}{4}$



$$\iint\limits_D f(x,y)\,dx\,dy = \int_{-\frac{1}{2}}^{\frac{1}{2}} dx \int_{\frac{1}{2}-\sqrt{\frac{1}{4}-x^2}}^{\frac{1}{2}+\sqrt{\frac{1}{4}-x^2}} f(x,y)\,dy$$

$\left(y-\frac{1}{2}\right)^2 = \frac{1}{4} - x^2$

$y = \frac{1}{2} \pm \sqrt{\frac{1}{4} - x^2}$

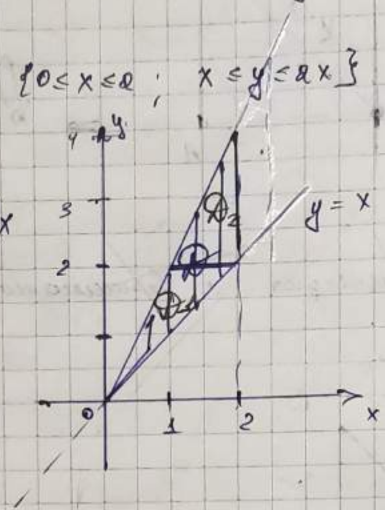
$= \int_0^1 dy \int_{-\sqrt{y-y^2}}^{\sqrt{y-y^2}} f(x,y)\,dx$

$x^2+y^2-y=0$

$x^2 = y-y^2 \Rightarrow x=\pm\sqrt{y-y^2}$

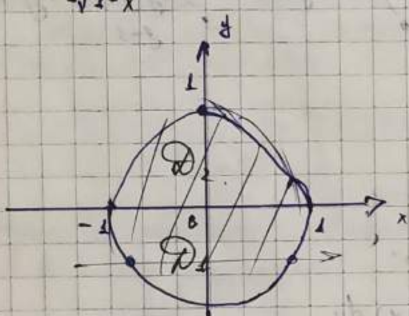
3924 $\int_0^2 dx \int_x^{2x} f(x,y)\,dy =$ $\qquad D=\{0\le x\le 2;\ x\le y\le 2x\}$

$=\int_0^2 dy \int_{\frac{y}{2}}^{y} f(x,y)\,dx + \int_2^4 dy \int_{\frac{y}{2}}^{2} f(x,y)\,dx$



3927. $\int_{-1}^{1} dx \int_{-\sqrt{1-x^2}}^{1-x^2} f(x,y)\,dy =$ $\qquad D: \{-1\le x\le 1;\ -\sqrt{1-x^2}\le y\le 1-x^2$

$= \int_{-1}^{0} dy \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} f(x,y)\,dx + \int_0^1 dy \int_{-\sqrt{1-y}}^{\sqrt{1-y}} f(x,y)\,dx$



$1-x^2=y$; $\quad 1-y=x^2$

$x=\pm\sqrt{1-y}$

3929 $\int_0^{2a} dx \int_{\sqrt{2ax-x^2}}^{\sqrt{2ax}} f(x,y)\,dy =$ $\qquad (a>0)$

$D: \{0\le x\le 2a;\ \sqrt{2ax-x^2}\le y\le\sqrt{2ax}\}$

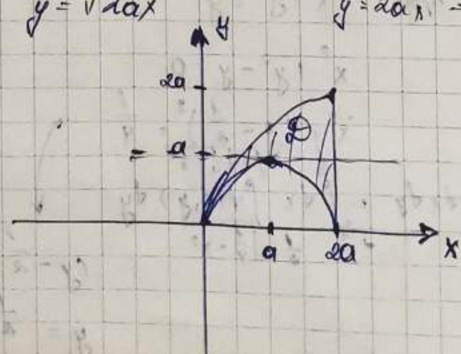
$y=\sqrt{2ax-x^2}\ge 0$ $\qquad y=\sqrt{2ax}$ $\qquad y^2=2ax \rightarrow x=\frac{y^2}{2a}$

$y^2=2ax-x^2$

$x^2-2ax+y^2=0$

$x^2-2ax+a^2+y^2=a^2$

$(x-a)^2+y^2=a^2,\ y\ge 0$



$= \int_0^a dy\left(\int_{\frac{y^2}{2a}}^{a-\sqrt{a^2-y^2}} f(x,y)\,dx + \int_{a+\sqrt{a^2-y^2}}^{2a} f(x,y)\,dx\right) +$

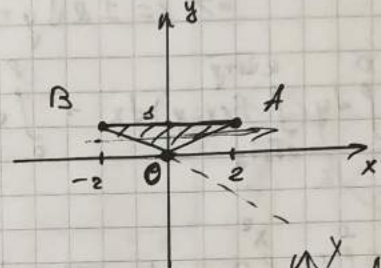
$$+\int_0^{2a} dy \int_{\frac{y^2}{2a}}^{2a} f(x,y)\,dx$$

Д/з: ~~3917~~, ~~3921~~, ~~3922~~, ~~3925~~, ~~3926~~, ~~3928~~, ~~3930~~, 3931?

Домашняя работа.

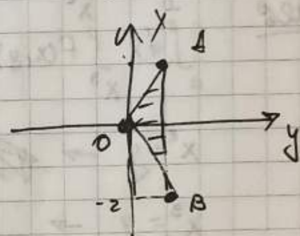
3917. $\iint f(x,y)\,dx\,dy =$

D - треугольник с вершинами $O(0,0)$ $A(2,1)$ $B(-2;1)$



$$= \int_{-2}^{0} dx \int_{-\frac{1}{2}x}^{1} f(x,y)\,dy + \int_0^2 dx \int_{\frac{1}{2}x}^{1} f(x,y)\,dy$$

$$= \int_0^1 dy \int_{-2y}^{2y} f(x,y)\,dx$$

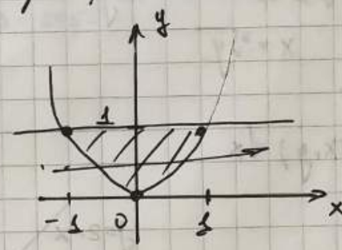


3921. D - параболический сегмент, огр. кривыми:

$x = \pm\sqrt{y} \leftarrow y = x^2$ и $y = 1$.

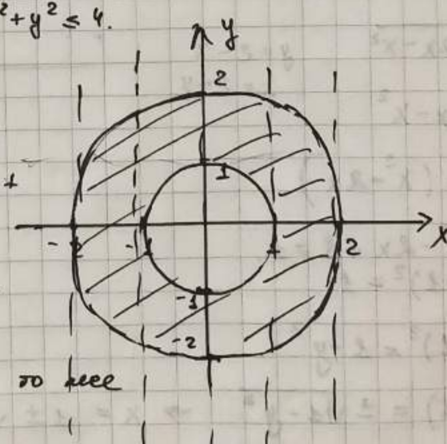
$$D = \int_{-1}^{1} dx \int_{x^2}^{1} f(x,y)\,dy =$$

$$= \int_0^1 dy \int_{-\sqrt{y}}^{\sqrt{y}} f(x,y)\,dx$$



3922. D - круговое кольцо $1 \le x^2+y^2 \le 4$.

$x^2+y^2=4$ $\quad x^2+y^2=1$.



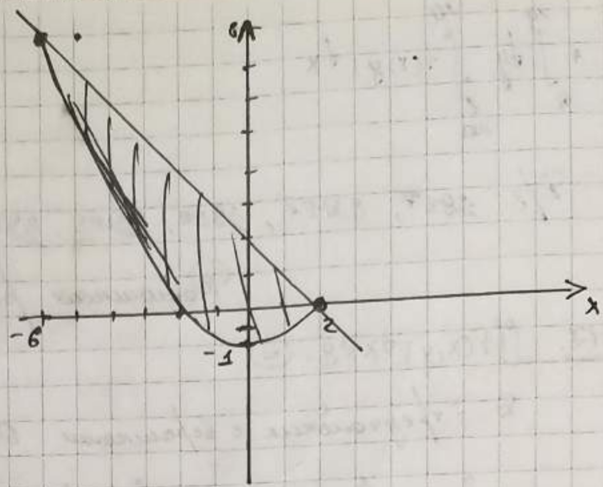
$$D = \int_{-2}^{-1} dx \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} f(x,y)\,dy + \int_{-1}^{1} dx \int_{-\sqrt{4-x^2}}^{-\sqrt{1-x^2}} f(x,y)\,dy +$$

$$+ \int_{-1}^{1} dx \int_{\sqrt{1-x^2}}^{\sqrt{4-x^2}} f(x,y)\,dy + \int_{1}^{2} dx \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} f(x,y)\,dy.$$

Кольцо симметрично - отн. y/x. то же самое:

$$D = \int_{-2}^{-1} dy \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} f(x,y)\,dx + \int_{-1}^{1} dy \int_{-\sqrt{4-y^2}}^{-\sqrt{1-y^2}} f(x,y)\,dx + \int_{-1}^{1} dy \int_{\sqrt{1-y^2}}^{\sqrt{4-y^2}} f(x,y)\,dx + \int_{1}^{2} dy \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} f(x,y)\,dx$$

3925. $$\int_{-6}^{2} dx \int_{\frac{x^2}{4}-1}^{2-x} f(x,y)\,dy =$$



$y = 2 - x$

$\to x = 2 - y.$

$y = \frac{x^2}{4} - 1 \quad | \cdot 4$

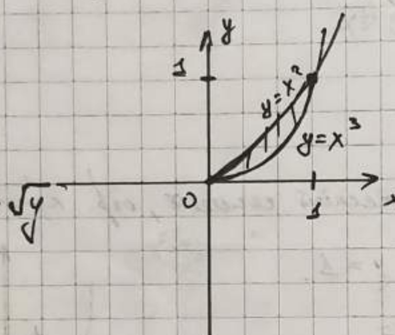
$4y = x^2 - 4$

$4(y+1) = x^2$

$\Rightarrow x = \pm 2\sqrt{y+1}.$

$$= \int_{-1}^{0} dy \int_{-2\sqrt{1+y}}^{2\sqrt{1+y}} f(x,y)\,dx + \int_{0}^{8} dy \int_{-2\sqrt{1+y}}^{2-y} f(x,y)\,dx$$

3926. $$\int_{0}^{1} dx \int_{x^3}^{x^2} f(x,y)\,dy =$$

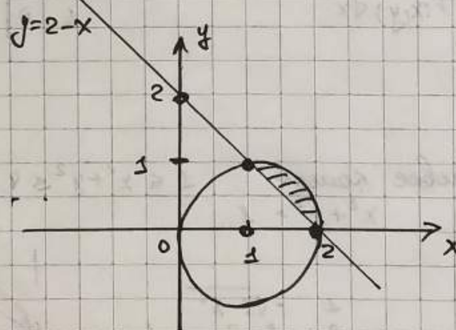


$x^2 = y \to x = \pm\sqrt{y}$

$x^3 = y \to x = \sqrt[3]{y}.$

$$= \int_{0}^{1} dy \int_{\sqrt{y}}^{\sqrt[3]{y}} f(x,y)\,dx$$

3928. $$\int_{1}^{2} dx \int_{2-x}^{\sqrt{2x-x^2}} f(x,y)\,dy. =$$



$y = \sqrt{2x - x^2}$ $\qquad$ $y = 2 - x$

$\qquad\qquad\qquad\qquad x = 2 - y$

$y^2 = 2x - x^2$

$y^2 = -(x^2 - 2x)$

$y^2 + x^2 - 2x + 1 = 1.$

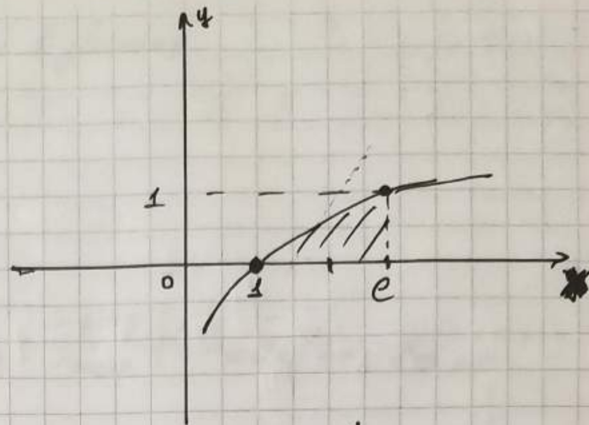
$y^2 + (x-1)^2 = 1$

$(x-1)^2 = 1 - y^2$

$(x-1) = \pm\sqrt{1-y^2} \to x = 1 \pm \sqrt{1-y^2}$

$$= \int_{0}^{1} dy \int_{2-y}^{1+\sqrt{1-y^2}} f(x,y)\,dx$$

3930. $\int_1^e dx \int_0^{\ln x} f(x,y)\,dy =$

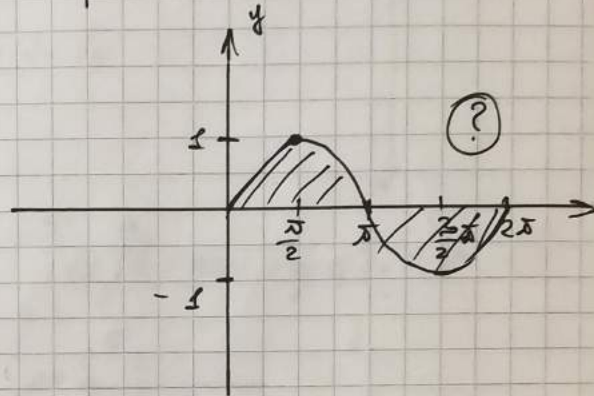


$y=\ln x \qquad y=0$

$x=e^y$

$= \int_0^1 dy \int_{e^y}^{e} f(x,y)\,dx$

3931. $\int_0^{2\pi} dx \int_0^{\sin x} f(x,y)\,dy =$



$y=0 \quad ; \quad y=\sin x \to x=\arcsin y$

$= \int dy \int f(x,y)\,dy$

Д/З: Берман: 3525-3530, 3533, 3558, 3540.
Демидович: 3944-3948, 3951, 3953-3955

Замена переменных в двойных интегралах.

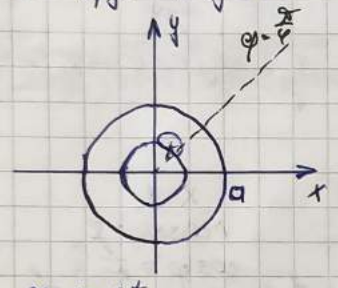
$$\begin{cases} x = r\cos\varphi \\ y = r\sin\varphi \end{cases} \qquad r \geq 0, \quad 0 \leq \varphi < 2\pi$$

Якобиан преобразования

$$J(r,\varphi) = \begin{vmatrix} x'_r & x'_\varphi \\ y'_r & y'_\varphi \end{vmatrix} = \begin{vmatrix} \cos\varphi & -r\sin\varphi \\ \sin\varphi & r\cos\varphi \end{vmatrix} = r\cos^2\varphi + r\sin^2\varphi = r$$

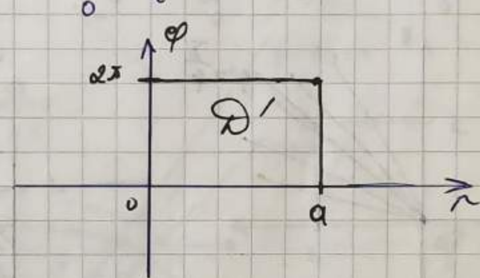
$$\iint\limits_D f(x,y)\,dx\,dy = \iint\limits_{D'} f(r\cos\varphi, r\sin\varphi)\cdot r\,dr\,d\varphi.$$

3937. D – круг $x^2+y^2 \leq a^2$



$\varphi = const$

$$\iint\limits_D f(x,y)\,dx\,dy = \int_0^{2\pi} d\varphi \int_0^a f(r\cos\varphi, r\sin\varphi)\cdot r\,dr$$



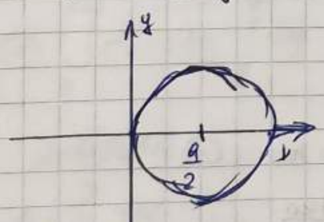
3938. (*) D – круг: $x^2+y^2 \leq ax \quad (a>0)$

$x^2+y^2 = ax$

$x^2 - ax + y^2 = 0$

$x^2 - ax + \frac{1}{4}a^2 + y^2 = \frac{a^2}{4}$

$\left(x - \frac{a}{2}\right)^2 + y^2 = \frac{a^2}{4} \rightarrow \left(x - \frac{a}{2}\right)^2 + y^2 \leq \frac{a^2}{4}$



$$\iint\limits_D f(x,y)\,dx\,dy = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\varphi \int_0^{a\cos\varphi} f(r\cos\varphi; r\sin\varphi)\cdot r\,dr$$

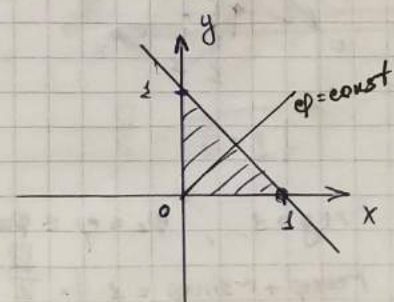
(*) $r^2\cos^2\varphi - a\cdot r\cos\varphi + r^2\sin^2\varphi = 0$

$r^2 - ar\cos\varphi = 0 \quad |:r$

$r = a\cos\varphi$ – в полярных сист. коорд.

3940. D – треугольник, $0 \leq x \leq 1, \quad 0 \leq y \leq 1-x$

$$\iint\limits_D f(x,y)\,dx\,dy = \int_0^{\frac{\pi}{2}} d\varphi \int_0^{\frac{\sqrt{2}}{2}\operatorname{cosec}(\varphi+\frac{\pi}{4})} r f(r\cos\varphi; r\sin\varphi)\,dr$$



$y = 1 - x$

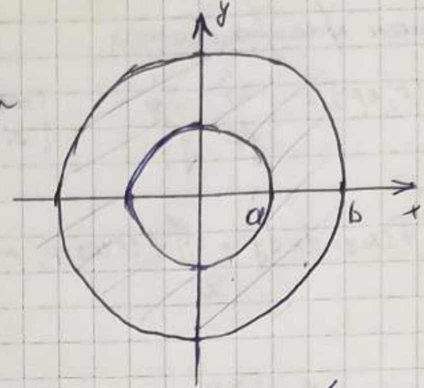
$r\sin\varphi = 1 - r\cos\varphi$

$r(\sin\varphi+\cos\varphi)=1$

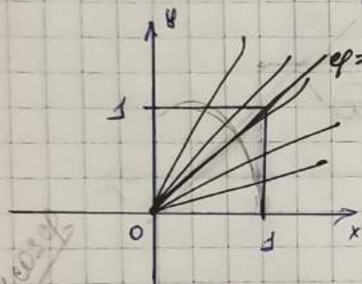
$$r=\frac{1}{\sin\varphi+\cos\varphi}=\frac{1}{\sqrt2\left(\frac{\sqrt2}{2}\sin\varphi+\frac{\sqrt2}{2}\cos\varphi\right)}=\frac{1}{\sqrt2\,\sin(\varphi+\frac{\pi}{4})}=\frac{\sqrt2}{2}\operatorname{cosec}(\varphi+\tfrac{\pi}{4})$$

3938. D - кольцо: $a\le x^2+y^2\le b^2$

$$\iint_D f(x,y)\,dx\,dy=\int_0^{2\pi}d\varphi\int_{|a|}^{|b|} r\,f(r\cos\varphi, r\sin\varphi)\,dr$$



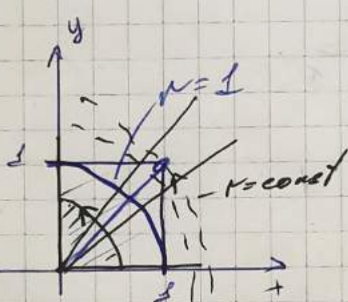
3943. $$\int_0^1 dx\int_0^1 f(x,y)\,dy=\int_0^{\frac{\pi}{4}}d\varphi\int_0^{\frac{1}{\cos\varphi}} r\,f(r\cos\varphi, r\sin\varphi)\,dr+\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}d\varphi\int_0^{\frac{1}{\sin\varphi}} r\,f(r\cos\varphi, r\sin\varphi)\,dr \;=$$



$x=1 \Rightarrow r\cos\varphi=1$, $r=\frac{1}{\cos\varphi}$

$y=1 \rightarrow r\sin\varphi=1$, $r=\frac{1}{\sin\varphi}$

$y=x$, $r\sin\varphi=r\cos\varphi$



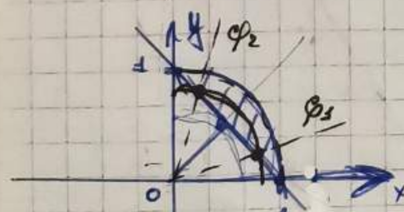
$x=1.$
$r\cos\varphi=1$
$\cos\varphi=\frac{1}{r}$
$\varphi=\arccos\frac{1}{r}$

$y=1. \rightarrow \varphi=\arcsin\frac{1}{r}$

$$=\int dr\int r\,f(r\cos\varphi, r\sin\varphi)\,d\varphi=$$
$$=\int_0^{\sqrt2} r\,dr\int_0^{\arcsin\frac1r} f(r\cos\varphi, r\sin\varphi)\,d\varphi+$$
$$+\int_1 r\,dr\int_{\arccos\frac1r} f(r\cos\varphi, r\sin\varphi)\,d\varphi$$

3944. $$\int_0^1 dx\int_{1-x}^{\sqrt{1-x^2}} f(x,y)\,dy=\int_0^{\frac{\pi}{2}}d\varphi\int_{\frac{\sqrt2}{2}\operatorname{cosec}(\varphi+\frac{\pi}{4})}^{1} r\,f(r\cos\varphi, r\sin\varphi)\,dr$$ — см. 3940

$$=\int_{\frac{\sqrt2}{2}}^{1} r\,dr\int_{\frac{\pi}{4}-\arccos\frac{1}{r\sqrt2}}^{\frac{\pi}{4}+\arccos\frac{1}{r\sqrt2}} f(r\cos\varphi, r\sin\varphi)\,d\varphi$$



$\sqrt{1-\frac{2}{4}}=\frac{\sqrt2}{2}$

$x+y=1$; $\varphi_1\le\varphi\le\varphi_2$.

$r\cos\varphi+r\sin\varphi=1 \;\Big|\cdot\frac{\sqrt2}{2}$ $\qquad a\sin x+b\cos x=c \quad \Big|:\sqrt{a^2+b^2}$

$$\underbrace{\frac{a}{\sqrt{a^2+b^2}}}_{\cos\varphi} \sin x + \underbrace{\frac{b}{\sqrt{a^2+b^2}}}_{\sin\varphi} \cos x = \frac{c}{\sqrt{a^2+b^2}}$$

$$\sin(x+\varphi) = \frac{c}{\sqrt{a^2+b^2}}$$

$$\cos\varphi + \sin\varphi = \frac{1}{r}$$

$$\sqrt{2}\left(\cos\frac{\pi}{4}\cos\varphi + \sin\frac{\pi}{4}\sin\varphi\right) = \frac{1}{r}$$

$$\cos\left(\varphi - \frac{\pi}{4}\right) = \frac{1}{r\sqrt{2}}$$

$$\varphi - \frac{\pi}{4} = \pm\arccos\left(\frac{1}{r\sqrt{2}}\right) \to \varphi = \frac{\pi}{4} \pm \arccos\left(\frac{1}{r\sqrt{2}}\right)$$

$$\Rightarrow \varphi_1 = \frac{\pi}{4} - \arccos\frac{1}{r\sqrt{2}}$$

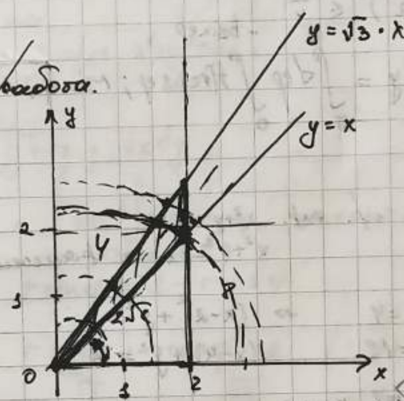
$$\varphi_2 = \frac{\pi}{4} + \arccos\frac{1}{r\sqrt{2}}$$

Бермант ~~3546 3530~~, ~~3580~~, 3558, 3540

Д/з: ~~3945~~, 3946, ~~3947~~, ~~3953 3955~~.

$y = r\sin\varphi.$
$x = r\cos\varphi.$

Домашняя работа.



3946.
$$\int_0^2 dx \int_x^{x\sqrt{3}} f(\sqrt{x^2+y^2})\,dy \;=$$

$$f(\sqrt{x^2+y^2}) = f(r)$$

$$= \int_{\pi/4}^{\pi/3} d\varphi \int_0^{\frac{2}{\cos\varphi}} r f(r)\,dr \;=$$

$$\sqrt{r^2\cos^2\varphi + r^2\sin^2\varphi} = r$$

$$r_1\cos\varphi = 2 \to r_1 = \frac{2}{\cos\varphi}; \quad r_2 = \frac{2}{\cos\varphi}.$$

$$= \int_0^{2\sqrt{2}} r f(r)\,dr \int_{\pi/4}^{\pi/3} d\varphi + \int_{2\sqrt{2}}^{4} r f(r)\,dr \int_{\arccos\frac{2}{r}}^{\pi/3} d\varphi$$

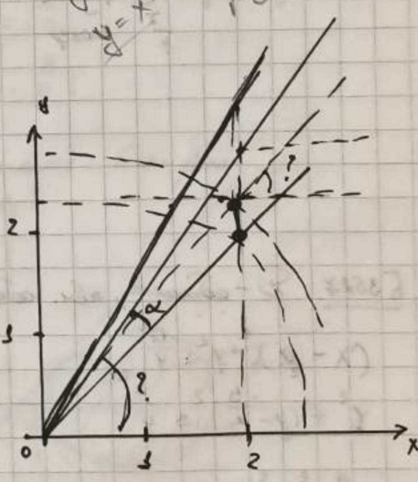
~~$2\ldots\sin\varphi$~~

$$\varphi_2 = \frac{\pi}{4} + \alpha$$

$$\frac{2}{r} = \cos\varphi_2 \Rightarrow \varphi_2 = \arccos\frac{2}{r}$$

?

$y = x\sqrt{3}$
$y = x$
$x = 2$
$r\cos\varphi = 2$



Б 3526. Д - круг 1) $x^2+y^2 \le R^2$; 2) $x^2+y^2 \le ax$; 3) $x^2+y^2 \le by$.

1) $x^2+y^2 \le R^2$

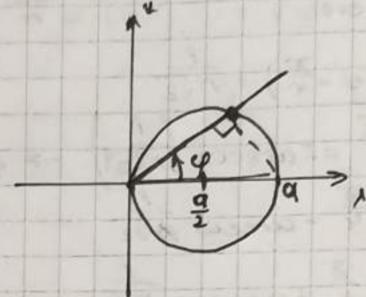
$$\iint f(x,y)\,dx\,dy = \int_0^{2\pi} d\varphi \int_0^R r f(r\cos\varphi; r\sin\varphi)\, r\,dr$$

2) $x^2+y^2 \le ax$

$x^2+y^2-ax \le 0$

$x^2-ax+\frac{a^2}{4}+y^2 \le \frac{a^2}{4}$

$(x-\frac{a}{2})^2+y^2 \le \frac{a^2}{4}.$

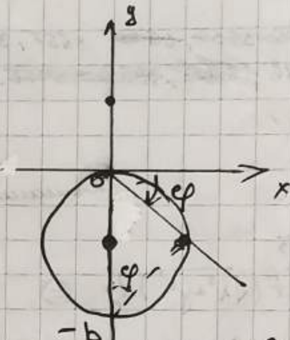


$$\iint f(x,y)\,dx\,dy = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\varphi \int_0^{a\cos\varphi} f(r\cos\varphi; r\sin\varphi)\, r\,dr$$

3) $x^2+y^2-by+\frac{b^2}{4} \le \frac{b^2}{4}$

$x^2+(y-\frac{b}{2})^2 \le \frac{b^2}{4}$

$$\iint f(x,y)\,dx\,dy = \int_{-\pi}^{0} d\varphi \int_0^{b\sin\varphi} f(r\cos\varphi; r\sin\varphi)\, r\,dr$$

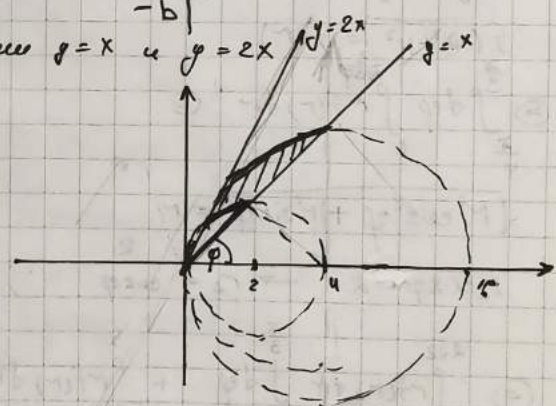


Б 3528. Д - область огр. окр. $x^2+y^2=4x$
$x^2+y^2=8x$ и прямыми $y=x$ и $y=2x$

$x^2+y^2-4x+4=4 \Rightarrow (x-2)^2+y^2=4$

$x^2-8x+16+y^2=16 \quad (x-4)^2+y^2=16$

$$\iint f(x,y)\,dx\,dy = \int_{\frac{\pi}{4}}^{\operatorname{arctg} 2} d\varphi \int_{4\cos\varphi}^{16\cos\varphi} f(r\cos\varphi; r\sin\varphi)\, r\,dr$$



$r_1 = 4\cos\varphi$

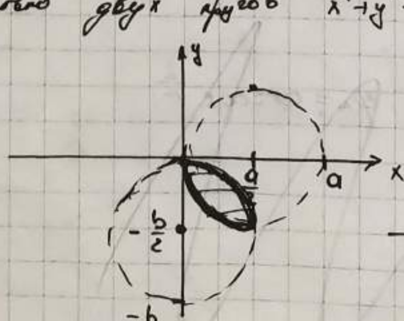
$r = \underline{16\cos\varphi - 4\cos\varphi}$

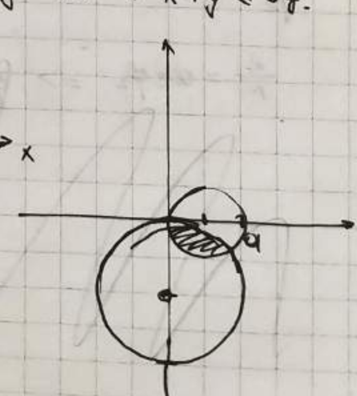
Б 3527. Д - область, явл. общей частью двух кругов $x^2+y^2 \le ax$ и $x^2+y^2 \le by$.

$(x-\frac{a}{2})^2+y^2 \le \frac{a^2}{4}$

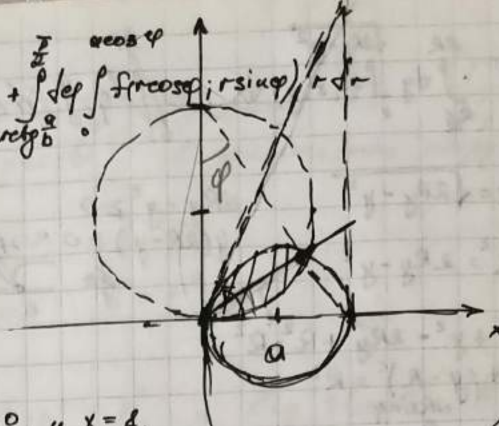
$x^2+(y-\frac{b}{2})^2 \le \frac{b^2}{4}$

$$\begin{cases} x^2+y^2 \le ax \\ x^2+y^2 \le by \end{cases}$$





$$\iint f(x,y)\,dx\,dy = \int_0^{\operatorname{arctg}\frac{a}{b}} d\varphi \int_0^{b\sin\varphi} f(r\cos\varphi; r\sin\varphi)\, r\,dr + \int_{\operatorname{arctg}\frac{a}{b}}^{\frac{\pi}{2}} d\varphi \int_0^{a\cos\varphi} f(r\cos\varphi; r\sin\varphi)\, r\,dr$$



$r = a\cos\varphi. = b\cos(90^\circ - \varphi)$

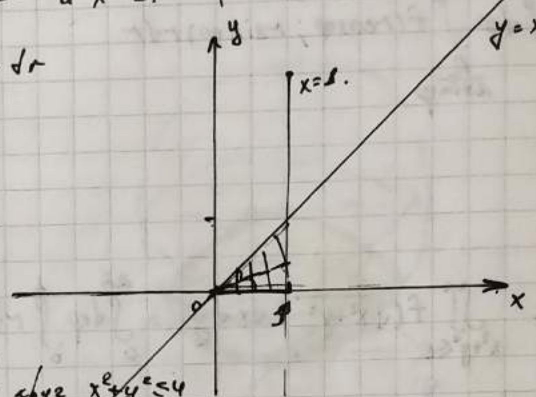
$a\cos\varphi = b\sin\varphi$

$\operatorname{tg}\varphi = \frac{a}{b} \Rightarrow \varphi = \operatorname{arctg}\frac{a}{b}$

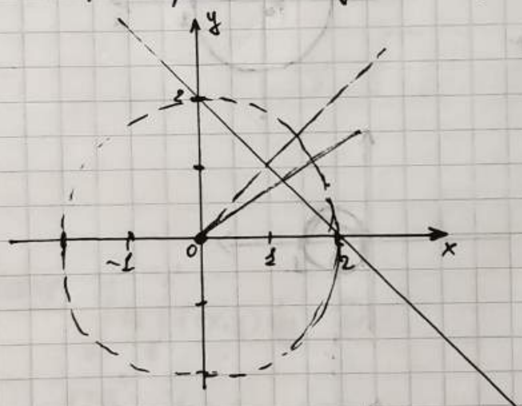
Б3528. Д-область, огр. прямыми $y=x$, $y=0$ и $x=1$.

$$\iint f(x,y)\,dx\,dy = \int_0^{\frac{\pi}{4}} d\varphi \int_0^{\sec\varphi} f(r\cos\varphi; r\sin\varphi)\, r\,dr$$

$r\cos\varphi = 1 \to r = \frac{1}{\cos\varphi} = \sec\varphi$



Б3529.
Д-меньший из двух сегментов, на которые прямая $y=2-x$ пересекает круг $x^2+y^2\le 4$



$$\iint f(x,y)\,dx\,dy = \int_0^{\frac{\pi}{2}} d\varphi \int_{\sqrt{2}\sec(\varphi-\frac{\pi}{4})}^{2} f(r\cos\varphi; r\sin\varphi)\, r\,dr$$

$r\sin\varphi = 2 - r\cos\varphi$

$r\sin\varphi + r\cos\varphi = 2 \quad \Big|\cdot\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$

$r\left(\frac{\sqrt{2}}{2}\sin\varphi + \frac{\sqrt{2}}{2}\cos\varphi\right) = \sqrt{2}$

$r\left(\cos\left(\varphi - \frac{\pi}{4}\right)\right) = \sqrt{2}$

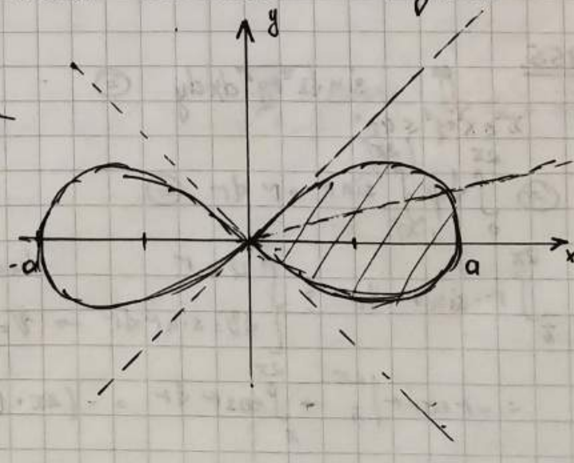
$r = \sqrt{2}\cdot\sec\left(\varphi - \frac{\pi}{4}\right)$

Б3530. Д-внутренняя часть правой петли лемнискаты Бернулли.

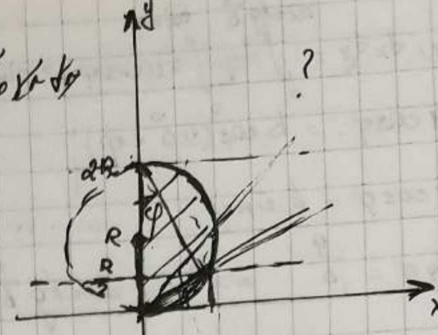
$(x^2+y^2)^2 = a^2(x^2-y^2)$

$$\iint f(x,y)\,dx\,dy = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} d\varphi \int_0^{a\sqrt{\cos 2\varphi}} f(r\cos\varphi; r\sin\varphi)\, r\,dr$$

$r^2 = a^2\cos 2\varphi \to r = a\sqrt{\cos 2\varphi}$

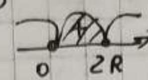


Б 3593. $\int\limits_{R/2}^{2R} dy \int\limits_{0}^{\sqrt{2Ry-y^2}} f(x,y)\,dx = \cancel{\int\limits_{0}^{\pi/2} d\varphi \int f(r\cos\varphi; r\sin\varphi)\, r\, d\varphi}$



$x = \sqrt{2Ry - y^2}$ $\quad 2Ry - y^2 \geq 0$

$\quad y(2R - y) \geq 0$



$x^2 = 2Ry - y^2$

$x^2 + y^2 - 2Ry + R^2 = R^2$

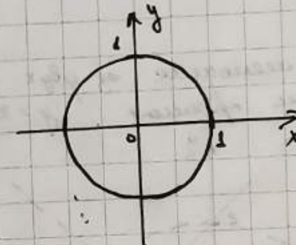
$x^2 + (y - R)^2 = R^2$

$\frac{R}{2r} = \sin\varphi$

$\Rightarrow r = \frac{R}{2\sin\varphi}$

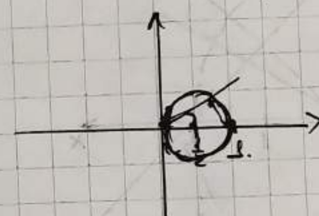
$= \int\limits_{\pi/6}^{\pi/2} d\varphi \int\limits_{\frac{R}{2\sin\varphi}}^{2R\sin\varphi} f(r\cos\varphi; r\sin\varphi)\, r\, dr$

Д 3951. $\iint\limits_{x^2+y^2\leq 1} f(\sqrt{x^2+y^2})\,dx\,dy = \int\limits_0^{2\pi} d\varphi \int\limits_0^1 r f(r)\,dr$



Д 3953. $\iint\limits_{x^2+y^2\leq x} f\left(\frac{y}{x}\right) dx\,dy = \int\limits_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\varphi \int\limits_0^{\cos\varphi} f(\operatorname{tg}\varphi)\, r\, dr$

$\frac{y}{x} = \frac{r\sin\varphi}{r\cos\varphi} = \operatorname{tg}\varphi$

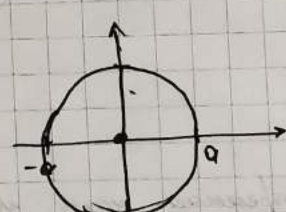


$x^2 - x + \frac{1}{4} + y^2 \leq \frac{1}{4}$

$(x - \frac{1}{2})^2 + y^2 \leq \frac{1}{4}$

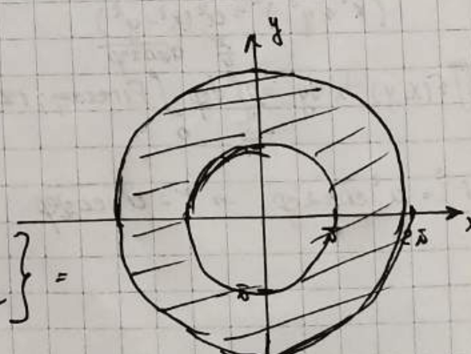
Д 3954. $\iint\limits_{x^2+y^2\leq a^2} \sqrt{x^2+y^2}\,dx\,dy = \int\limits_0^{2\pi} d\varphi \int\limits_0^{a} r^2\,dr =$

$= \int\limits_0^{2\pi} d\varphi \cdot \frac{a^3}{3} = \frac{2\pi a^3}{3}$



Д 3955. $\iint\limits_{\pi^2\leq x^2+y^2\leq 4\pi^2} \sin\sqrt{x^2+y^2}\,dx\,dy =$

$= \int\limits_0^{2\pi} d\varphi \int\limits_{\pi}^{2\pi} \sin r \cdot r\,dr =$



$\int\limits_{\pi}^{2\pi} r \cdot \sin r\,dr = \left\{ \begin{array}{ll} u = r & du = dr \\ dv = \sin r\,dr \to & v = -\cos r \end{array} \right\} =$

$= -r\cos r \Big|_{\pi}^{2\pi} + \int\limits_{\pi}^{2\pi} \cos r\,dr = -(2\pi \cdot (+1) + \pi) + \underbrace{(\sin 2\pi - \sin\pi)}_{=0} =$

$= -3\pi$

$= 2\pi \cdot (-3\pi) = -6\pi^2$

$\iiint\limits_D f(x,y,z)\,dx\,dy\,dz$

$(x = r\cos\varphi;\ y = r\sin\varphi;\ z = z)$ – цилиндр. с.к.

$(x = \rho\cos\varphi\sin\theta,\ y = \rho\sin\varphi\sin\theta,\ z = \rho\cos\theta)$ – в сферических коорд-х

Для цилиндрической системы координат: $I = r$

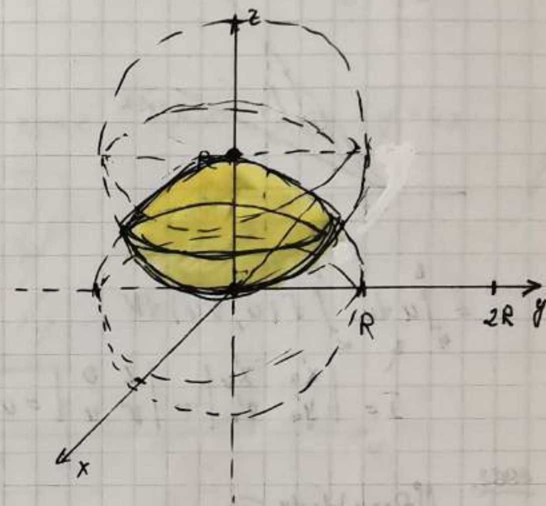
Для сферической системы координат: $I = r^2\cos\theta$

Б 3551. D – общая часть двух шаров $x^2+y^2+z^2 \le R^2$ и $x^2+y^2+(z-R)^2 \le R^2$

В цилиндрической с.к.:

$\iiint\limits_D f(x,y,z)\,dx\,dy\,dz =$

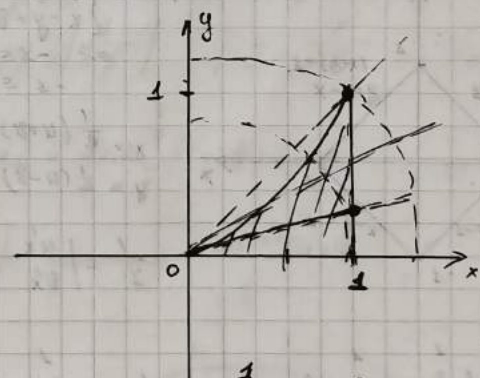
$= \int\limits_0^{2\pi} d\varphi \int r\,dr \int f(r\cos\varphi; r\sin\varphi; z)\,dz$



Д3946. $\int\limits_0^1 dx \int\limits_0^{x^2} f(x,y)\,dy \ominus$

$y = x^2$

$\ominus \int\limits_0^{\frac{\pi}{4}} d\varphi \int^{\frac{1}{\cos\varphi}} f(r\cos\varphi; r\sin\varphi)\,r\,dr$



$\frac{1}{r_2} = \cos\varphi \Rightarrow r_2 = \frac{1}{\cos\varphi}$

6.04.2021.

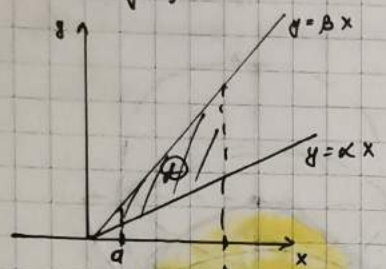
Замена переменных в двойных интегралах (общий случай).

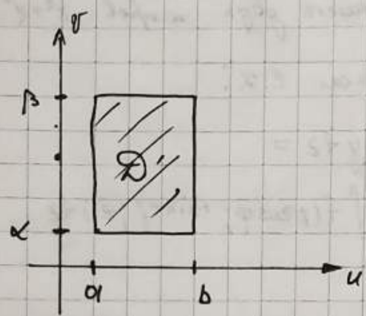
3957. Вместо x и y ввести новые переменные u и v и определить пределы интегрирования в следующих двойных интегралах:

$$\int_a^b dx \int_{\alpha x}^{\beta x} f(x,y)\,dy \quad \text{, если } u=x,\ v=\frac{y}{x}; \quad 0<a<b.$$

$a \le x \le b$

$\alpha x \le y \le \beta x$





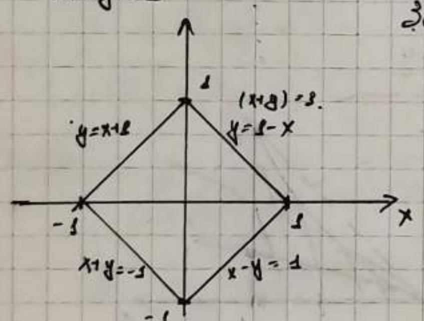
$$= \int_a^b u\,du \int_\alpha^\beta f(u;\, vu)\,dv$$

$$\begin{cases} x = u \\ y = uv \end{cases}$$

$$I = \begin{vmatrix} x'_u & x'_v \\ y'_u & y'_v \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ v & u \end{vmatrix} = u \quad ; \quad |I| = |u|$$

3962.

$$\iint\limits_{|x|+|y|\le 1} f(x+y)\,dx\,dy =$$



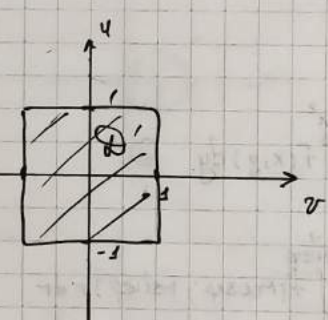
Замена $\begin{cases} x+y=u \\ x-y=v \end{cases}$

$\Rightarrow -1 \le u \le 1$

$-1 \le v \le 1.$

$x = \frac{1}{2}(u+v)$

$y = \frac{1}{2}(u-v)$



$$\frac{1}{I} = \begin{vmatrix} u'_x & u'_y \\ v'_x & v'_y \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -2 \quad \Rightarrow I = -\frac{1}{2}$$

$$|I| = \frac{1}{2}$$

$$= \frac{1}{2}\int_{-1}^{1} dv \int_{-1}^{1} f(u)\,du = \frac{1}{2}\cdot(2)\int_{-1}^{1} f(u)\,du = \int_{-1}^{1} f(u)\,du$$

$v|_{-1}^{1} = 2$

3965. $\iint\limits_D (x+y)\,dx\,dy$, D огр. кривой $x^2+y^2 = x+y$

$x^2 - x + \frac{1}{4} + y^2 - y + \frac{1}{4} = \frac{1}{2}$

$(x-\frac{1}{2})^2 + (y-\frac{1}{2})^2 = \frac{1}{2}$

Замена

$x = \frac{1}{2} + r\cos\varphi$

$y = \frac{1}{2} + r\sin\varphi$

$D' = \{(r;\varphi):\ 0 \le r \le \frac{1}{\sqrt{2}};\ 0 \le \varphi \le 2\pi\}$

$I = r$ (проверить самостоятельно).

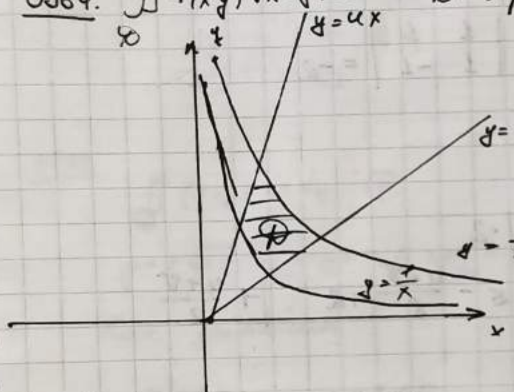
$$= \int_0^{2\pi} d\varphi \int_0^{\frac{1}{\sqrt{2}}} r(1 + r\cos\varphi + r\sin\varphi)\,dr = \int_0^{2\pi} d\varphi \left(\int_0^{\frac{1}{\sqrt{2}}} r\,dr + \int_0^{\frac{1}{\sqrt{2}}} r^2(\cos\varphi + \sin\varphi)\,dr \right) =$$

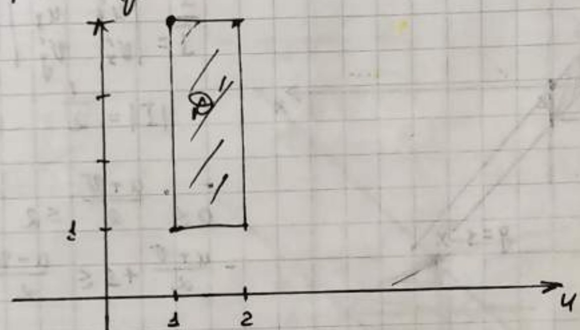
$$= \int_0^{2\pi} d\varphi \cdot \frac{r^2}{2}\Big|_0^{\frac{1}{\sqrt{2}}} + \int_0^{2\pi} (\cos\varphi + \sin\varphi)\,d\varphi \int_0^{\frac{1}{\sqrt{2}}} r^2\,dr =$$

$$= 2\pi \cdot \frac{1}{4} + \frac{r^3}{3}\Big|_0^{\frac{1}{\sqrt{2}}} \cdot \int_0^{2\pi} (\cos\varphi + \sin\varphi)\,d\varphi = \frac{\pi}{2} + \frac{r^3}{3}\Big|_0^{\frac{1}{\sqrt{2}}} \cdot \underbrace{(\sin\varphi - \cos\varphi)\Big|_0^{2\pi}}_{=0} =$$

$$= \frac{\pi}{2}$$

3964. $\iint\limits_D f(xy)\,dx\,dy = $ D огр. кривыми $xy = 1$; $xy = 2$, $y = x$; $y = 4x$





$$\begin{cases} u = xy \\ v = \frac{y}{x} \end{cases} \rightarrow \begin{matrix} 1 \le u \le 2 \\ 1 \le v \le 4 \end{matrix}$$

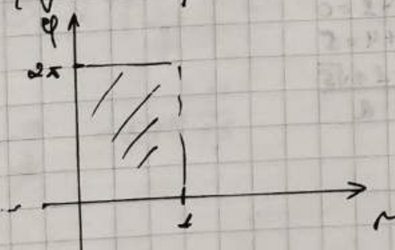
$$\frac{1}{I} = \begin{vmatrix} y & x \\ -\frac{y}{x^2} & \frac{1}{x} \end{vmatrix} = \frac{y}{x} + \frac{y}{x} = \frac{2y}{x} = 2v.$$

$$\Rightarrow |I| = \left|\frac{1}{2v}\right| ; \quad I = \frac{1}{2v} \quad (v > 0)$$

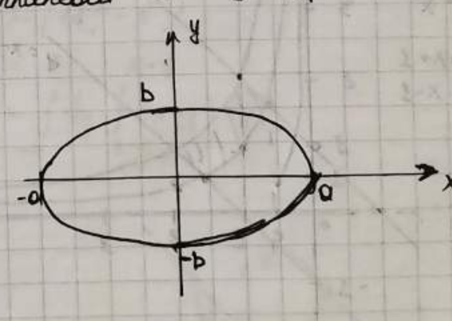
$$= \frac{1}{2} \int_1^4 \frac{dv}{v} \int_1^2 f(u)\,du = \frac{1}{2} \cdot \ln 4 \cdot \int_1^2 f(u)\,du = \ln 2 \int_1^2 f(u)\,du$$

3967. $\iint\limits_D \sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}\,dx\,dy$; D огр. эллипсом $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

$$\begin{cases} x = ar\cos\varphi \\ y = br\sin\varphi \end{cases}$$



$0 \le r \le 1$
$0 \le \varphi \le 2\pi$



$$I = \begin{vmatrix} a\cos\varphi & -ar\sin\varphi \\ b\sin\varphi & br\cos\varphi \end{vmatrix} = abr \quad ; \quad |I| = abr$$

$$= \int_0^{2\pi} d\varphi \int_0^1 abr\sqrt{1 - r^2}\,dr = \frac{\cancel{2}\pi ab}{\cancel{2}} \int_0^1 \sqrt{1 - r^2}\,d(r^2) = -\pi ab \int_0^1 \sqrt{1 - r^2}\,d(1 - r^2) =$$

$= -\pi ab\ (1-r^2)^{\frac{3}{2}}\Big|_0^1 \cdot \frac{2}{3} = \frac{2\pi ab}{3}$

Д/з: Дошуовем ?
~~3958~~, ~~61~~, 66, 68, 69, 70.

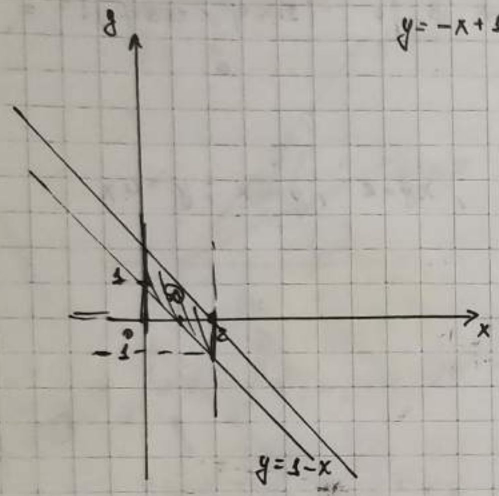
Домашняя работа.

3958. $\int\limits_0^2 dx \int\limits_{1-x}^{2-x} f(x,y)\,dy \ominus$, если $u = x+y$, $v = x-y$.

$\begin{cases} x+y = u \\ x-y = v \end{cases}$

$x = \frac{u+v}{2}$; $y = \frac{u-v}{2}$

$y = -x+1$



$0 \le x \le 2$

$-x+1 \le y \le -x+2$

$\frac{1}{I} = \begin{vmatrix} u'_x & u'_y \\ v'_x & v'_y \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -2$

$|I| = \frac{1}{2}$

$0 \le \frac{u+v}{2} \le 2 \quad |\cdot 2$

$-\frac{u+v}{2} + 1 \le \frac{u-v}{2} \le -\frac{u+v}{2} + 2 \quad \Big| + \frac{u+v}{2}$

$0 \le u+v \le 4$

$1 \le \frac{u - \cancel{v} + u + \cancel{v}}{2} \le 2 \quad \Rightarrow \quad 1 \le u \le 2$

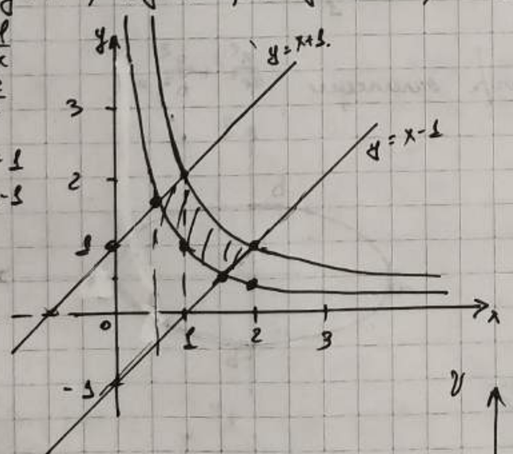
$-u \le v \le 4-u$

$\ominus\ \frac{1}{2}\int\limits_1^2 du \int\limits_{-u}^{4-u} f\left(\frac{u+v}{2}; \frac{u-v}{2}\right) dv$

3961. $xy = 1$, $xy = 2$, $x-y+1=0$, $x-y-1=0$ $(x>0,\ y>0)$

$\to$ $y = \frac{1}{x}$
$y = \frac{2}{x}$

$y = x+1$
$y = x-1$



Пусть $\begin{cases} u = xy \\ v = x-y \end{cases}$

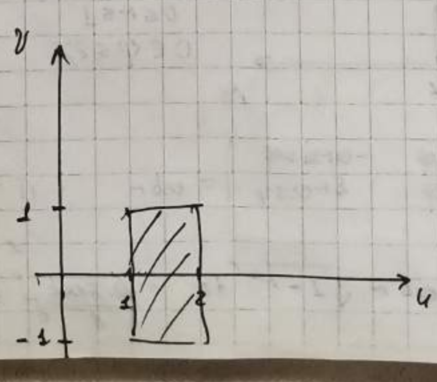
$\frac{1}{x} = x+1 \quad |\cdot x$

$x^2 + x - 1 = 0$

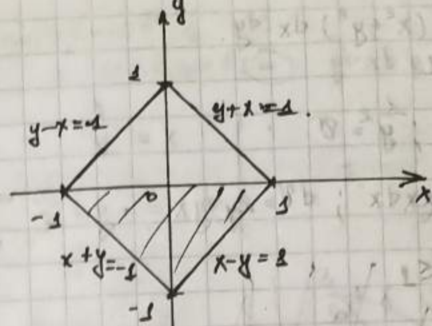
$D = 1+4 = 5$

$x_1 = \frac{-1+\sqrt{5}}{2}$

$\begin{cases} u = xy \\ v = x-y \end{cases} \Rightarrow \begin{matrix} 1 \le u \le 2 \\ -1 \le v \le 1 \end{matrix}$



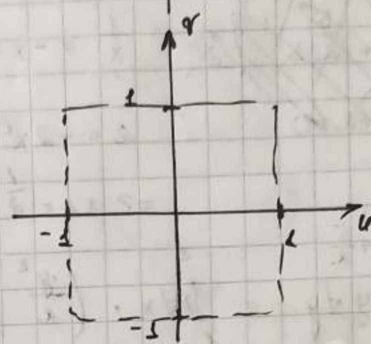
3968. $\iint\limits_{|x|+|y|\le 1} (|x|+|y|)\,dx\,dy =$



$\begin{cases} x+y=u \\ x-y=v \end{cases} \Rightarrow \begin{matrix} -1\le u\le 1. \\ -1\le v\le 1. \end{matrix}$

$x=\dfrac{u+v}{2} \to |x|=\dfrac{|u+v|}{2}$

$y=\dfrac{u-v}{2} \to |y|=\dfrac{|u-v|}{2}$



$|I|=\dfrac{1}{2}$

$= \dfrac{1}{2}\int\limits_{-1}^{1} du \int$

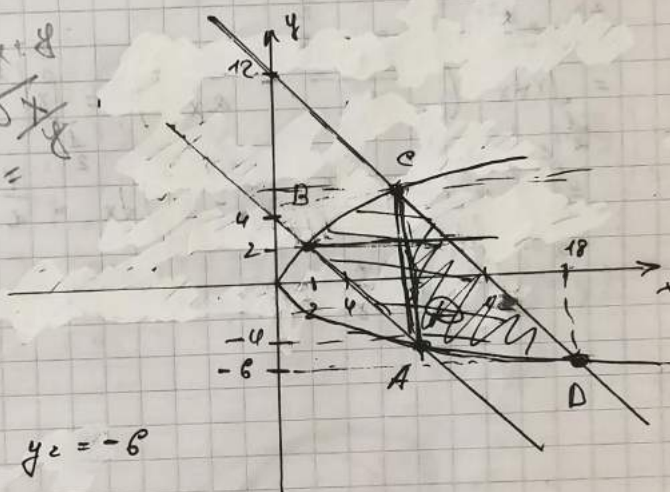
3969. $\iint\limits_{D} (x+y)\,dx\,dy =$

$u=x+y$

$v=\sqrt{\dfrac{x}{y}}$

$y=\pm\sqrt{2x}$; $x+y=4$; $x+y=12$ $u=$

Как заменить?



$y^2=2x$; $y=-x+12$

$y^2=x^2-24x+144$

$x^2-24x+144=2x$

$x^2-26x+144=0$

$x_1=8$; $x_2=18$ $\to y_1=4$; $y_2=-6$

$y^2=2x$; $y=-x+4$

$y^2=x^2-8x+16$

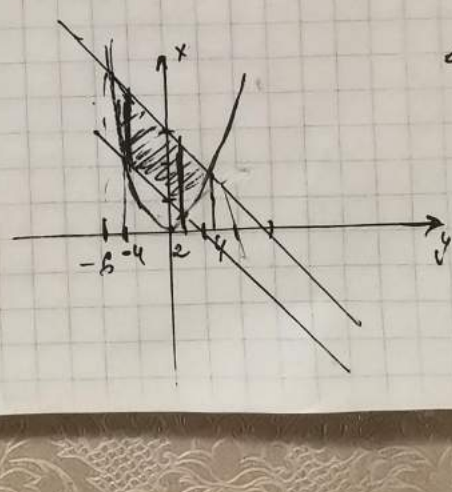
$x^2-10x+16=0$

$x_1=2$; $x_2=8$ $\to y_1=2$; $y_2=-4$

$B(2;2)$; $C(8;4)$; $A(8;-4)$; $D(18;-6)$

$= \int\limits_{-6}^{-4} dy \int\limits_{\frac{y^2}{2}}^{12-y} (x+y)\,dx + \int\limits_{2}^{4} dy \int\limits_{\frac{y^2}{2}}^{12-y} (x+y)\,dx + \int\limits_{-4}^{2} dy \int\limits_{4-y}^{12-y} (x+y)\,dx =$

$= 543\dfrac{11}{15}$

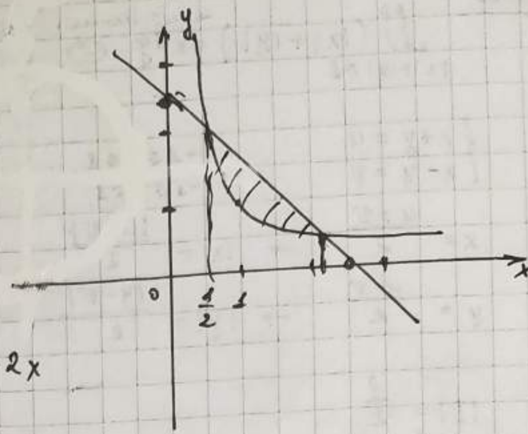


3270.

$\iint\limits_{D} xy\,dx\,dy$ $=$

$y = \frac{1}{x}$ | $x = \frac{1}{y}$

$x + y = \frac{5}{2}$ | $x = -y + \frac{5}{2}$



$\cancel{\frac{1}{x} = -x + \frac{5}{2} \quad |\cdot x}$

$\cancel{1 = -x^2 + \frac{5}{2}x \quad |\cdot 2}$

$\cancel{x^2 - 5x + 2 = 0}$

$\frac{1}{x} = -x + \frac{5}{2} \quad |\cdot 2x$

$2 = -2x^2 + 5x$

$\Rightarrow x_1 = \frac{1}{2}; \quad x_2 = 2.$

$$= \int_{\frac{1}{2}}^{2} x\,dx \int_{\frac{1}{x}}^{-x+\frac{5}{2}} y\,dy = \int_{\frac{1}{2}}^{2} x\,dx \cdot \left(\frac{y^2}{2} \Bigg|_{\frac{1}{x}}^{-x+\frac{5}{2}} \right) =$$

$$= \int_{\frac{1}{2}}^{2} x\,dx \cdot \left(\frac{\left(\frac{5}{2} - x\right)^2}{2} - \frac{1}{2x^2} \right) = \int_{\frac{1}{2}}^{2} x \left(\frac{\frac{25}{4} - 5x + x^2}{2} - \frac{1}{2x^2} \right) dx =$$

$$= \frac{1}{2} \int_{\frac{1}{2}}^{2} \left(\frac{25}{4}x - 5x^2 + x^3 - \frac{1}{x} \right) dx = \frac{1}{2} \left(\frac{25x^2}{8} - \frac{5x^3}{3} + \frac{x^4}{4} - \ln x \Bigg|_{\frac{1}{2}}^{2} \right) =$$

$$= 1\frac{87}{128} - \ln 2$$

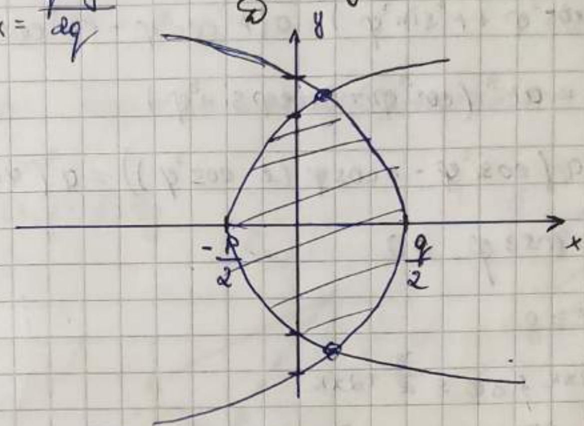
Приложение двойных интегралов к нахождению площадей и объёмов.

3985. Найти площади, ограниченные следующими кривыми.

$y^2 = 2px + p^2 \Rightarrow x = \dfrac{y^2 - p^2}{2p}$

$S = \iint\limits_D \overbrace{dx\,dy}^{f(x,y)}$

$y^2 = -2qx + q^2 \quad (p>0,\ q>0) \quad x = \dfrac{q^2 - y^2}{2q}$



$2px + p^2 = -2qx + q^2$

$2px + 2qx = q^2 - p^2$

$2x(p+q) = (q-p)(q+p) \quad | : (p+q)$

$x = \dfrac{q-p}{2} \Rightarrow y = \pm\sqrt{pq}$

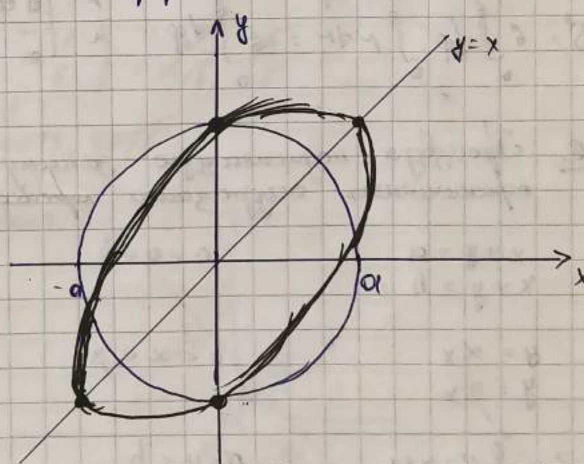
$$S = \int\limits_{-\sqrt{pq}}^{\sqrt{pq}} dy \int\limits_{\frac{y^2-p^2}{2p}}^{\frac{q^2-y^2}{2q}} dx =$$

$$= 2\int\limits_0^{\sqrt{pq}} dy \left(\frac{q^2-y^2}{2q} - \frac{y^2-p^2}{2p} \right) = \int\limits_0^{\sqrt{pq}} \frac{pq^2 - py^2 - qy^2 + p^2q}{pq}\, dy =$$

$$= \int\limits_0^{\sqrt{pq}} \frac{-y^2(p+q) + pq(p+q)}{pq}\, dy = \frac{p+q}{pq}\int\limits_0^{\sqrt{pq}} (pq - y^2)\,dy = \frac{p+q}{pq}\left(pqy - \frac{y^3}{3} \Big|_0^{\sqrt{pq}} \right) =$$

$$= \frac{p+q}{pq}\cdot\left(pq\cdot\sqrt{pq} - \frac{pq\sqrt{pq}}{3} \right) = \frac{(p+q)\cdot 2\cancel{pq}\sqrt{pq}}{3\cancel{pq}} = \frac{2}{3}\sqrt{pq}\,(p+q)$$

3986.



$(x-y)^2 + x^2 = a^2 \quad (a>0)$

$(x-y)^2 = a^2 - x^2$

$a^2 - x^2 \geq 0$

$-a \leq x \leq a$

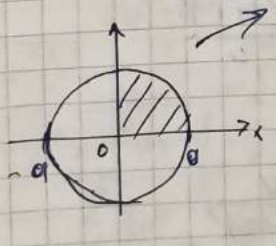
$x - y = \pm\sqrt{a^2 - x^2}$

$y = x \pm \sqrt{a^2 - x^2}$

~~$S = \int\limits_{-a}^{a} dx \,(x + \sqrt{a^2-x^2} -$~~

$$S = \int\limits_{-a}^{a} dx \int\limits_{x-\sqrt{a^2-x^2}}^{x+\sqrt{a^2-x^2}} dy = \int\limits_{-a}^{a} dx\left(\cancel{x} + \sqrt{a^2-x^2} - \cancel{x} + \sqrt{a^2-x^2}\right) = \int\limits_{-a}^{a} 2\sqrt{a^2-x^2}\,dx = 4\int\limits_0^{a}\sqrt{a^2-x^2}\,dx =$$

$$= 4\cdot\frac{1}{4}S_{кр.} = \pi a^2$$



3889. Переходя к полярным координатам вычислить площадь, огр. следующими кривыми

$$(x^2+y^2)^2 = a(x^3-3xy^2) \quad a>0$$

$$\begin{cases} x = r\cos\varphi \\ y = r\sin\varphi \end{cases}$$

$$(r^2\cos^2\varphi + r^2\sin^2\varphi)^2 = a(r^3\cos^3\varphi - 3r^3\cos\varphi\sin^2\varphi)$$

$$r^4 = ar^3(\cos^3\varphi - 3\cos\varphi\sin^2\varphi)$$

$$r = a(\cos^3\varphi - 3\cos\varphi(1-\cos^2\varphi)) = a(4\cos^3\varphi - 3\cos\varphi) = a\cdot\cos 3\varphi.$$

$$r = a\cos 3\varphi. \geqslant 0$$

$$\cos 3\varphi \geqslant 0$$

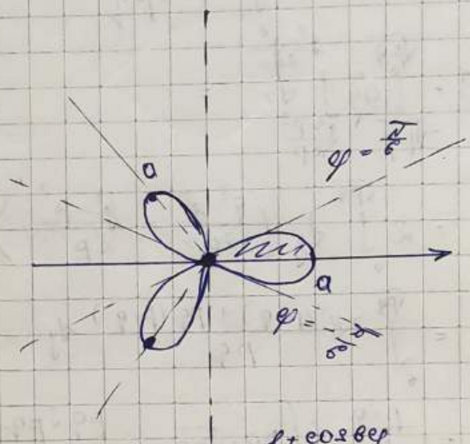
$$-\frac{\pi}{2} + 2\pi k \leq 3\varphi \leq \frac{\pi}{2} + 2\pi k.$$

$$-\frac{\pi}{6} + \frac{2\pi}{3}k \leq \varphi \leq \frac{\pi}{6} + \frac{2\pi}{3}k$$

$k=0 \qquad -\frac{\pi}{6} \leq \varphi \leq \frac{\pi}{6}.$

$k=1 \qquad \frac{\pi}{2} \leq \varphi \leq \frac{5\pi}{6}$

$k=2 \qquad \frac{7\pi}{6} \leq \varphi \leq \frac{3\pi}{2}$

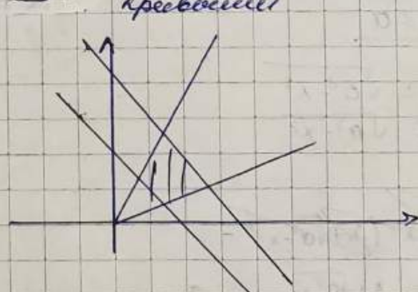


$$S = 6\int_0^{\frac{\pi}{6}} d\varphi \int_0^{a\cos 3\varphi} r\,dr = 6\int_0^{\frac{\pi}{6}} d\varphi \left.\frac{r^2}{2}\right|_0^{a\cos 3\varphi} = 3a^2\int_0^{\frac{\pi}{6}} \underbrace{\cos^2 3\varphi}_{\frac{1+\cos 6\varphi}{2}}\,d\varphi = \frac{\pi a^2}{4}$$

3986. Производя надлежащую замену переменных, найти площади фигур, ограниченных следующими кривыми

$x+y=a \qquad\qquad 0<a<b.$
$x+y=b$

$y=\alpha x \qquad\qquad 0<\alpha<\beta.$
$y=\beta x$



$$\Rightarrow \begin{cases} u = x+y. \\ v = \frac{y}{x} \end{cases} \qquad \begin{matrix} a\leq u\leq b \\ \alpha\leq v\leq\beta. \end{matrix}$$

$$\frac{1}{I} = \begin{vmatrix} u'_x & u'_y \\ v'_x & v'_y \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ -\frac{y}{x^2} & \frac{1}{x} \end{vmatrix} = \frac{x+y}{x^2} = \frac{u}{\frac{u^2}{(1+v)^2}} = \frac{(1+v)^2}{u}$$

$$y = u-x$$
$$v = \frac{u-x}{x} \quad\rightarrow\quad u-x = vx, \quad x+vx = u, \quad x(v+1) = u, \quad x = \frac{u}{v+1}.$$

$$\Rightarrow I = \frac{u}{(1+v)^2}$$

$$|I| = \frac{4}{(1+v)^2}$$

$$S = \int_a^b u\,du \int_\alpha^\beta \frac{dv}{(1+v)^2} = \int_a^b u\,du \cdot \int_\alpha^\beta \frac{dv}{(1+v)^2} = \frac{a^2}{2}\Big|_a^b \cdot \left(-\frac{1}{v+1}\Big|_\alpha^\beta\right) =$$

$$= \frac{1}{2}(b^2 - a^2)\left(-\frac{1}{\beta+1} + \frac{1}{\alpha+1}\right)$$

4007. Найти объём тела, огранич. следующими поверхностями.

$z = 1 + x + y$

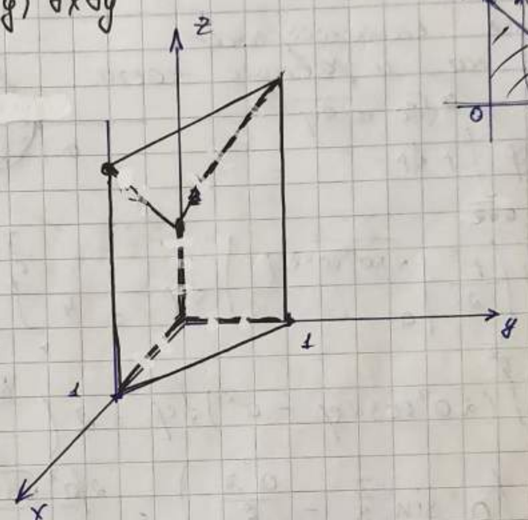
$z = 0$

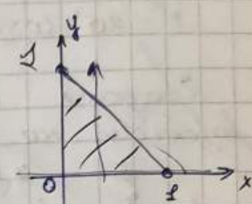
$x + y = 1$

$x = 0$

$y = 0$

$$V = \iint_D \overset{=z}{f(x,y)}\,dx\,dy$$





$$V = \int_0^1 dx \int_0^{1-x} (1 + x + y)\,dy =$$

$$= \int_0^1 \left(1 - x + x(1-x) + \frac{1}{2}(1-x)^2\right) dx$$

$$= \int_0^1 \left(1 - \cancel{x} + \cancel{x} - x^2 + \frac{1}{2}(1 - 2x + x^2)\right) dx =$$

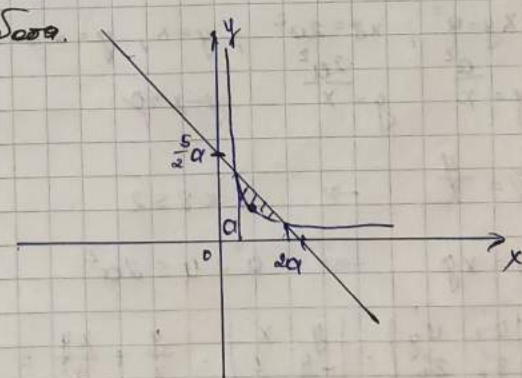
$$= \int_0^1 \left(1 + \frac{1}{2} - x + \frac{x^2}{2}\right) dx =$$

$$= \int_0^1 \left(\frac{3}{2} - x - \frac{x^2}{2}\right) dx =$$

$$= \left(\frac{3}{2}x - \frac{x^2}{2} - \frac{x^3}{6}\Big|_0^1\right) = \left(\frac{3}{2} - \frac{1}{2} - \frac{1}{6}\right) = \frac{5}{6}$$

Д/з: ~~3984~~, ~~3987~~, ~~3997~~, ~~4008~~, ~~4010~~, ~~4013~~, ~~4015~~

Домашняя работа.



3984. $y = \frac{a^2}{x}, \quad y = -x + \frac{5}{2}a$

$$\frac{a^2}{x} = -x + \frac{5}{2}a \quad \Big| \cdot 2x$$

$2a^2 = -2x^2 + 5ax$

$2x^2 - 5ax + 2a^2 = 0$

$D = 25a^2 - 16a^2 = 9a^2 \rightarrow \sqrt{D} = 3a$

$$x_1 = \frac{5a + 3a}{4} = 2a; \quad x_2 = \frac{5a - 3a}{4} = \frac{a}{2}.$$

$$S = \int_{a/2}^{2a} dx \int_{a^2/x}^{-x+\frac{5}{2}a} dy = \int_{a/2}^{2a} \left(-x + \frac{5}{2}a - \frac{a^2}{x}\right) dx = \left(-\frac{x^2}{2} + \frac{5a}{2}x - a^2 \ln x \Big|_{a/2}^{2a}\right) =$$

$$= -\frac{4a^2}{2} + \frac{10a^2}{2} - a^2 \ln 2a + \frac{a^2}{8} - \frac{5a^2}{4} + a^2 \ln\frac{a}{2} = \frac{6a^2}{2} + \frac{a^2}{8} - \frac{10a^2}{8} -$$

$$- a^2 \ln 2 - \cancel{a^2 \ln a} + \cancel{a^2 \ln a} - a^2 \ln 2 = \frac{15a^2}{8} - a^2 \ln 2$$

3987. $(x^2+y^2)^2=2a^2(x^2-y^2)$; $x^2+y^2\geq a^2$

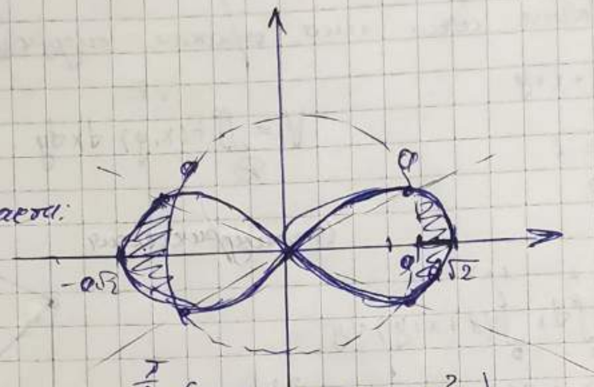
$$\begin{cases} x=r\cos\varphi \\ y=r\sin\varphi \end{cases}$$

$$(r^2\cos^2\varphi+r^2\sin^2\varphi)^2=2a^2(r^2\cos^2\varphi-r^2\sin^2\varphi)$$

$$r^4=2a^2\cdot r^2(\cos^2\varphi-\sin^2\varphi) \quad \Big| : r^2\neq 0$$

$$r^2=2a^2(\cos^2\varphi-\sin^2\varphi)$$

$$r^2=2a^2\cos2\varphi.$$



Графики симметричен, разобьем на 4 равные части.

$$S=4\int_0^{\pi/6}d\varphi\int_a^{\sqrt{2a^2\cos2\varphi}} r\,dr=$$

$$=4\int_0^{\pi/6}\left(\frac{r^2}{2}\Big|_a^{\sqrt{2a^2\cos2\varphi}}\right)d\varphi=4\int_0^{\pi/6}\left(\frac{2a^2\cos2\varphi}{2}-\frac{a^2}{2}\right)d\varphi=$$

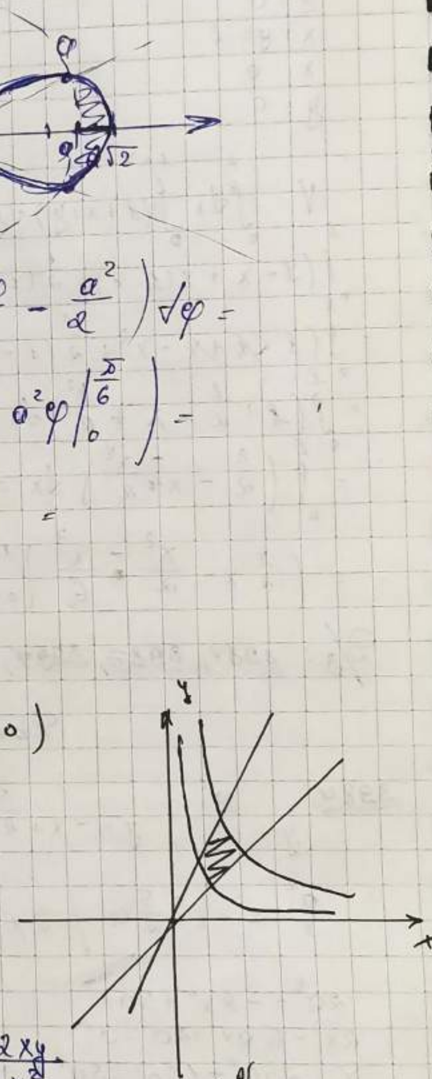
$$=2\int_0^{\pi/6}(2a^2\cos2\varphi-a^2)\,d\varphi=2\left(\frac{2a^2\sin2\varphi}{2}-a^2\varphi\Big|_0^{\pi/6}\right)=$$

$$=2\left(a^2\sin\frac{\pi}{3}-\frac{a^2\pi}{6}\right)=2\left(\frac{a^2\sqrt3}{2}-\frac{a^2\pi}{6}\right)=$$

$$=\sqrt3a^2-\frac{a^2\pi}{3}=\frac{3\sqrt3-\pi}{3}\cdot a^2$$

3997. $xy=a^2$, $xy=2a^2$, $y=x$, $y=2x$ $(x>0;\ y>0)$

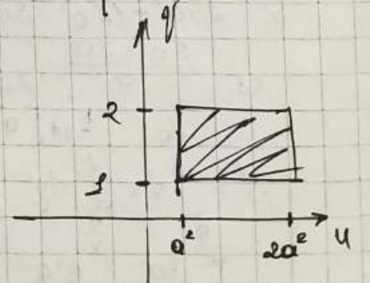
$\to y=\frac{a^2}{x}$ $\quad y=\frac{2a^2}{x}$ $\quad \cancel{y-x=0}$



$$\begin{cases} v=\frac{y}{x} \to 1\leq v\leq 2 \\ u=xy \to a^2\leq u\leq 2a^2 \end{cases}$$

$$\frac{1}{I}=\begin{vmatrix} u'_x & u'_y \\ v'_x & v'_y \end{vmatrix}=\begin{vmatrix} y & x \\ -\frac{y}{x^2} & \frac{1}{x} \end{vmatrix}=\frac{y}{x}+\frac{xy}{x^2}=\frac{xy+xy}{x^2}=\frac{2xy}{x^2}$$

$vx=y.$
$\frac{u}{x}=y. \to$ $\quad vx=\frac{u}{x} \to vx^2=u \to x^2=\frac{u}{v}$

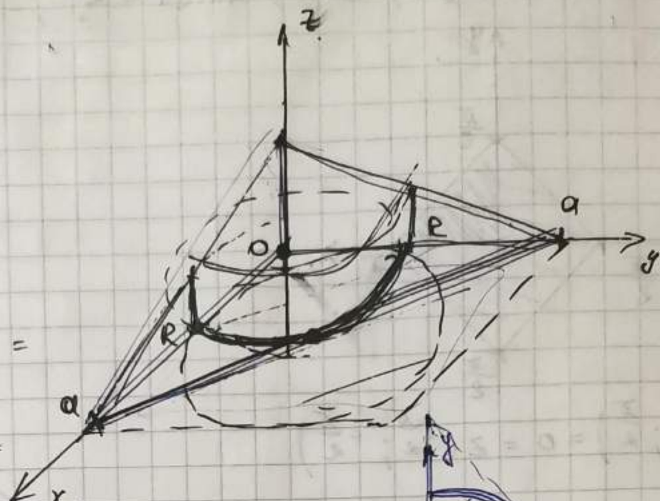


$$I=\frac{x^2}{2xy}=\frac{u}{2v\cdot u}=\frac{1}{2v}$$

$$\Rightarrow S=\frac{1}{2}\int_{a^2}^{2a^2}du\int_1^2\frac{1}{v}\,dv=\frac{a^2}{2}\ln2$$

4008. $x+y+z=a$, $x^2+y^2=R^2$, $x=0$, $y=0$, $z=0$ $(a \geq R\sqrt{2})$

$z = a - x - y$

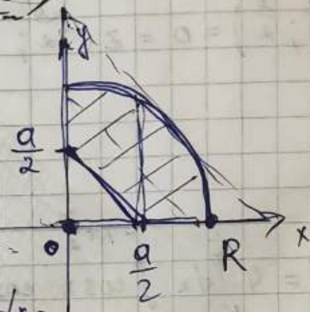


$$V = \int_0^R dx \int_0^{\sqrt{R^2-x^2}} (a-x-y)\,dy =$$

$$= \int_0^R \left(ay - xy - \frac{y^2}{2} \Big|_0^{\sqrt{R^2-x^2}}\right) dx =$$

$$= \int_0^R \left(a\sqrt{R^2-x^2} - x\sqrt{R^2-x^2} - \frac{R^2-x^2}{2}\right) dx =$$

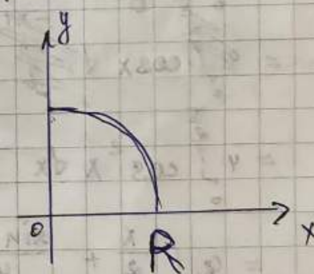
$$= \int_0^R \left(\sqrt{R^2-x^2}\,(a-x) - \frac{R^2-x^2}{2}\right) dx =$$



$$= \int_0^R a\sqrt{R^2-x^2}\,dx - \int_0^R \left(x\sqrt{R^2-x^2} + \frac{R^2-x^2}{2}\right) dx =$$

$$= a\left(\frac{x}{2}\sqrt{R^2-x^2} + \frac{R^2}{2}\arcsin\frac{x}{R}\Big|_0^R\right) - \int_0^R \left(x\sqrt{R^2-x^2} + \frac{R^2-x^2}{2}\right) dx =$$

$$= a\left(\underbrace{\frac{R}{2}\sqrt{R^2-R^2}}_{=0} + \frac{R^2}{2}\cdot\underbrace{\arcsin(1)}_{\frac{\pi}{2}}\right) - \int_0^R \left(x\sqrt{R^2-x^2} + \frac{R^2-x^2}{2}\right) dx =$$



$$= \frac{\pi a R^2}{4} - \int_0^R \left(x\sqrt{R^2-x^2} + \frac{R^2-x^2}{2}\right) dx =$$

$$= \frac{\pi a R^2}{4} - \int_0^R x\sqrt{R^2-x^2}\,dx - \int_0^R \frac{R^2-x^2}{2}\,dx =$$

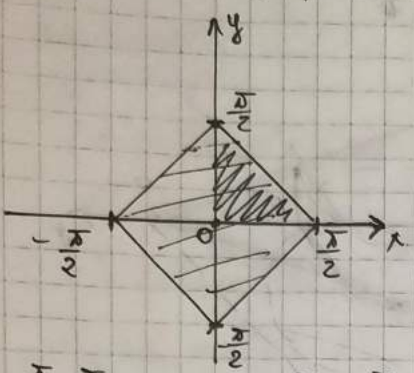
$$= \frac{\pi a R^2}{4} - \int_0^R \frac{R^2}{2}\,dx + \int_0^R \frac{x^2}{2}\,dx - \int_0^R x\sqrt{R^2-x^2}\,dx =$$

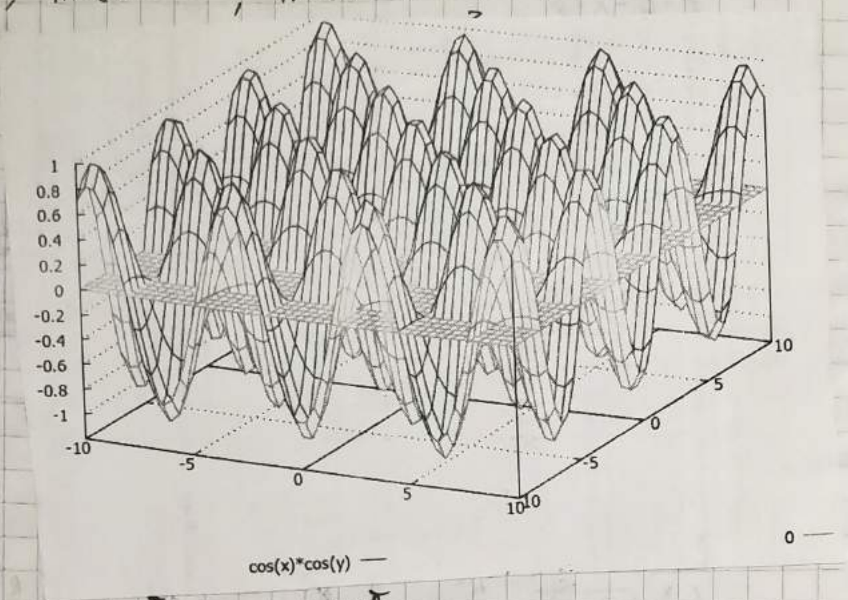
$$= \frac{\pi a R^2}{4} - \frac{R^3}{2} + \frac{R^3}{6} - \left(-\frac{(R^2-x^2)\sqrt{R^2-x^2}}{3}\Big|_0^R\right) = \frac{\pi a R^2}{4} \underbrace{- \frac{R^3}{2} + \frac{R^3}{6} + \frac{R^3}{3}}_{=0} = \frac{\pi a R^2}{4}$$

4

4010.

$z=\cos x\cdot\cos y$, $z=0$, $|x+y|\le\frac{\pi}{2}$, $|x-y|\le\frac{\pi}{2}$





$z(\frac{\pi}{2};\frac{\pi}{2})=0=z(-\frac{\pi}{2};-\frac{\pi}{2})$

$$V=4\int_0^{\frac{\pi}{2}}dx\int_0^{-x+\frac{\pi}{2}}\cos x\cdot\cos y\,dy=4\int_0^{\frac{\pi}{2}}\cos x\,dx\int_0^{-x+\frac{\pi}{2}}\cos y\,dy=$$

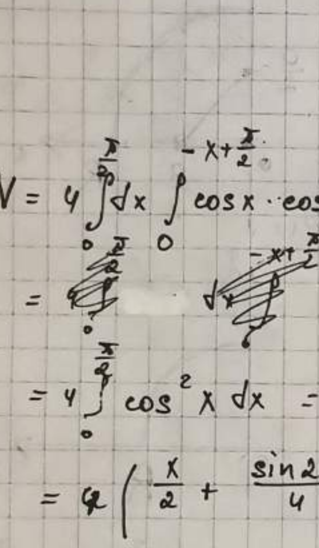
$$=4\int_0^{\frac{\pi}{2}}\cos x\,dx\cdot\left(\sin y\Big|_0^{-x+\frac{\pi}{2}}\right)=4\int_0^{\frac{\pi}{2}}\cos x\cdot\sin\left(-x+\frac{\pi}{2}\right)dx=$$

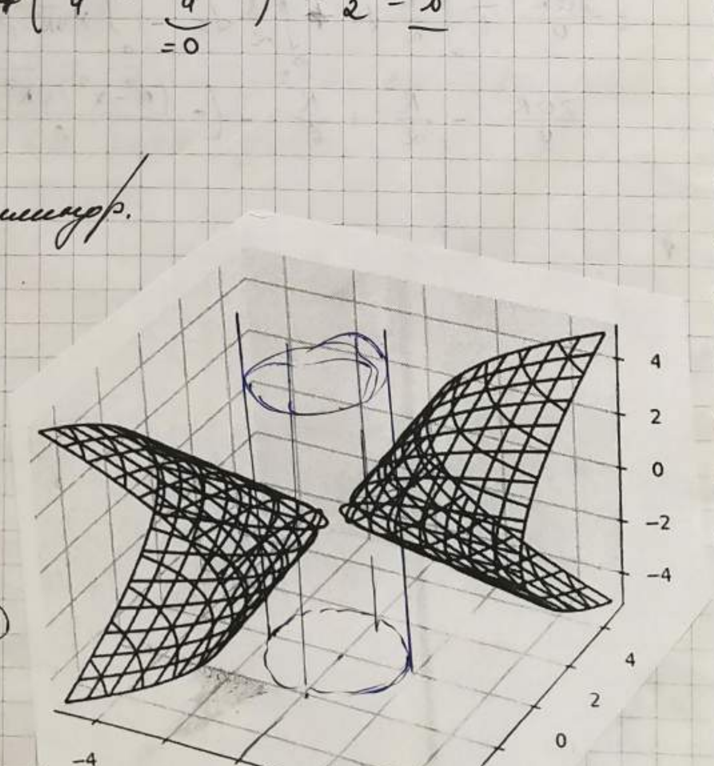
$$=4\int_0^{\frac{\pi}{2}}\cos^2 x\,dx=4\int_0^{\frac{\pi}{2}}\frac{(1+\cos 2x)}{2}dx=4\int_0^{\frac{\pi}{2}}\left(\frac{1}{2}+\frac{\cos 2x}{2}\right)dx=$$

$$=4\left(\frac{x}{2}+\frac{\sin 2x}{4}\Big|_0^{\frac{\pi}{2}}\right)=4\left(\frac{\pi}{4}+\underbrace{\frac{\sin\pi}{4}}_{=0}\right)=\frac{2\pi}{2}=\underline{\pi}$$

4013.

$z^2=xy$, $x^2+y^2=a^2$ — в x, y, z - цилиндр.





Кусочки симметричные.

$x=r\cos\varphi$; $y=r\sin\varphi$; $z=h$

$$V=4\int_0^a dr\int_0^{\frac{\pi}{2}}\sqrt{r^2\cos\varphi\sin\varphi}\cdot r\,d\varphi=$$

$$\ominus\ 4\int_0^a dr \int_0^{\frac{\pi}{2}} \sqrt{\cos\varphi\ \sin\varphi}\ r^2\, d\varphi = 4\int_0^a \frac{r^2}{1} dr \int_0^{\frac{\pi}{2}} \cos^{\frac{1}{2}}\varphi \cdot \sin^{\frac{1}{2}}\varphi\ d\varphi =$$

$$= \frac{4a^3}{8} \cdot \int_0^{\frac{\pi}{2}} \cos^{\frac{1}{2}}\varphi \cdot \sin^{\frac{1}{2}}\varphi\ d\varphi\ \ ?$$

4015.

$z = x^2 + y^2$, $x^2 + y^2 = x$, $x^2 + y^2 = 2x$, $z = 0$

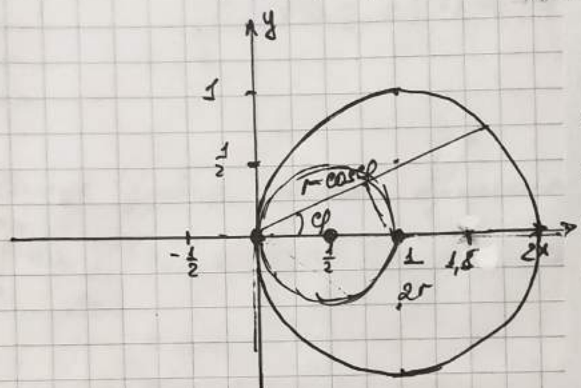
$$x^2 - x + y^2 = 0$$

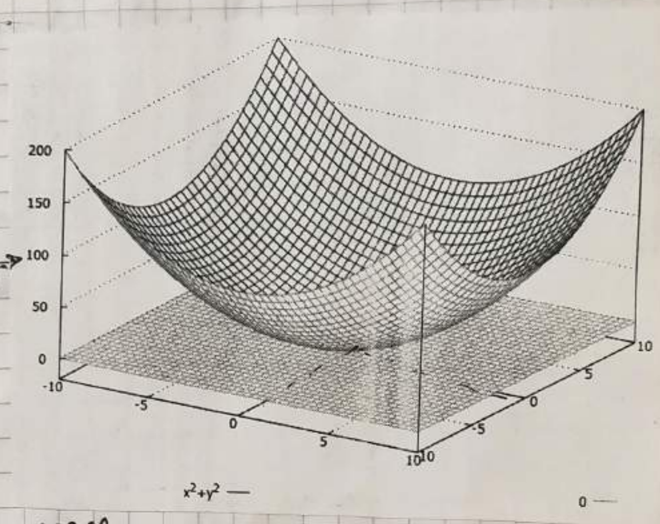
$$x - \frac{1}{2} \cdot 2x + \frac{1}{4} + y^2 = \frac{1}{4}$$

$$\left(x - \frac{1}{2}\right)^2 + y^2 = \frac{1}{4}$$

$$x^2 - 2x + 1 + y^2 = 1$$

$$(x-1)^2 + y^2 = 1.$$





$$z = r^2\cos^2\varphi + r^2\sin^2\varphi = r^2$$

$$r_1^2\cos^2\varphi + r_1^2\sin^2\varphi = r_1\cos\varphi. \rightarrow r_1 = \cos\varphi$$

$$r_2^2\cos^2\varphi + r_2^2\sin^2\varphi = 2r_2\cos\varphi \rightarrow r_2 = 2\cos\varphi$$

$$V = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\varphi \int_{\cos\varphi}^{2\cos\varphi} r^2 \cdot r\, dr = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\varphi \cdot \left(\frac{r^4}{4}\ \Bigg|_{\cos\varphi}^{2\cos\varphi}\right) =$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{16\cos^4\varphi}{4} - \frac{\cos^4\varphi}{4}\right) d\varphi = \frac{15}{4}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^4\varphi\ d\varphi\ \ominus$$

$$\cos^4\varphi = \frac{1 + \cos^2 2\varphi}{2} =$$

$$\textcircled{=}\ \frac{15}{4}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1+\cos^2 2\varphi}{2}\,d\varphi = \frac{15}{8}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (1+\cos^2 2\varphi)\,d\varphi =$$

$$= \frac{15}{8}\left(\varphi\,\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\right) + \frac{15}{8}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1+\cos 4\varphi}{2}\,d\varphi =$$

$$= \frac{15}{8}\left(\frac{\pi}{2}+\frac{\pi}{2}\right) + \frac{15}{16}\left(\varphi\,\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\right) + \frac{15}{16}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos 4\varphi\,d\varphi =$$

$$= \frac{15\pi}{8} + \frac{15\pi}{16} + \frac{15}{16}\underbrace{\left(\frac{\sin 4\varphi}{4}\,\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\right)}_{=0} = \frac{15}{8}\pi + \frac{15}{16}\pi = \frac{45}{16}\pi$$

15.04.2021. Тройные интегралы.
Расстановка пределов интегрирования.

Ауд: Дем 4076, 4081, 4083, 4085, 4092
Д/з: Дем 4077, 4082, 4084, 4093

Если ф-я $F(x,y,z)$ непрерывна, а область V ограничена и определяется следующими неравенствами:

$$x_1 \le x \le x_2; \quad y_1(x) \le y \le y_2(x); \quad z_1(x,y) \le z \le z_2(x,y),$$ где

$y_1(x)$, $y_2(x)$, $z_1(x,y)$, $z_2(x,y)$ непрерывные функции, то тройной интеграл от ф-ии $F(x,y,z)$, распространённый на область V, может быть вычислен по формуле:

$$\iiint\limits_V F(x,y,z)\,dx\,dy\,dz = \int_{x_1}^{x_2} dx \int_{y_1(x)}^{y_2(x)} dy \int_{z_1(x,y)}^{z_2(x,y)} F(x,y,z)\,dz$$

(или в другом порядке.)

Иногда удобно также применять формулу $\iiint\limits_V F(x,y,z)\,dx\,dy\,dz = \int_{x_1}^{x_2} dx \iint\limits_{S(x)} F(x,y,z)\,dy\,dz$; $S(x)$ - сечение V плоскостью $x = const$.

4081.

$$\underbrace{\int_0^1 dx \int_0^{1-x} dy}_{\iint\limits_D dx\,dy} \int_0^{x+y} f(x,y,z)\,dz$$

$0 \le x \le 1.$

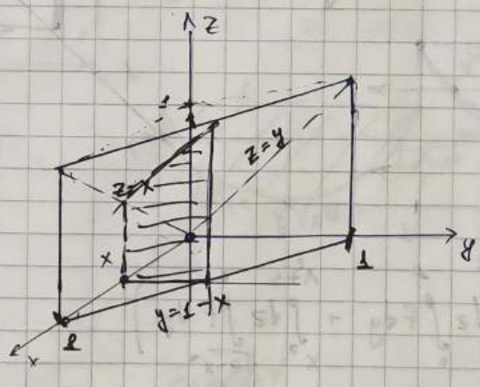
$0 \le y \le 1-x$

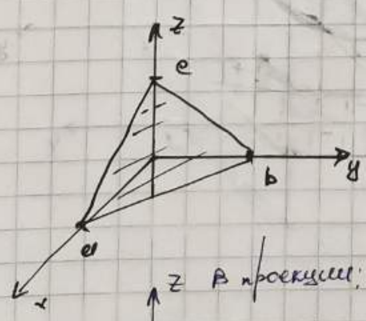
$0 \le z \le x+y \quad \rightarrow \quad x+y-z=0$

$\Downarrow$

$y = z - x$

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$



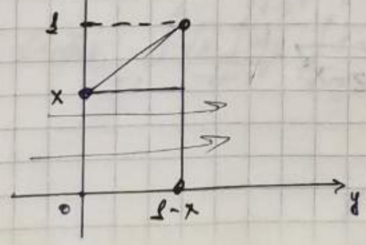


$y = 1-x \qquad z = x+y.$

$z = x+1-x = z = 1.$

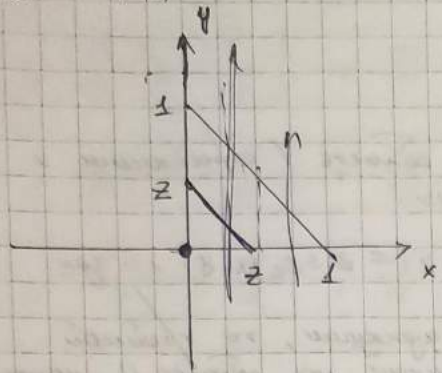
$$I = \int_{x_1}^{x_2} dx \iint\limits_{S(x)} f\,dy\,dz = \int_0^1 dx \left(\int_0^x dz \int_0^{1-x} f\,dy + \int_x^1 dz \int_{z-x}^{1-x} f\,dy \right)$$

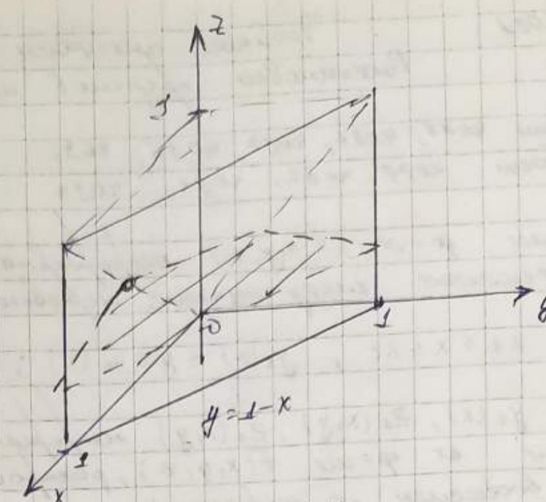
z в проекции:



$y = 0 \rightarrow z = x$

$I = \int_0^1 dz \iint_{S(z)} f\, dx\, dy$





$I = \int_0^1 dz \iint_{S(z)} f\, dx\, dy = \int_0^1 dz \left(\int_0^z dx \int_{z-x}^{1-x} f\, dy + \int_z^1 dx \int_0^{1-x} f\, dy \right)$

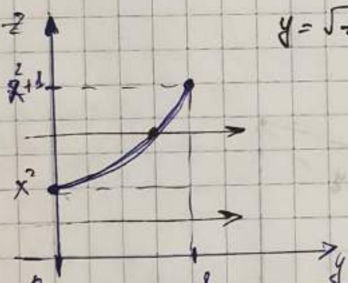
4083. $\int_0^1 dx \int_0^1 dy \int_0^{x^2+y^2} f\, dz$

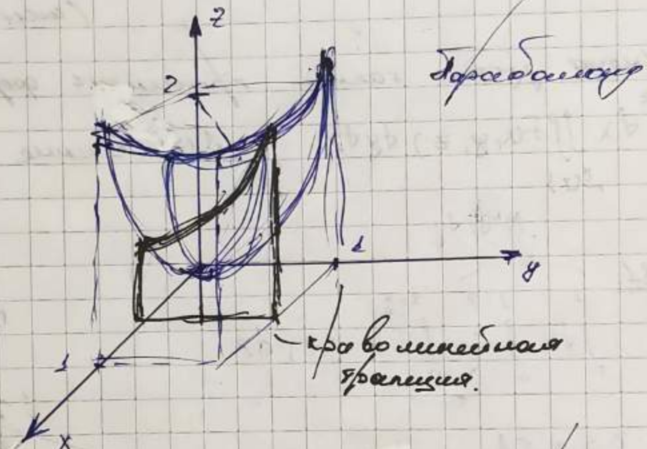
$0 \le x \le 1$

$0 \le y \le 1$

$0 \le z \le x^2+y^2 \quad \to y^2 = z - x^2$

$y = \sqrt{z-x^2}$

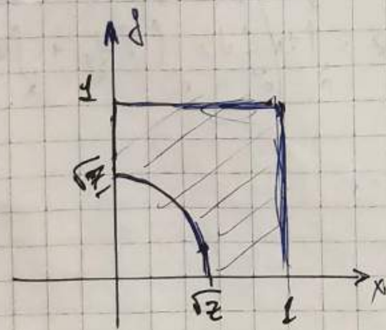


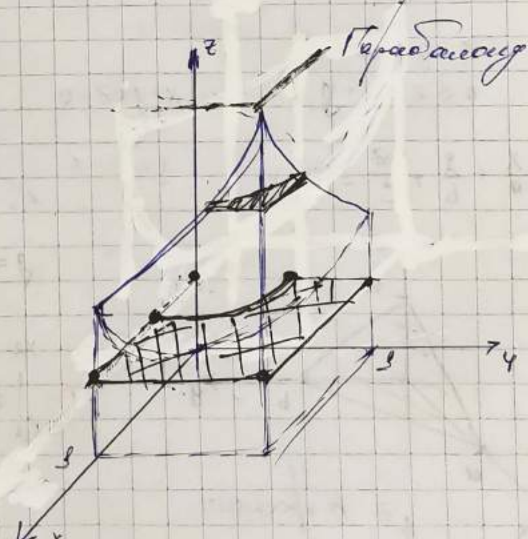


$I = \int_0^1 dx \left(\int_0^{x^2} dz \int_0^1 f\, dy + \int_{x^2}^{x^2+1} dz \int_{\sqrt{z-x^2}}^1 f\, dy \right)$

Когда $z = const$

$0 \le z \le 1$

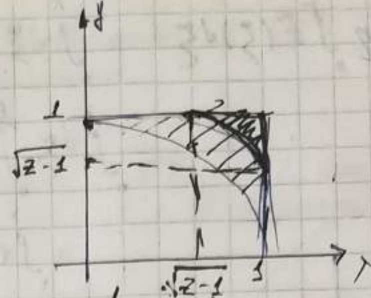




$y = \sqrt{z - x^2}$

$1 \le z \le 2$

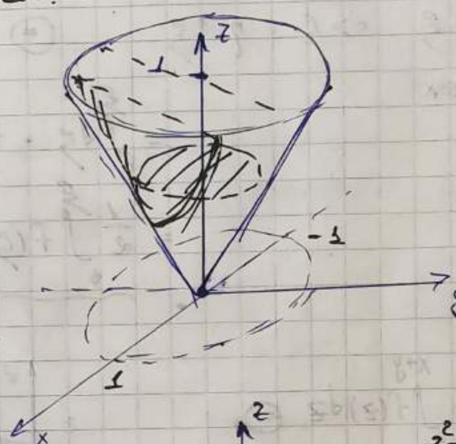
$x = 1 \to y = \sqrt{z-1}$



$$I = \int_0^1 dz \left(\int_0^{\sqrt{z}} dx \int_{\sqrt{z-x^2}}^1 f\,dy + \int_{\sqrt{z}}^1 dx \int_0^1 f\,dy \right) + \int_1^2 dz \int_{\sqrt{z-1}}^1 dx \int_{\sqrt{z-x^2}}^1 f\,dy$$

Д/з: ~~4082~~, 4084; 4085.

Домашняя работа.



4082. $$\int_{-1}^1 dx \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dy \int_{\sqrt{x^2+y^2}}^1 f(x,y,z)\,dz =$$

$-1 \le x \le 1$

$-\sqrt{1-x^2} \le y \le \sqrt{1-x^2}$

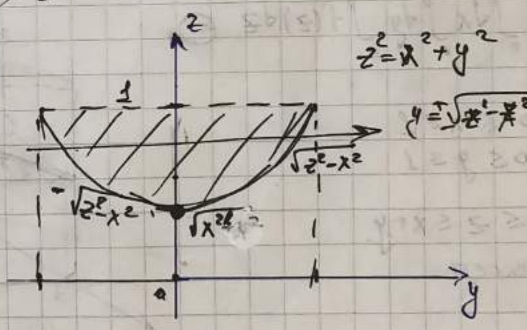
$\sqrt{x^2+y^2} \le z \le 1$ $\quad z = \sqrt{x^2+y^2}$

$z^2 = x^2+y^2$ $\quad \sqrt{z^2-x^2}$

$x = const$

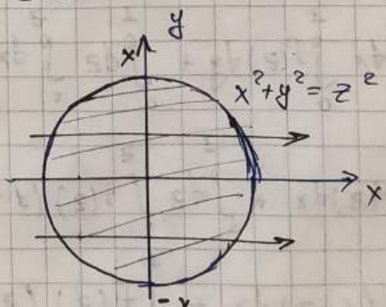
$$= \int_{-1}^1 dx \int_{|x|}^1 dz \int_{-\sqrt{z^2-x^2}}^{\sqrt{z^2-x^2}} f\,dy =$$

$$= \int_0^1 dz \int_{-x}^{x} dy \int_{-\sqrt{z^2-y^2}}^{\sqrt{z^2-y^2}} f\,dx$$



$x = y$

$z = const.$



Ответ. $$\int_{-1}^1 dx \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dy \int_{\sqrt{x^2+y^2}}^1 f\,dz = \int_{-1}^1 dx \int_{|x|}^1 dz \int_{-\sqrt{z^2-x^2}}^{\sqrt{z^2-x^2}} f\,dy = \int_0^1 dz \int_{-x}^{x} dy \int_{-\sqrt{z^2-y^2}}^{\sqrt{z^2-y^2}} f\,dx$$

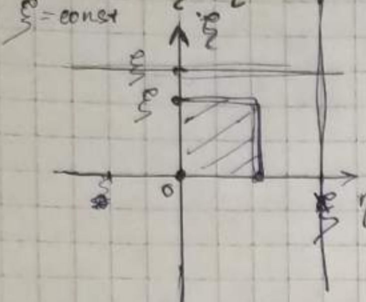
4084. $\int_0^x d\xi \int_0^\xi d\eta \int_0^\eta f(\zeta)\,d\zeta = \int_0^x d\xi \int_0^\xi d\zeta \int_\zeta^\xi f\,d\eta =$

$0 \le \xi \le x$

$0 \le \eta \le \xi$

$0 \le \zeta \le \eta$

$\xi = const$



$= \int_0^x d\xi \int_0^\xi d\zeta \int_\zeta^\xi f(\zeta)\,d\eta =$

$= \int_0^x d\xi \int_0^\xi d\zeta \cdot f(\zeta)\cdot(\xi-\zeta) =$

$0 \le \xi \le x$

$0 \le \xi-\zeta \le \xi \quad -\xi \le -\zeta \le 0$

$0 \le \zeta \le \xi$

$\Rightarrow \zeta \le \xi \le x \to 0 \le \zeta \le \xi$

Если $0 \le \zeta \le \xi \Rightarrow \zeta \le \eta \le \xi$

$= \int_0^\xi d\zeta \int_\zeta^x f(\zeta)(\xi-\zeta)\,d\xi =$

$= \int_0^\xi d\zeta \cdot f(\zeta)\cdot\left(\frac{x^2}{2} - x\zeta - \frac{\zeta^2}{2} + \zeta^2\right) =$

$= \frac{1}{2}\int_0^\xi f(\zeta)(x-\zeta)^2\,d\zeta$

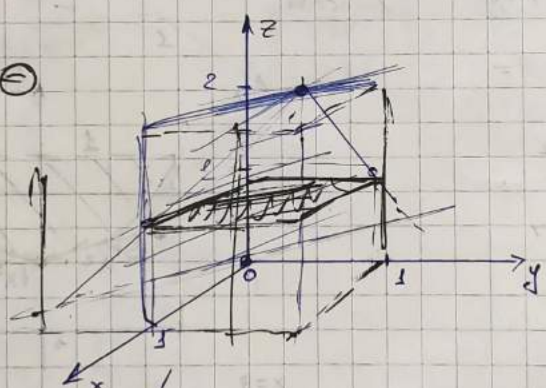
4085. $\int_0^1 dx \int_0^1 dy \int_0^{x+y} f(z)\,dz =$

$0 \le x \le 1$

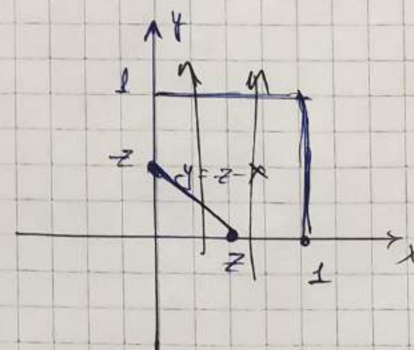
$0 \le y \le 1$

$0 \le z \le x+y.$

плоскость



Если $z = const$, в сечении треугольники:



$= \int_0^1 dz \int_z^1 dx \int_0^1 f(z)\,dy + \int_0^1 dz \int_0^z dx \int_{z-x}^1 f(z)\,dy =$

$= \int_0^1 dz \int_z^1 f(z)\,dx + \int_0^1 dz \int_0^z f(z)(1-z+x)\,dx =$

$= \int_0^1 f(z)(1-z)\,dz + \int_0^1 f(z)\left(z - z^2 + \frac{z^2}{2}\right)dz = \int_0^1 f(z)(1-z)\,dz +$

$+ \int_0^1 F(z)\left(z - \frac{z^2}{2}\right)dz = \int_0^1 f(z)(1-z)\,dz + \int_0^1 f(z)\left(1-\frac{z}{2}\right)z\,dz$

Участок. $1 \le z \le 2.$

$\Rightarrow \int_1^2 dz \int^1$

Замена переменных в тройном интеграле (цилиндрические координаты).

Цилиндрическими координатами точки $M(x, y, z)$ наз-ся тройка чисел (φ, r, z), связанная с числами x, y, z формулами:

$$\begin{cases} x = r\cos\varphi \\ y = r\sin\varphi \\ z = z \end{cases}$$ — цилиндрические координаты.

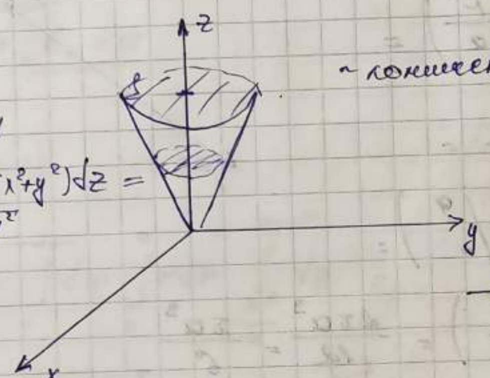
$$\iiint\limits_V f(x,y,z)\,dx\,dy\,dz = \int\limits_{z_1}^{z_2} dz \iint\limits_{\Omega_z} f(x,y,z)\,dx\,dy$$

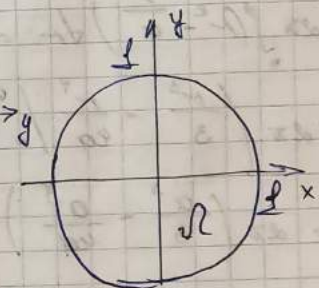
$$\iiint\limits_V f(x,y,z)\,dx\,dy\,dz = \iint\limits_{\Omega} dx\,dy \int\limits_{z_1(x,y)}^{z_2(x,y)} f(x,y,z)\,dz$$

$|J| = r$

① $V:\ S_1:\ z^2 = x^2 + y^2$
$S_2:\ z = 1.$

— коническая поверхность





$$\iiint\limits_V (\overset{r^2}{x^2+y^2})\,dx\,dy\,dz = \iint\limits_{\Omega} dx\,dy \int\limits_{\sqrt{x^2+y^2}}^{1} (x^2+y^2)\,dz =$$

$$= \int\limits_0^{2\pi} d\varphi \int\limits_0^1 \underbrace{r}_{\text{якобиан}}\,dr \cdot r^2 \cdot z\Big|_r^1 =$$

$$= 2\pi \int\limits_0^1 r^3(1-r)\,dr =$$

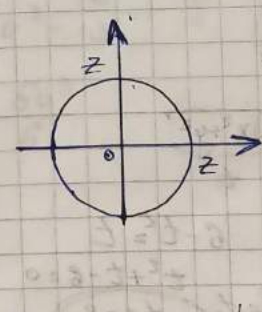
$$= 2\pi \cdot \left(\frac{r^4}{4} - \frac{r^5}{5}\Big|_0^1\right) = 2\pi\left(\frac{1}{4} - \frac{1}{5}\right) = \frac{\pi}{10}$$

2 способ:

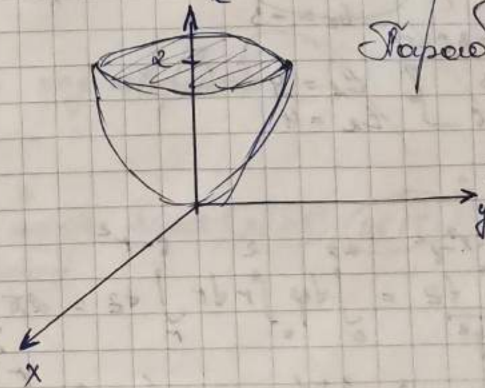
$$\iiint\limits_V (x^2+y^2)\,dx\,dy\,dz = \int\limits_0^1 dz \iint\limits_{\Omega_z} (x^2+y^2)\,dx\,dy =$$

$$= \int\limits_0^1 dz \int\limits_0^{2\pi} d\varphi \int\limits_0^z r^2 \cdot r\,dr = 2\pi \cdot \int\limits_0^1 \frac{z^4}{4}\,dz =$$

$$= \frac{\pi}{2} \cdot \left(\frac{z^5}{5}\Big|_0^1\right) = \frac{\pi}{10}$$



Параболоид.

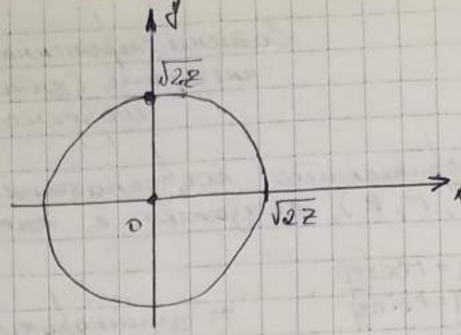


4081. $$\iiint\limits_V (x^2+y^2)\,dx\,dy\,dz$$

$x^2 + y^2 = 2z$
$z = 2$

$$\int\limits_0^2 dz \iint\limits_{\Omega_z} (x^2+y^2)\,dx\,dy =$$

$$= \int_0^2 dz \int_0^{2\pi} d\varphi \int_0^{\sqrt{2z}} r^3 dr =$$



$$2\pi \int_0^2 dz \cdot \frac{r^4}{4}\Big|_0^{\sqrt{2z}} =$$

$$= 2\pi \int_0^2 z^2 dz = 2\pi \frac{z^3}{3}\Big|_0^2 =$$

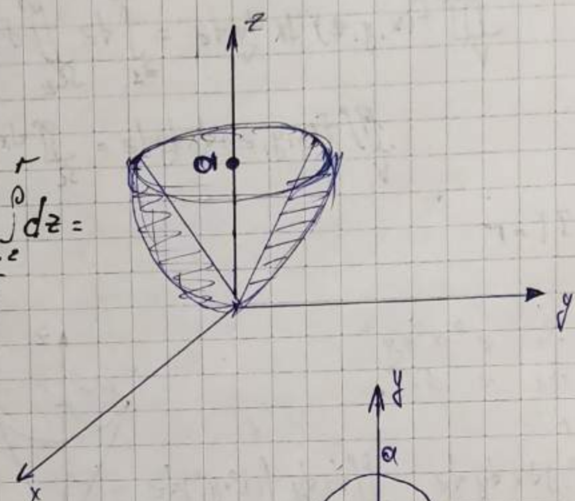
$$= \frac{2 \cdot 8 \cdot \pi}{3} = \frac{16\pi}{3}$$

4104.

$az = x^2+y^2 \to z = \frac{x^2+y^2}{a}$

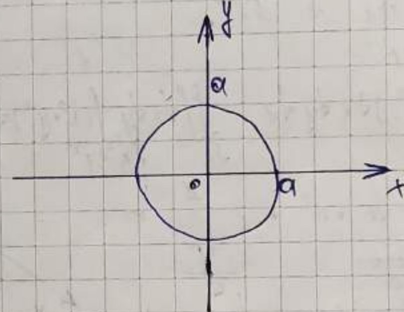
$z = \sqrt{x^2+y^2}$

$a > 0$



$$\iint_\Omega dx\,dy \int_{\frac{x^2+y^2}{a}}^{\sqrt{x^2+y^2}} dz = \int_0^{2\pi} d\varphi \int_0^a r\,dr \int_{\frac{r^2}{a}}^{r} dz =$$

$$= 2\pi \int_0^a r\,dr \left(r - \frac{r^2}{a}\right) =$$



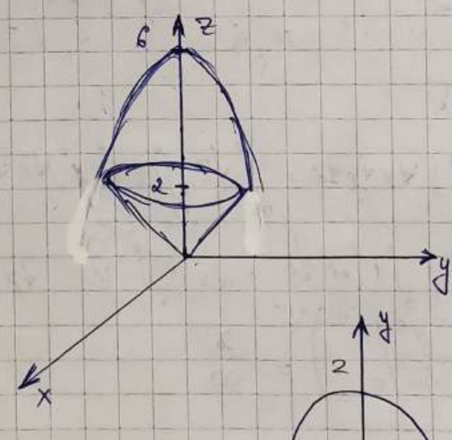
$$= 2\pi \int_0^a \left(r^2 - \frac{r^3}{a}\right) dr =$$

$$= 2\pi \left(\frac{r^3}{3} - \frac{r^4}{4a}\Big|_0^a\right) =$$

$$= 2\pi \left(\frac{a^3}{3} - \frac{a^3}{4}\right) = \frac{2\pi a^3}{12} = \frac{\pi a^3}{6}$$

4106.

$z = 6 - x^2 - y^2$



$z = \sqrt{x^2+y^2}$

$6 - x^2 - y^2 = \underbrace{\sqrt{x^2+y^2}}_{t}$

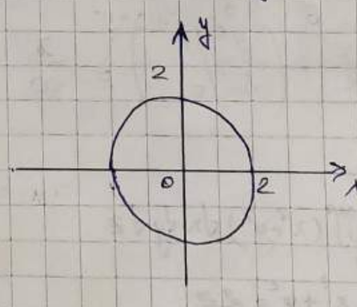
$x^2+y^2 = t$

$6 - t^2 = t$

$6 - t = \sqrt{t}$ $\quad t^2 + t - 6 = 0$

$36 - 12t + t^2 = t$ $\quad t_1 = 2$

$t_2 = -3$



$t^2 - 13t + 36 = 0$

$t_1 + t_2 = 13$ $\quad t_1 = 9$?

$t_1 \cdot t_2 = 36$ $\quad t_2 = 4$

$\sqrt{x^2+y^2} = 2$

$$\iint_\Omega dx\,dy \int_{\sqrt{x^2+y^2}}^{6-x^2-y^2} dz = \int_0^{2\pi} d\varphi \int_0^2 r\,dr \int_r^{6-r^2} dz = 2\pi \int_0^2 r(6 - r^2 - r)\,dr =$$

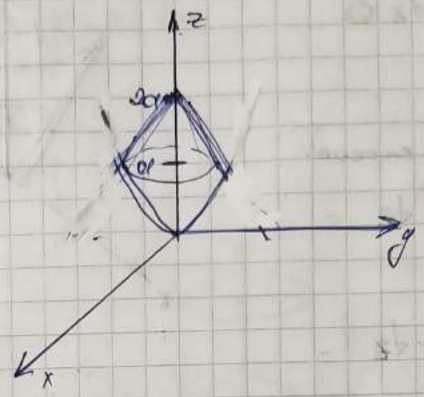
$$= 2\pi \int_0^2 (6r - r^3 - r^2)\,dr = 2\pi \left(3r^2 - \frac{r^4}{4} - \frac{r^3}{3}\Big|_0^2\right) =$$

$$= 2\pi\left(12 - \frac{16}{4} - \frac{8}{3}\right) = 2\pi\left(8 - \frac{8}{3}\right) = 2\pi\left(\frac{16}{3}\right) = \frac{32\pi}{3}$$

4126

$x^2+y^2 = az, \quad a>0$

$z = 2a - \sqrt{x^2+y^2}$



$2a - z = \sqrt{x^2+y^2}$

$\frac{x^2+y^2}{a} = 2a - \underbrace{\sqrt{x^2+y^2}}_{=t}$

$\frac{t^2}{a} = 2a - t \quad \Big|\cdot a$

$t^2 = 2a^2 - ta$

$t^2 + ta - 2a^2 = 0$

$\left.\begin{array}{l} t_1+t_2 = -a \\ t_1\cdot t_2 = -2a^2 \end{array}\right\} t_1 = -2a; \; t_2 = a$

$\sqrt{x^2+y^2} = a$

$z = 2a - a = a$

$$\iint\limits_{\Omega} dx\,dy \int\limits_{\frac{x^2+y^2}{a}}^{2a-\sqrt{x^2+y^2}} dz = \int\limits_0^{2\pi} d\varphi \int\limits_0^{a} r\,dr \int\limits_{\frac{r^2}{a}}^{2a-r} dz =$$

$$= 2\pi \int\limits_0^a r\left(2a - r - \frac{r^2}{a}\right) dr = 2\pi \int\limits_0^a \left(2ar - r^2 - \frac{r^3}{a}\right) dr =$$

$$= 2\pi\left(ar^2 - \frac{r^3}{3} - \frac{r^4}{4a}\Big|_0^a\right) = 2\pi\left(a^3 - \frac{a^3}{3} - \frac{a^3}{4}\right) =$$

$$= 2\pi\left(\frac{2a^3}{3} - \frac{a^3}{4}\right) = 2\pi \cdot \frac{8a^3 - 3a^3}{12} = \frac{5a^3}{6}$$

Площадь поверхности:

$$S = \iint\limits_{\Omega} \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dx\,dy.$$

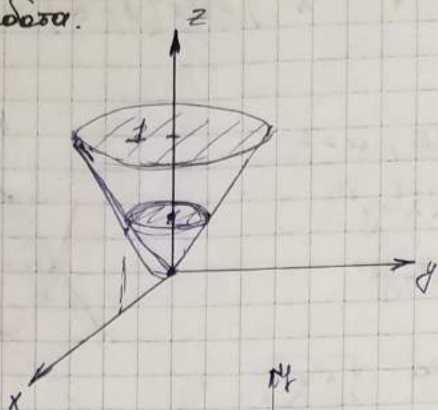
Ω - проекция на xoy поверхности S

Д/З: ~~4126 площадь поверхн.~~, 8

Бермант ~~3544~~, ~~3548~~, ~~3552~~, ~~3553~~, 3603, 3614.

~~Демидович 4080 (2 способа)~~

Домашняя работа.



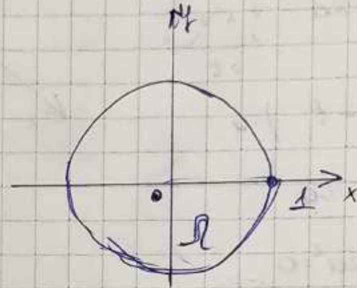
Д. 4080.

$$\iiint_V \sqrt{x^2+y^2}\,dx\,dy\,dz =$$

$x^2+y^2=z^2$

$z=1.$ 1 способ:

$$=\iint_{\Omega} dx\,dy \int_{\sqrt{x^2+y^2}}^{1} \sqrt{x^2+y^2}\,dz =$$



$$=\int_0^{2\pi} d\varphi \int_0^1 r\,dr \int_r^1 r\,dz =$$

$$= 2\pi \int_0^1 r^2 dr \cdot z \Big|_r^1 =$$

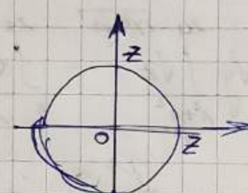
$$= 2\pi \int_0^1 r^2(1-r)\,dr = 2\pi \int_0^1 (r^2-r^3)\,dr =$$

$$= 2\pi \left(\frac{r^3}{3} - \frac{r^4}{4} \Big|_0^1 \right) = 2\pi \left(\frac{1}{3} - \frac{1}{4} \right) = \frac{\pi}{6}$$

2 способ: $\iiint_V \sqrt{x^2+y^2}\,dx\,dy\,dz = \int_0^1 dz \iint_{\Omega_z} \sqrt{x^2+y^2}\,dx\,dy =$

$$= \int_0^1 dz \int_0^{2\pi} d\varphi \int_0^z r^2 dr = 2\pi \int_0^1 dz \cdot \frac{z^3}{3} =$$

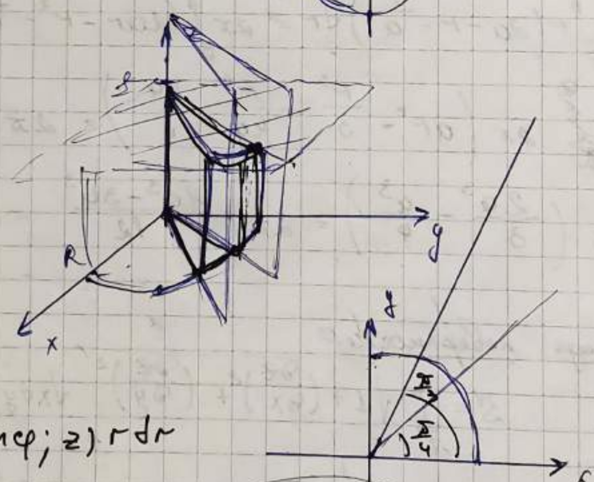
$$= 2\pi \cdot \frac{z^4}{12} \Big|_0^1 = \frac{\pi}{6}$$



Б 3547.

$x^2+y^2=R^2$
$z=0,\ z=1.$

$y=x\ ;\ y=x\sqrt{3}$



$$\iiint_V f(x,y,z)\,dx\,dy\,dz =$$

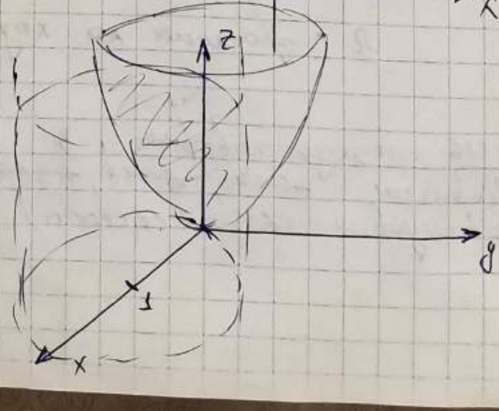
$$= \int_0^1 dz \iint_{\Omega_z} f(x,y,z)\,dx\,dy =$$

$$= \int_0^1 dz \int_{\pi/4}^{\pi/3} d\varphi \int_0^R f(r\cos\varphi;\, r\sin\varphi;\, z)\, r\,dr$$

Б 3548. $x^2+y^2=2x\ ;\ z=0\ ;\ z=x^2+y^2$

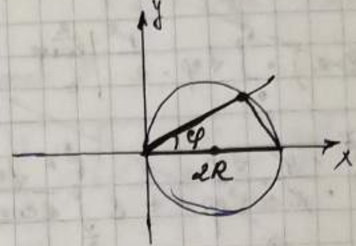
$x^2-2x+1+y^2=1$
$(x-1)^2+y^2=1.$

Участок, пересекающийся цилиндром и параболоидом



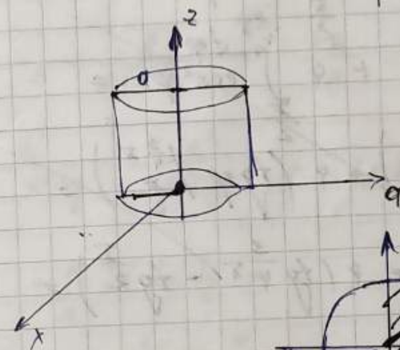
$\iiint_V f(x,y,z)\,dx\,dy\,dz = \cancel{\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\varphi} \iint_\Omega dx\,dy \int_0^{x^2+y^2} f(r\cos\varphi; r\sin\varphi; z)\,dz =$

$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\varphi \int_0^{2\cos\varphi} r\,dr \int_0^{r^2} f(r\cos\varphi; r\sin\varphi; z)\,dz$



Б 3552.

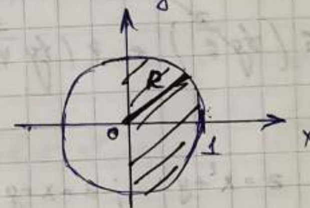
$\int_0^1 dx \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dy \int_0^a dz =$



$0 \le x \le 1$

$-\sqrt{1-x^2} \le y \le \sqrt{1-x^2}$

$0 \le z \le a$

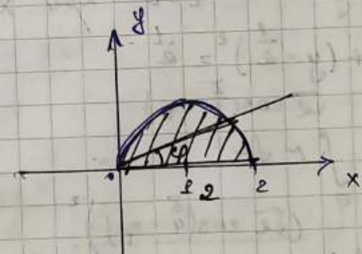


$\cancel{= \int_0^a dz \int_0^{2\pi} d\varphi \int_0^1 r\,dr =}$

$\cancel{= a \cdot 2\pi \cdot \frac{1}{2}}$

$= \int_0^a dz \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\varphi \int_0^1 r\,dr = \frac{1}{2} \cdot a \cdot \pi = \frac{\pi a}{2}.$

Б 3553. $\int_0^2 dx \int_0^{\sqrt{2x-x^2}} dy \int_0^a z\sqrt{x^2+y^2}\,dz =$



$0 \le x \le 2$

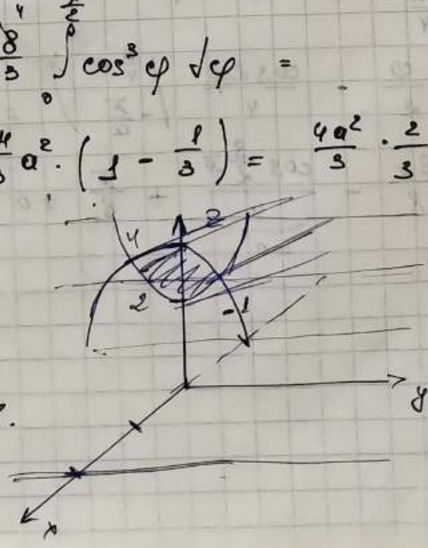
$0 \le y \le \sqrt{2x-x^2}$

$0 \le z \le a$

$= \int_0^a z\,dz \int_0^{\frac{\pi}{2}} d\varphi \int_0^{2\cos\varphi} r^2\,dr = \frac{a^2}{2} \int_0^{\frac{\pi}{2}} d\varphi \cdot \frac{r^3}{3}\Big|_0^{2\cos\varphi} =$

$= \frac{a^2}{2} \int_0^{\frac{\pi}{2}} d\varphi \cdot \left(\frac{8\cos^3\varphi}{3}\right) = \frac{a^2}{2} \cdot \frac{8}{3} \int_0^{\frac{\pi}{2}} \cos^3\varphi\,d\varphi =$

$= \frac{4}{3} a^2 \cdot \left(\sin\varphi - \frac{\sin^3\varphi}{3}\Big|_0^{\frac{\pi}{2}}\right) = \frac{4}{3} a^2 \cdot \left(1 - \frac{1}{3}\right) = \frac{4a^2}{3} \cdot \frac{2}{3} = \frac{8a^2}{9}$



Б. 3609. $z = 4 - y^2$ и $z = y^2 + 2$

$x = -1$ и $x = 2$.

$4 - y^2 = y^2 + 2 \to 2y^2 = 2 \to y = \pm 1.$

$z = 3$

$(1; 3)$ и $(1; -3)$ точки пересечения

$$4\int_2^4 dz \int_0^{\frac{\pi}{6}} d\varphi \int_0^{\frac{2}{\cos\varphi}} r\,dr + 4\int_2^4 dz \int_{\frac{\pi}{2}}^{\frac34\pi} d\varphi \int_0^{\frac{1}{\cos\varphi}} r\,dr =$$

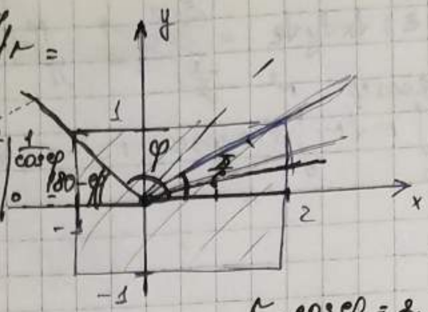
$$= 4(4-2)\cdot\int_0^{\frac{\pi}{6}} d\varphi \cdot \frac{r^2}{2}\Big|_0^{\frac{2}{\cos\varphi}} + 8\int_{\frac{\pi}{2}}^{\frac34\pi} d\varphi\cdot\frac{r^2}{2}\Big|_0^{\frac{1}{\cos\varphi}}$$

$$= 8\cdot\int_0^{\frac{\pi}{6}}\cdot\frac{4\,d\varphi}{\cancel{2}\cos^2\varphi} + 8\int_{\frac{\pi}{2}}^{\frac34\pi}\frac{d\varphi}{2\cos^2\varphi} =$$

$$= 16\cdot\int_0^{\frac{\pi}{6}}\frac{d\varphi}{\cos^2\varphi} + 4\int_{\frac{\pi}{2}}^{\frac34\pi}\frac{d\varphi}{\cos^2\varphi} =$$

$$= 16\left(tg\varphi\Big|_0^{\frac{\pi}{6}}\right) + 4\left(tg\varphi\Big|_{\frac{\pi}{2}}^{\frac34\pi}\right) =$$

$$= 16\left(tg\frac{\pi}{6}\right) + 4\left(tg\frac34\pi - tg\frac{\pi}{2}\right) =$$



$r\cdot\cos\varphi = 2$

$r = \frac{2}{\cos\varphi}$

$r\cdot\cos(180-\varphi) = -1$

$r = \frac{1}{\cos\varphi}$

Б.3614.

$z = x^2+y^2$; $z = x+y$ Если x =

$x^2+y^2 = x+y$

$x^2 - x + \frac14 + y^2 - y + \frac14 = \frac12$

$\begin{cases} z = x^2+y^2 \\ z = x+y \end{cases}$ $z=2$ при $x=1$, $y=1$. $z=0$ при $x=0$, $y=0$!

$(x-\frac12)^2+(y-\frac12)^2=\frac12$

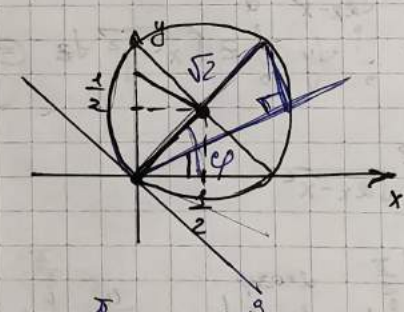
$$\int_0^2 dz \int_{-\frac{\pi}{4}}^{\frac34\pi} d\varphi \int_0^{\sqrt2\cos(\frac\pi4-\varphi)} r\,dr =$$

$$= 2\int_{-\frac{\pi}{4}}^{\frac34\pi} d\varphi\cdot\frac{\left(\sqrt2\cos(\frac\pi4-\varphi)\right)^2}{\cancel{2}} =$$

$$= 2\int_{-\frac{\pi}{4}}^{\frac34\pi} \cos^2\left(\frac\pi4-\varphi\right)d\varphi =$$

$$= 2\left(\frac\varphi2 - \frac{\cos2\varphi}{4}\Big|_{-\frac\pi4}^{\frac34\pi}\right) =$$

$$= 2\left(\frac{3\pi}{8} - \underbrace{\frac{\cos\frac32\pi}{4}}_{=0} + \frac\pi8 + 0\right) = 2\cdot\left(\frac\pi2\right) = \pi$$ – неверно



$-\frac\pi4 \le \varphi \le \frac34\pi$

$r = \sqrt2\cos\left(\frac\pi4-\varphi\right)$

Д. 4126 Вычисление площади поверхности.

$$S = \iint\limits_{\Omega} \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dx\, dy \qquad x^2 + y^2 = az \to z = \frac{x^2}{a} + \frac{y^2}{a}$$

Ω - проекция на xoy поверхности S

$$\sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} = \sqrt{1 + \left(\frac{2x}{a}\right)^2 + \left(\frac{2y}{a}\right)^2} = \sqrt{1 + \frac{4x^2}{a^2} + \frac{4y^2}{a^2}} =$$

$$= \frac{1}{a}\sqrt{a^2 + 4x^2 + 4y^2}$$

$$z = 2a - \sqrt{x^2 + y^2} \qquad \frac{\partial z}{\partial x} = -\frac{2x}{2\sqrt{x^2+y^2}} \qquad \frac{\partial z}{\partial y} = \frac{-y}{\sqrt{x^2+y^2}}$$

$$\sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} = \sqrt{1 + \frac{x^2}{x^2+y^2} + \frac{y^2}{x^2+y^2}} = \sqrt{1+1} = \sqrt{2}$$

$$S = \iint\limits_{\Omega} \frac{1}{a}\sqrt{a^2 + 4x^2 + 4y^2}\, dx\, dy + \iint\limits_{\Omega} \sqrt{2}\, dx\, dy =$$

$$= \int_0^{2\pi} d\varphi \int_0^a \frac{1}{a}\sqrt{a^2 + 4x^2 + 4y^2}\, r\, dr + \int_0^{2\pi} d\varphi \int_0^a \sqrt{2}\, r\, dr =$$

$$= \frac{1}{a}\int_0^{2\pi} d\varphi \int_0^a \sqrt{a^2 + 4r^2}\, r\, dr + 2\pi \cdot \frac{\sqrt{2}\, a^2}{2} =$$

$$= \frac{1}{a} \cdot 2\pi \cdot \left(\frac{(a^2 + 4r^2)\sqrt{a^2+4r^2}}{12} \Bigg|_0^a \right) + \sqrt{2}\pi a^2 =$$

$$= \frac{2\pi}{a}\left(\frac{5a^2 \cdot \sqrt{5a^2}}{12} - \frac{a^2 \cdot \sqrt{a^2}}{12} \right) + \sqrt{2}\pi a^2 =$$

$$= \frac{2\pi}{a}\left(\frac{5a^3 \cdot \sqrt{5}}{12} - \frac{a^3}{12} \right) + \sqrt{2}\pi a^2 =$$

$$= \frac{\pi a^2}{6}\left(5\sqrt{5} - 1\right) + \sqrt{2}\pi a^2 = \frac{\pi a^2}{6}\left(6\sqrt{2} + 5\sqrt{5} - 1\right)$$

Вычисление тройных интегралов в сферических координатах.

$$\begin{cases} x = r\cos\theta\cos\varphi \\ y = r\cos\theta\sin\varphi \\ z = r\sin\theta \end{cases}$$

$-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

$0 \le \varphi \le 2\pi$

$0 \le r < +\infty$

$|J| = r^2\cos\theta$

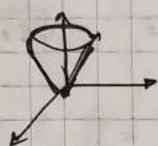
$|y| = r^2\cos\theta$

Координатные поверхности

1) $r = const$

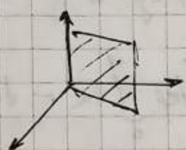
$r = const$ – сфера

2) $\theta = const$



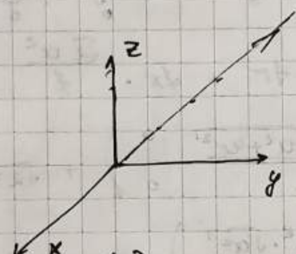
– конус.

3) $\varphi = const$

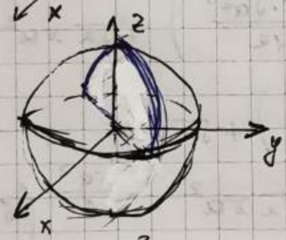


– плоскость

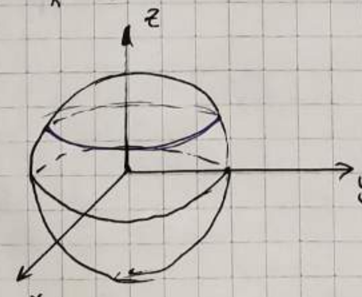
Координатные линии



r-линия $\begin{cases} \varphi = const \\ \theta = const \end{cases}$



θ-линия $\begin{cases} \varphi = const \\ r = const \end{cases}$



φ-линия $\begin{cases} r = const \\ \theta = const \end{cases}$

4089 $I = \iiint\limits_V F(\sqrt{x^2+y^2+z^2})\,dx\,dy\,dz.$

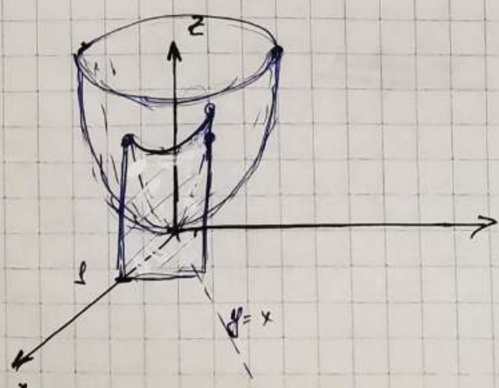
V: $z = x^2+y^2$

$x = y$

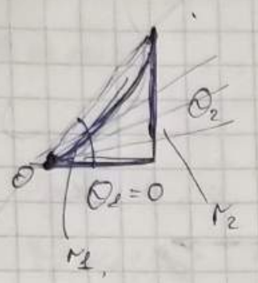
$x = 1$

$y = 0$

$z = 0$



$$= \int_0^{\frac{\pi}{4}} d\varphi \int_0^{\operatorname{arctg}\frac{1}{\cos\varphi}} d\theta \int_{\frac{\sin\theta}{\cos^2\theta}}^{\left(1/\cos\theta\cos\varphi\right)} f(r)\, r^2 \cdot \cos\theta\, dr = \int_0^{\frac{\pi}{4}} d\varphi \int_0^{\operatorname{arctg}\frac{1}{\cos\varphi}} \cos\theta\, d\theta \int_{\frac{\sin\theta}{\cos^2\theta}}^{\left(1/\cos\theta\cos\varphi\right)} F(r)\cdot r^2 dr$$



r_1: $z = x^2 + y^2$

$$r\sin\theta = r^2\cos^2\theta\cos^2\varphi + r^2\cos^2\theta\sin^2\varphi = r^2\cos^2\theta$$

$$\sin\theta = r\cos^2\theta$$

$$\boxed{r_1 = \frac{\sin\theta}{\cos^2\theta}}$$

r_2: $x = 1$

$$r\cos\theta\cos\varphi = 1 \rightarrow r = \frac{1}{\cos\theta\cos\varphi}$$

$$\Rightarrow r_2 = r = \frac{1}{\cos\theta\cos\varphi}$$

θ_2 – пересечение r_1 и r_2

$$\theta_2 : r_1 = r_2 \rightarrow \frac{\sin\theta}{\cos^2\theta} = \frac{1}{\cos\theta\cos\varphi}$$

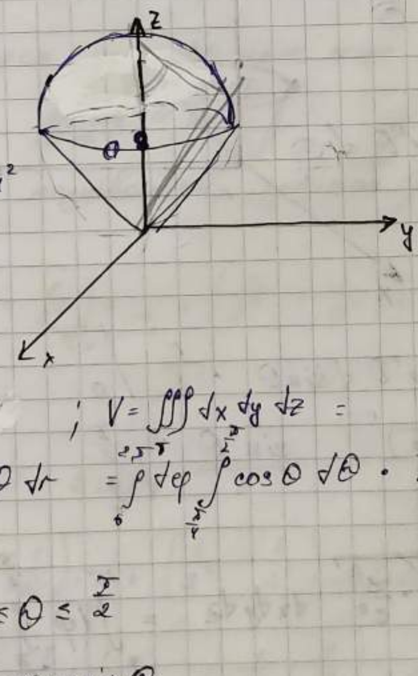
$$\operatorname{tg}\theta = \frac{1}{\cos\varphi} \rightarrow \theta = \operatorname{arctg}\left(\frac{1}{\cos\varphi}\right)$$

4107.

$$x^2 + y^2 + z^2 = 2az$$
$$x^2 + y^2 = z^2$$

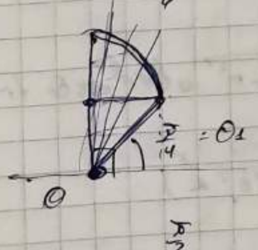
$$x^2 + y^2 + z^2 - 2az + a^2 = a^2$$

$$x^2 + y^2 + (z - a)^2 = a^2$$



$; \; V = \iiint dx\, dy\, dz =$

$$= \int_0^{2\pi} d\varphi \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} d\theta \int_0^{2a\sin\theta} r^2 \cos\theta\, dr = \int_0^{2\pi} d\varphi \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos\theta\, d\theta \cdot \frac{r^3}{3}\Big|_0^{2a\sin\theta} =$$



$$\frac{\pi}{4} \le \theta \le \frac{\pi}{2}$$

$$r^2 = 2ar\sin\theta$$

$$r = 2a\sin\theta$$

$$= 2\pi \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos\theta \cdot \frac{8a^3\sin^3\theta}{3}\, d\theta = \frac{16\pi a^3}{3} \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos\theta \sin^3\theta\, d\theta =$$

$$= \frac{16\pi a^3}{3} \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin^3\theta\, d\sin\theta =$$

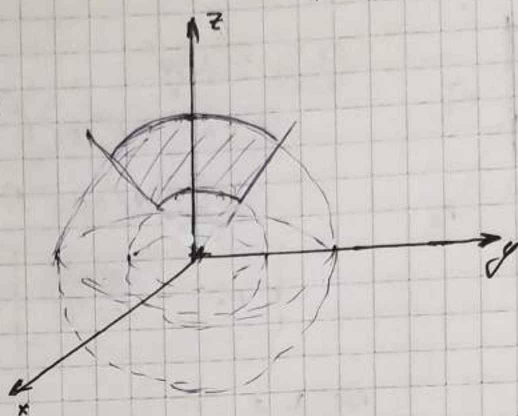
$= \frac{16\pi a^3}{3}\cdot\frac{\sin^4\theta}{4}\Big|_{\frac{\pi}{4}}^{\frac{\pi}{2}} = \frac{4\pi a^3}{3}\left(1-\frac{1}{4}\right) = \pi a^3$

4120.

$x^2+y^2+z^2=a^2$ Пусть $b>a$

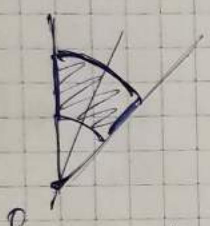
$x^2+y^2+z^2=b^2$

$x^2+y^2=z^2 \quad (z\geq 0)$



$I = \iiint\limits_V dx\,dy\,dz =$

$= \int_0^{2\pi} d\varphi \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} d\theta \int_a^b r^2\cos\theta\,dr =$



$= 2\pi\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\cos\theta\,d\theta\cdot\frac{r^3}{3}\Big|_a^b =$

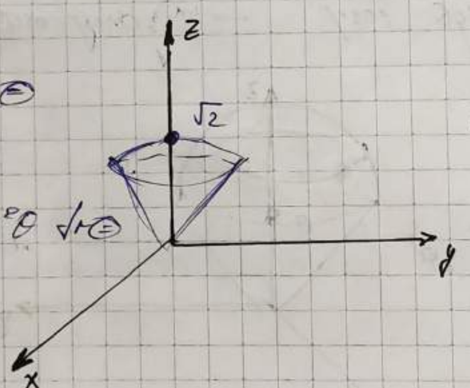
$= 2\pi\cdot\frac{b^3-a^3}{3}\cdot\sin\theta\Big|_{\frac{\pi}{4}}^{\frac{\pi}{2}} =$

$= \frac{2\pi}{3}\cdot\frac{b^3-a^3}{\cancel{3}}\cdot\left(1-\frac{\sqrt2}{2}\right) = \frac{2\pi(b^3-a^3)}{3}\left(1-\frac{\sqrt2}{2}\right)$

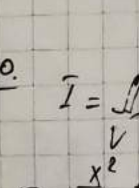
4088.

$\int_0^1 dx \int_0^{\sqrt{1-x^2}} dy \int_{\sqrt{x^2+y^2}}^{\sqrt{2-x^2-y^2}} z^2\,dz =$

$0\leq y\leq\sqrt{1-x^2}$



$= \int_0^{2\pi}d\varphi\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}d\theta\int_0^{\sqrt2} r^2\cos\theta\, r^2\sin^2\theta\,dr =$



$= 2\pi\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\sin^2\theta\,d\sin\theta\cdot\frac{r^5}{5}\Big|_0^{\sqrt2} =$

$= 2\pi\cdot\frac{(\sqrt2)^5}{5}\cdot\frac{\sin^3\theta}{3}\Big|_{\frac{\pi}{4}}^{\frac{\pi}{2}}$

4090.

$I=\iiint\limits_V\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}}\,dx\,dy\,dz = \int_0^{2\pi}d\varphi\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}d\theta\int_0^1\sqrt{1-r^2}\,abc\,r^2\cos\theta\,dr$

V: $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$

$r=\sin t \qquad t\in\left(0;\frac{\pi}{2}\right)$

$\int_0^{\frac{\pi}{2}}\cos^2 t\sin^2 t\,dt$

$\begin{cases} x = ar\cos\theta\cos\varphi \\ y = br\cos\theta\sin\varphi \\ z = cr\sin\theta \end{cases}$

$|I| = abcr^2\cos\theta$

Д/З: Берман ~~[illegible]~~, ~~3551~~, ~~3554~~ - ~~3556~~ ~~3555~~; 3616, 3625 ?
Кашир: ~~4078~~, ~~4087~~, ~~4079~~

30.04.2021.

Числовые ряды. Сходимость по определению, признаки сравнения.

Пусть $u_1, u_2, \ldots, u_n, \ldots$ - бесконечная последовательность чисел.

$(*)$ $u_1 + u_2 + \ldots + u_n + \ldots$ называется рядом, а элементы последовательности $u_1, u_2, \ldots, u_n, \ldots$ - членами ряда.

$$\sum_{n=1}^{\infty} u_n.$$
$$\sum_{n=0}^{\infty} u_n$$

Опр. Сумма n первых членов ряда $(*)$

$S_n = u_1 + u_2 + \ldots + u_n$ называется n-й **частичной суммой ряда.**

$S_1 = u_1$
$S_2 = u_1 + u_2$
$S_3 = u_1 + u_2 + u_3$
$\ldots\ldots\ldots$

Опр. Ряд $(*)$ называется сходящимся, если последовательность $S_1, S_2, \ldots, S_n, \ldots$

$$\lim_{n\to\infty} S_n = S$$

- значение S называется суммой ряда.

2547. $\left(\frac{1}{2} + \frac{1}{3}\right) + \left(\frac{1}{2^2} + \frac{1}{3^2}\right) + \ldots + \left(\frac{1}{2^n} + \frac{1}{3^n}\right) + \ldots$

$$S_1 = \underbrace{\frac{1}{2} + \frac{1}{3}}_{a_1}$$

$$S_2 = \underbrace{\frac{1}{2} + \frac{1}{3}}_{a_1} + \underbrace{\frac{1}{2^2} + \frac{1}{3^2}}_{a_2} = a_1 + a_2$$

$$\ldots\ldots S_n = \left(\frac{1}{2} + \frac{1}{3}\right) + \left(\frac{1}{2^2} + \frac{1}{3^2}\right) + \ldots + \left(\frac{1}{2^n} + \frac{1}{3^n}\right) = a_1 + a_2 + \ldots + a_n$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(\frac{1}{2} + \frac{1}{3} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots + \frac{1}{2^n} + \frac{1}{3^n}\right) = \lim_{n\to\infty}\left(\frac{1}{2} + \frac{1}{2^2} + \ldots + \frac{1}{2^n} + \frac{1}{3} + \frac{1}{3^2} + \ldots + \frac{1}{3^n}\right) =$$

$$S_n = \frac{b_1(1-q^n)}{1-q}.$$

$$= \lim_{n\to\infty} \frac{\frac{1}{2}}{1-\frac{1}{2}}\left(1 - \left(\frac{1}{2}\right)^n\right) + \frac{\frac{1}{3}}{1-\frac{1}{3}}\left(1 - \left(\frac{1}{3}\right)^n\right) =$$

$$= \lim_{n\to\infty}\left(1 - \underset{\downarrow 0}{\frac{1}{2^n}} + \frac{1}{2}\left(1 - \underset{\downarrow 0}{\frac{1}{3^n}}\right)\right) = 1 + \frac{1}{2} = \frac{3}{2}.$$

— док-ли сходимость ряда и нашли сумму.

2550. $\frac{1}{1\cdot 4}+\frac{1}{4\cdot 7}+\ldots+\frac{1}{(3n-2)(3n+1)}+\ldots$

$$S_n=\frac{1}{1\cdot 4}+\frac{1}{4\cdot 7}+\ldots+\frac{1}{(3n-2)(3n+1)}=\frac{1}{3}\left(\frac{4-1}{1\cdot 4}+\frac{7-4}{4\cdot 7}+\ldots+\frac{(3n+1)-(3n-2)}{(3n-2)(3n+1)}\right)=$$

$$=\frac{1}{3}\left(1-\cancel{\frac{1}{4}}+\cancel{\frac{1}{4}}-\cancel{\frac{1}{7}}+\ldots+\cancel{\frac{1}{3n-2}}-\frac{1}{3n+1}\right)=\frac{1}{3}\left(1-\frac{1}{3n+1}\right)$$

$$S=\lim_{n\to\infty}S_n=\lim_{n\to\infty}\frac{1}{3}\left(1-\underbrace{\frac{1}{3n+1}}_{\to 0}\right)=\frac{1}{3}$$

2548. $\frac{1}{2}+\frac{3}{2^2}+\frac{5}{2^3}+\ldots+\frac{2n-1}{2^n}+\ldots$

$$S_n=\frac{1}{2}+\frac{3}{2^2}+\frac{5}{2^3}+\ldots+\frac{2n-1}{2^n}$$

$$\frac{1}{2}S_n=\frac{1}{2^2}+\frac{3}{2^3}+\frac{5}{2^4}+\ldots+\frac{2n-1}{2^{n+1}}$$

$$\frac{1}{2}S_n=\frac{1}{2}+\frac{3}{2^2}-\frac{1}{2^2}+\underbrace{\frac{5}{2^3}-\frac{3}{2^3}}_{\frac{1}{2^2}}+\ldots+\frac{1}{2^{n-1}}$$

$$\frac{1}{2^3}(5-3)\qquad \frac{1}{2^2}\cdot 2=\frac{1}{2}$$

$$\frac{2n-1}{2^n}-\frac{2n-3}{2^n}=\frac{1}{2^{n-1}}$$

$$\frac{1}{2}S_n=\frac{1}{2}\left(1+\underbrace{1+\frac{1}{2}+\frac{1}{2^2}+\ldots+\frac{1}{2^{n-2}}}-\frac{2n-1}{2^n}\right)$$

$$S_n=1+\frac{1}{1-\frac{1}{2}}\left(1-\frac{1}{2^{n-1}}\right)-\frac{2n-1}{2^n}$$

$$S_n=1+2\left(1-\underbrace{\frac{1}{2^{n-1}}}_{\to 0}\right)-\underbrace{\frac{2n-1}{2^n}}_{\to 0}$$

$$\lim_{n\to\infty}S_n=3$$

Необходимое условие сходимости ряда.

– Если ряд $\sum_{n=1}^{\infty}a_n$ сходится, то $\lim_{n\to\infty}a_n=0$

2552. $0{,}001+\sqrt{0{,}001}+\sqrt[3]{0{,}001}+\ldots$

$$\boxed{\lim_{n\to\infty}\sqrt[n]{a}=1}$$

$$a_n=\sqrt[n]{0{,}001}$$

$$\lim_{n\to\infty}a_n=\lim_{n\to\infty}\sqrt[n]{0{,}001}=1$$

$\sum_{n=1}^{\infty}\sqrt[n]{0{,}001}$ – расходится

Предельный признак сравнения

Если ряды $\sum\limits_{n=1}^{\infty} a_n$ и $\sum\limits_{n=1}^{\infty} b_n$ $(\forall n \quad a_n > 0, \; b_n > 0)$ и

$\lim\limits_{n\to\infty} \frac{a_n}{b_n} = c, \quad c \in (0; +\infty)$, то ряды сходятся или расходятся одновременно.

2560. $\frac{1}{1001} + \frac{1}{2001} + \frac{1}{3001} + \ldots + \frac{1}{1000n+1} + \ldots$

$\sum\limits_{n=1}^{\infty} \frac{1}{n}$ – гармонический ряд – пример расходящегося ряда.

$$\lim_{n\to\infty} \frac{\frac{1}{1000n+1}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{n(1000+\frac{1}{n})} = \frac{1}{1000}$$

$\sum\limits_{n=1}^{\infty} \frac{1}{n}$ расходится, значит $\sum\limits_{n=1}^{\infty} \frac{1}{1000n+1}$ тоже расходится по предельному признаку сравнения.

Признаки сравнения.

Если ряды (1) $\sum\limits_{n=1}^{\infty} a_n$ и (2) $\sum\limits_{n=1}^{\infty} b_n$ положительные и $a_n \le b_n$ $\forall n > n_0 \in \mathbb{N}$, то из сходимости ряда (2) следует сходимость ряда (1), а из расходимости ряда (1) следует расходимость ряда (2).

2562. $1 + \frac{1}{3^2} + \frac{1}{5^2} + \ldots + \frac{1}{(2n-1)^2}$

$\sum\limits_{n=1}^{\infty} \frac{1}{n^p}$

$p > 1 \quad \sum\limits_{n=1}^{\infty} \frac{1}{n^p}$ сходится.

$p \le 1 \quad \sum\limits_{n=1}^{\infty} \frac{1}{n^p}$ расх.

$\cancel{\frac{1}{(2n-1)^2} > \frac{1}{(2n)^2}}$

$b_n \ge \frac{1}{(2n-1)^2}$

$\frac{1}{kn^2} \ge \frac{1}{(2n-1)^2}$

$kn^2 \le (2n-1)^2$ (дописать)

$kn^2 \le 4n^2 - 4n + 1.$

$n^2(4-k) - 4n + 1 \ge 0$

При каком k, нерав-во выполнено $\forall n$

$\cancel{D = 16 - 4(4-k) = 16 - 16 + 4k \le 0}$

Д/з: ~~2546~~, ~~2549~~, !2551 (а, б), ~~2556~~, ~~2559~~, ~~2552~~, ~~2563~~, ~~2564~~.

$$\frac{b_1(1-q^n)}{(1-q)}$$

Домашняя работа.

2546. $1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\ldots+\frac{(-1)^{n-1}}{2^{n-1}}+\ldots$

$\lim\limits_{n\to\infty}\frac{(-1)^{n-1}}{2^{n-1}}=\lim\limits_{n\to\infty}\frac{(-1)^n\,2}{2^n\,(-1)}=0$ — необходимое условие сходимости выполнено.

$S_n=1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\ldots+\frac{(-1)^{n-1}}{2^{n-1}}=\ldots+\frac{(-1)^n\cdot 2}{(-1)\cdot 2^n}=\ldots+\frac{-2\cdot(-1)^n}{2^n}=$

$=\ldots+-$

$S_n=1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\ldots+\frac{(-1)^{n-1}}{2^{n-1}}=1+\frac{1}{4}+\frac{1}{16}-\frac{1}{2}-\frac{1}{8}-\frac{1}{32}+\frac{1}{64}+\ldots+$

$+\frac{(-1)^{n-1}}{2^{n-1}}=1+\frac{\frac{1}{4}\left(1-\left(\frac{1}{4}\right)^n\right)}{\left(1-\frac{1}{4}\right)}-\frac{\left(\frac{1}{2}\right)\left(1-\left(\frac{1}{4}\right)^n\right)}{\left(1-\frac{1}{4}\right)}$

$\lim\limits_{n\to\infty}S_n=1+\frac{\frac{1}{4}\left(1-\left(\frac{1}{4}\right)^n\right)}{\frac{3}{4}}-\frac{\frac{1}{2}\left(1-\left(\frac{1}{4}\right)^n\right)}{\frac{3}{4}}=1+\frac{1\cdot 4}{3\cdot 4}-\frac{1\cdot 4}{2\cdot 3}=$

$=1-\frac{1}{3}=\frac{2}{3}$

Ответ: $\lim\limits_{n\to\infty}S_n=\frac{2}{3}$

2549. $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\ldots+\frac{1}{n(n+1)}+\ldots$

$\lim\limits_{n\to\infty}\frac{1}{n(n+1)}=0$ — необходимое условие сходимости выполнено.

$S_n=\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\ldots+\frac{1}{n(n+1)}=\frac{2-1}{1\cdot 2}+\frac{3-2}{2\cdot 3}+\ldots+\frac{(n+1)-n}{n(n+1)}=$

$=1\underbrace{-\frac{1}{2}+\frac{1}{2}-\frac{1}{3}+\frac{1}{3}-\frac{1}{4}+\ldots+\frac{1}{n}}_{\text{зачеркнётся}}-\frac{1}{(n+1)}$

$\Rightarrow\lim\limits_{n\to\infty}S_n=1-\frac{1}{n+1}\to 0=1.$

Ответ: $\lim\limits_{n\to\infty}S_n=1.$

2556. $1-1+1-1+1-1+\ldots$

$1-1+1-1+1-1+\ldots+(-1)^{n-1}+\ldots$

$\lim\limits_{n\to\infty}(-1)^{n-1}\ \nexists$, т.е. не выполняется необходимое условие сходимости.

$\Rightarrow$ последовательность расходится

2559. $1+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\ldots+\frac{1}{2n-1}+\ldots$

$\lim\limits_{n\to\infty}\frac{1}{2n-1}=0$ $\to$ выполнено необходимое условие сходимости посл-ти

$S_n = 1+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\ldots+\frac{1}{2n-1} = 1+\frac{1}{8}+\ldots+\frac{1}{n+n-1}$

$\sum\limits_{n=1}^{\infty}\frac{1}{n}$ - гармонический ряд (расходится)

$$\lim_{n\to\infty}\frac{\frac{1}{2n-1}}{\frac{1}{n}} = \lim_{n\to\infty}\frac{n}{2n-1} = \lim_{n\to\infty}\frac{n}{n(2-\frac{1}{n})} = \lim_{n\to\infty}\frac{1}{(2-\underbrace{\tfrac{1}{n}}_{\to 0})} = \frac{1}{2}$$

$\Rightarrow$ ряд $1+\frac{1}{3}+\ldots+\frac{1}{2n-1}+\ldots$ так же расходится.

2552. $\sum\limits_{n=1}^{\infty}(\sqrt{n+2}-2\sqrt{n+1}+\sqrt{n})$

$S_n = (\sqrt{3}-2\sqrt{2}+1)+(\sqrt{4}-2\sqrt{3}+\sqrt{2})+(\sqrt{5}-2\sqrt{4}+\sqrt{3})+(\sqrt{6}-2\sqrt{5}+\sqrt{4})+\ldots$
$+(\sqrt{n+2}-2\sqrt{n+1}+\sqrt{n}) =$
$=\sqrt{3}-2\sqrt{2}+1+2-2\sqrt{3}+\sqrt{2}+\sqrt{5}-4+\sqrt{3}+\sqrt{6}-2\sqrt{5}+2+\ldots+(\sqrt{n+2}-2\sqrt{n+1}+\sqrt{n}) =$

все члены с корнями сократятся кроме $\sqrt{2}$ и 1.

$= 1-\sqrt{2}+\sqrt{n+2}-\sqrt{n+1} = 1-\sqrt{2}+\frac{(\sqrt{n+2}-\sqrt{n+1})(\sqrt{n+2}+\sqrt{n+1})}{(\sqrt{n+2}+\sqrt{n+1})}$

$$\lim_{n\to\infty}S_n = \lim_{n\to\infty}\left(1-\sqrt{2}+\underbrace{\frac{n+2-n-1}{(\sqrt{n+2}+\sqrt{n+1})}}_{\to 0}\right) = 1-\sqrt{2}$$

Ответ: $\lim\limits_{n\to\infty}S_n = 1-\sqrt{2}$

2563. $\frac{1}{\sqrt{2}}+\frac{1}{2\sqrt{3}}+\frac{1}{3\sqrt{4}}+\ldots+\frac{1}{n\sqrt{n+1}}+\ldots$

$\sum\limits_{n=1}^{\infty}\frac{1}{n^p}$ - сходится при $p>1$.

Пусть $p=\frac{3}{2}$ $\to \sum\limits_{n=1}^{\infty}\frac{1}{n^{\frac{3}{2}}} = \sum\limits_{n=1}^{\infty}\frac{1}{\sqrt{n^3}}$

Исходная посл-ть положительна >0

$$\frac{1}{n\sqrt{n+1}} \le \frac{1}{n^{\frac{3}{2}}}$$
$$\frac{1}{\sqrt{n^2(n+1)}} \le \frac{1}{\sqrt{n^3}}$$
$$\frac{1}{\sqrt{n^3+n^2}} \le \frac{1}{\sqrt{n^3}}$$

Т.к. $n\ge1$, то ~~левая~~ знаменатель левой дроби > правой.
$\Rightarrow$ левая дробь меньше, а следовательно, по признаку сравнения $\to$ последовательность сходится.

2564. $\frac{1}{\sqrt{1\cdot 3}} + \frac{1}{\sqrt{3\cdot 5}} + \dots + \frac{1}{\sqrt{(2n-1)(2n+1)}} + \dots$

$\sum_{n=1}^{\infty} \frac{1}{n}$ - гармонический ряд (расходится)

$\lim_{n\to\infty} \frac{\frac{1}{2n}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{2n} = \frac{1}{2}$, т.е. $\sum_{n=1}^{\infty} \frac{1}{an}$, $a \in R$ тоже расходится.

$\frac{1}{an} \leq \frac{1}{\sqrt{(2n-1)(2n+1)}}$

$\frac{1}{an} < \frac{1}{\sqrt{4n^2-1}}$

Пусть $a = 2$.

$\frac{1}{2n} < \frac{1}{\sqrt{4n^2-1}} \longrightarrow \frac{1}{\sqrt{4n^2}} < \frac{1}{\sqrt{4n^2-1}}$

Знаменатель правой дроби < левой, тогда дробь слева меньше правой.

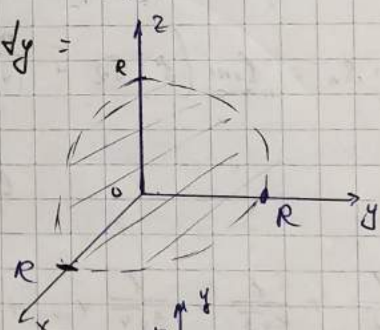
Следовательно, по признаку сравнения исходная последовательность расходится.

Б 3549. $\begin{cases} x = r\cos\varphi \\ y = r\sin\varphi \\ z = z \end{cases}$ - цилиндрические координаты. $|I| = r$

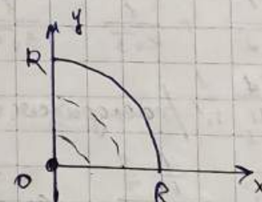
Ω - часть шара $x^2+y^2+z^2 \leq R^2$, лежащая в первом октанте.

$$\iiint_\Omega f(x,y,z)\,dx\,dy\,dz = \int_0^R dz \iint_{\Omega_z} f(x,y,z)\,dx\,dy =$$

$$= \int_0^R dz \int_0^{\frac{\pi}{2}} d\varphi \int_0^R f(r\cos\varphi, r\sin\varphi, z)\cdot r\,dr$$



$\begin{cases} x = \rho\cos\varphi\sin\theta \\ y = \rho\sin\varphi\sin\theta \\ z = \rho\cos\theta \end{cases}$ - сферические координаты.



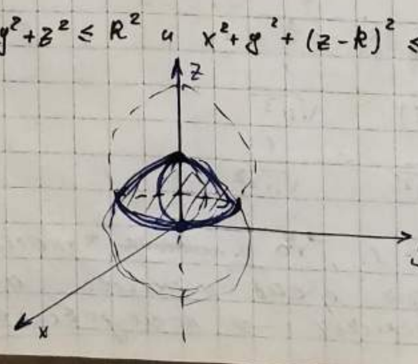
$|I| = \rho^2\sin\theta$

$$\iiint_\Omega f(x,y,z)\,dx\,dy\,dz = \int_0^{\frac{\pi}{2}} d\varphi \int_0^{\frac{\pi}{2}} \sin\theta\,d\theta \int_0^R f(\rho\cos\varphi\sin\theta, \rho\sin\varphi\sin\theta, \rho\cos\theta)\rho^2\,d\rho$$

Б 3551.

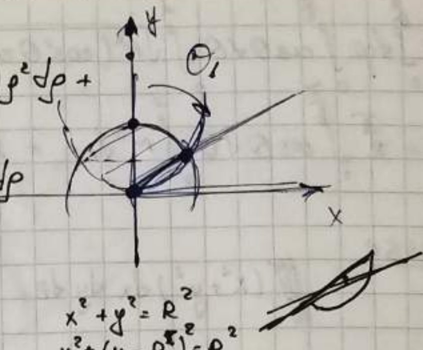
Ω - общая часть двух шаров $x^2+y^2+z^2 \leq R^2$ и $x^2+y^2+(z-R)^2 \leq R^2$

$$\iiint_\Omega f(x,y,z)\,dx\,dy\,dz =$$



$$= \int_0^{2\pi} d\varphi \int_0^{\frac{\pi}{3}} \sin\Theta\, d\Theta \int_0^{R} f(\rho\cos\varphi\sin\Theta, \rho\sin\varphi\sin\Theta, \rho\cos\Theta)\rho^2 d\rho +$$

$$+ \int_0^{2\pi} d\varphi \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \sin\Theta\, d\Theta \int_0^{2R\cos\Theta} f(\rho\cos\varphi\sin\Theta, \rho\sin\varphi\sin\Theta, \rho\cos\Theta)\rho^2 d\rho$$



$x^2+y^2=R^2$

$x^2+(y-R)^2=R^2$

$R^2-y^2=R^2-(y-R)^2$

$-y^2=-(y^2-2yR+R^2)$

$\cancel{-y^2}=\cancel{-y^2}+2yR+R^2$

$2yR=-R^2$

$y=-\frac{R}{2}$

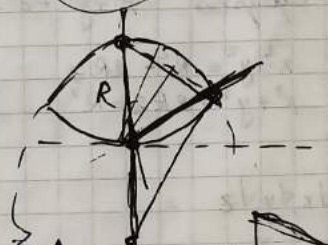
$y=\frac{R}{2}$

$x^2=R^2-\frac{R^2}{4}=\frac{3R^2}{4} \to x=\frac{R\sqrt{3}}{2}$

$tg\,\Theta_1=\sqrt{3} \Rightarrow \Theta_1=\frac{\pi}{3}$

Б 3554.

$$\int_{-R}^{R} dx \int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} dy \int_0^{\sqrt{R^2-x^2-y^2}} (x^2+y^2)\,dz \ominus$$

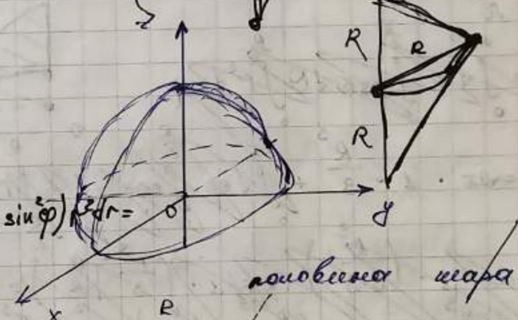


$-R \le x \le R$

$-\sqrt{R^2-x^2} \le y \le \sqrt{R^2-x^2}$

$0 \le z \le \sqrt{R^2-x^2-y^2} \to z^2=R^2-x^2-y^2$

$x^2+y^2+z^2=R^2$



$$\begin{cases} x=r\cos\Theta\cos\varphi \\ y=r\cos\Theta\sin\varphi \\ z=r\sin\Theta \end{cases} \quad I=r^2\cos\Theta$$

$$\ominus \int_0^{2\pi} d\varphi \int_0^{\frac{\pi}{2}} \cos\Theta\, d\Theta \int_0^R (r^2\cos^2\Theta\cos^2\varphi + r^2\cos^2\Theta\sin^2\varphi) r^2 dr =$$

половина шара.

$$= 2\pi \int_0^{\frac{\pi}{2}} \cos\Theta\, d\Theta \int_0^R (r^2\cos^2\Theta) \cdot r^2 dr =$$

$$= 2\pi \int_0^{\frac{\pi}{2}} \cos^3\Theta\, d\Theta \int_0^R r^4 dr = 2\pi \cdot \frac{R^5}{5} \int_0^{\frac{\pi}{2}} \cos^2\Theta \cdot \cos\Theta\, d\Theta =$$

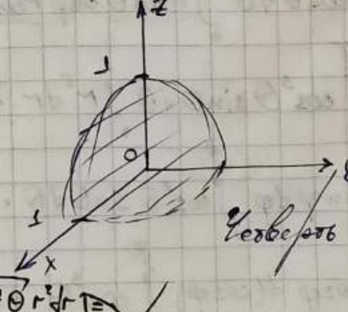
$$= \frac{2\pi R^5}{5} \int_0^{\frac{\pi}{2}} (1-\sin^2\Theta)\, d(\sin\Theta) = \frac{2\pi R^5}{5} \cdot \left(\sin\Theta \Big|_0^{\frac{\pi}{2}} - \frac{\sin^3\Theta}{3} \Big|_0^{\frac{\pi}{2}} \right) =$$

$$= \frac{2\pi R^5}{5} \cdot \left(1-\frac{1}{3}\right) = \frac{2\pi R^5}{5} \cdot \frac{2}{3} = \frac{4\pi R^5}{15}$$

$d\varphi$

Б 3555.

$$\int_0^1 dx \int_0^{\sqrt{1-x^2}} dy \int_0^{\sqrt{1-x^2-y^2}} \sqrt{x^2+y^2+z^2}\, dz \ominus$$



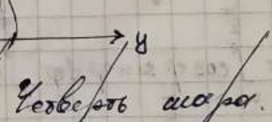
Четверть шара.

$0 \le x \le 1.$

$0 \le y \le \sqrt{1-x^2}$

$0 \le z \le \sqrt{1-x^2-y^2} \to x^2+y^2+z^2=1.$

$$\ominus \int_0^{\frac{\pi}{2}} d\varphi \int_0^{\frac{\pi}{2}} \cos\Theta\, d\Theta \int_0^1 \sqrt{r^2\cos^2\Theta\cos^2\varphi + r^2\cos^2\Theta\sin^2\varphi + r^2\sin^2\Theta}\; r^2 dr =$$



$$= \int_0^{\frac{\pi}{2}} d\varphi \int_0^{\frac{\pi}{2}} \cos\theta\, d\theta \int_0^1 \sqrt{r^2(\cos^2\theta\cos^2\varphi + \cos^2\theta\sin^2\varphi + \sin^2\theta)}\; r^2 dr =$$

$$= \frac{\pi}{2}\int_0^{\frac{\pi}{2}} \cos\theta\, d\theta \int_0^1 r^3 dr = \frac{\pi}{2}\cdot\frac{1}{4}\cdot\left(\sin\theta \Big|_0^{\frac{\pi}{2}}\right) = \cancel{\frac{\pi}{16}}\ \underline{\underline{\frac{\pi}{8}}}$$

Б <u>3556.</u> $\iiint\limits_{\Omega} (x^2+y^2)\,dx\,dy\,dz \quad ; \quad \Omega:\ z \geq 0\ ;\ r^2 \leq x^2+y^2+z^2 \leq R^2$

r из большого шара вырезали маленький.

$$\int_0^{2\pi} d\varphi \int_0^{\frac{\pi}{2}} \cos\theta\, d\theta \int_r^R r^4 \cos^2\theta\, dr = \frac{2\pi(R^5-r^5)}{5}\cdot\left(\sin\theta - \frac{\sin^3\theta}{3}\Big|_0^{\frac{\pi}{2}}\right) =$$

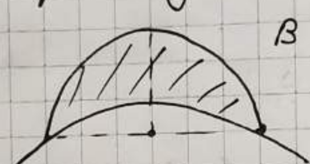
$$= \frac{4\pi(R^5-r^5)}{15} \qquad \text{(аналогия с Б3554)}$$

<u>Б3616.</u> Сферой $x^2+y^2+z^2=R^2$ и параболоидом $x^2+y^2=R(R-2z)$, $z \geq 0$

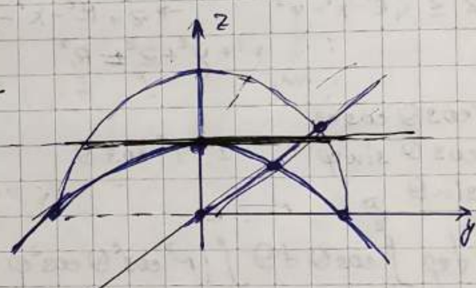
В сечении

$x^2+y^2 = R^2-2Rz$

$x^2+y^2+2Rz = R^2$



$\iiint\limits_{\Omega} dx\,dy\,dz =$



$$\cancel{= \int_0^{2\pi} d\varphi \int_0^{\frac{\pi}{2}} \cos\theta\, d\theta \int_0^{\frac{R}{2}} r^2 dr} \cancel{+ \int_0^{2\pi} d\varphi \int_0^{\frac{\pi}{2}} \cos\theta\, d\theta \int_{\frac{R}{2}}^{R} r^2 dr} =$$

$$\cancel{= 2\pi\cdot 1\cdot\left(\frac{R^3}{8\cdot 3}\right) + 2\pi\cdot 1\cdot\left(\frac{r^3}{3}\Big|_{\frac{R}{2}}^{R}\right)} = \cancel{\frac{2\pi R^3}{24} + 2\pi\cdot\left(\frac{R^3}{3} - \frac{R^3}{24}\right)} = \cancel{\frac{2\pi R^3}{24} + \frac{2\pi\cdot 7 R^3}{24} = \frac{8\cdot 2\pi R^3}{24_3}}$$

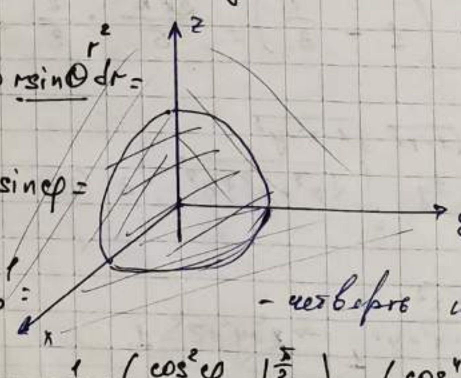
Сферические коорд-ы не <u>подходят.</u>

<u>Д 4078.</u> $\iiint\limits_V xyz\,dx\,dy\,dz \qquad V:\ x^2+y^2+z^2=1\ ,\ x=0,\ y=0,\ z=0$

$$\int_0^{\frac{\pi}{2}} d\varphi \int_0^{\frac{\pi}{2}} \cos\theta\, d\theta \int_0^1 \underline{r\cos\theta\cos\varphi}\ \underline{r\cos\theta\sin\varphi}\ \underline{r\sin\theta}\, r^2 dr =$$

$$= \int_0^{\frac{\pi}{2}} d\varphi \int_0^{\frac{\pi}{2}} \cos^3\theta\sin\theta\, d\theta \int_0^1 r^5 dr\cdot\cos\varphi\sin\varphi =$$

$$= \int_0^{\frac{\pi}{2}} \cos\varphi\sin\varphi\, d\varphi \int_0^{\frac{\pi}{2}} \cos^3\theta\sin\theta\, d\theta\cdot\frac{r^6}{6}\Big|_0^1 =$$



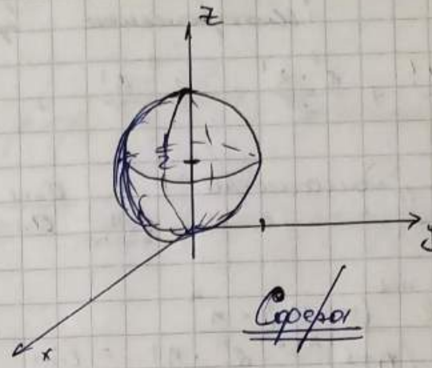
– четверть шара.

$$= \frac{1}{6}\int_0^{\frac{\pi}{2}} \cos\varphi\, d(\cos\varphi) \int_0^{\frac{\pi}{2}} \cos^3\theta\, d(\cos\theta) = \frac{1}{6}\cdot\left(\frac{\cos^2\varphi}{2}\Big|_0^{\frac{\pi}{2}}\right)\cdot\left(\frac{\cos^4\theta}{4}\Big|_0^{\frac{\pi}{2}}\right) =$$

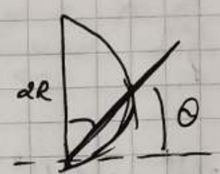
$$= \frac{1}{6}\cdot\frac{1}{2}\cdot\frac{1}{4} = \underline{\underline{\frac{1}{48}}}$$

№4087. $\iiint\limits_V \sqrt{x^2+y^2+z^2}\,dx\,dy\,dz \;=$

$$V:\; x^2+y^2+z^2=z$$
$$x^2+y^2+z^2-2\cdot\frac{1}{2}z+\frac{1}{4}=\frac{1}{4}$$
$$x^2+y^2+\left(z-\frac{1}{2}\right)^2=\frac{1}{4}$$



$$= \int_0^{2\pi} d\varphi \int_0^{\frac{\pi}{2}} \cos\theta\, d\theta \int_0^{\sin\theta} r^3\, dr =$$
$$= 2\pi \int_0^{\frac{\pi}{2}} \cos\theta\, d\theta \cdot \frac{\sin^4\theta}{4} =$$
$$= \frac{\pi}{2}\int_0^{\frac{\pi}{2}} \sin^4\theta\, d(\sin\theta) = \frac{\pi}{2}\cdot\frac{1}{5} = \underline{\underline{\frac{\pi}{10}}}$$



$$r = 2R\cos(90^\circ-\theta) = \underset{\frac{1}{2}}{\underset{\parallel}{2R}}\cdot\sin\theta$$

№4079. $\iiint\limits_V \left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\right)dx\,dy\,dz \;=$ $\qquad V:\; \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1.$

$$\begin{cases} x = ar\cos\varphi\cos\theta \\ y = br\sin\varphi\cos\theta \\ z = cr\sin\theta \end{cases} \qquad I = abc\, r^2\cos\theta$$

$$= \int_0^{2\pi} d\varphi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\theta\, d\theta \int_0^1 \left(\frac{a^2r^2\cos^2\theta\cos^2\varphi}{a^2}+\frac{b^2r^2\cos^2\theta\sin^2\varphi}{b^2}+\frac{c^2r^2\sin^2\theta}{c^2}\right)abc\,r^2\,dr =$$
$$= \int_0^{2\pi} d\varphi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\theta\, d\theta \int_0^1 abc\, r^4\, dr = 2\pi abc\cdot\frac{1}{5}\cdot\left(\sin\theta\,\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\right) = \underline{\underline{\frac{4\pi abc}{5}}}$$

13.05.2021.

Признаки Даламбера и Коши.
Интегральные признаки.

2581. $\frac{2\cdot 1!}{1} + \frac{2^2\cdot 2!}{2^2} + \frac{2^3\cdot 3!}{3^3} + \dots + \frac{2^n n!}{n^n} + \dots$

Признак Даламбера: $\lim\limits_{n\to\infty}\frac{a_{n+1}}{a_n} = \begin{cases}1, & ?\\ <1, & \text{сходится}\\ >1, & \text{расходится}\end{cases}$

$$\lim_{n\to\infty}\frac{2^{n+1}(n+1)!\cdot n^n}{(n+1)^{n+1}\cdot 2^n\cdot n!} = \lim_{n\to\infty}\frac{2(n+1)\cdot n^n}{(n+1)^n\cdot(n+1)} = 2\lim_{n\to\infty}\frac{n^n}{n^n\left(1+\frac{1}{n}\right)^n} =$$

$$=\frac{2}{e} < 1. \Rightarrow \text{Ряд сходится.}$$

2582. $\frac{(1!)^2}{2} + \frac{(2!)^2}{2^4} + \frac{(3!)^2}{2^9} + \dots + \frac{(n!)^2}{2^{n^2}} + \dots$

$$\lim_{n\to\infty}\frac{((n+1)!)^2\cdot 2^{n^2}}{2^{(n+1)^2}\cdot(n!)^2} = \lim_{n\to\infty}\frac{(n+1)!\,(n+1)!\cdot 2^{n^2}}{n!\cdot n!\cdot 2^{(n+1)^2}} = \lim_{n\to\infty}\frac{(n+1)^2\cdot 2^{n^2}}{2^{(n+1)^2}} =$$

$$2^{n^2-(n+1)^2} = 2^{\cancel{n^2}-\cancel{n^2}-2n-1} = 2^{-2n-1}.$$

$$= \lim_{n\to\infty}\frac{(n+1)^2\cdot\cancel{2^{n^2}}}{\cancel{2^{n^2}}\cdot 2^{2n}\cdot 2} = \lim_{n\to\infty}\frac{(n+1)^2}{2^{2n}\cdot 2} = \frac{1}{2}\lim_{n\to\infty}\frac{(n+1)^2}{2^{2n}} =$$

$$= \frac{1}{2}\lim_{n\to\infty}\frac{n^2+2n+1}{2^{2n}} = \frac{1}{2}\lim_{n\to\infty}\frac{2n+2}{2\cdot 2^{2n}\cdot\ln 2} = \lim_{n\to\infty}\frac{n+1}{2^{2n+1}\ln 2} =$$

$$= \lim_{n\to\infty}\frac{1}{\ln^2 2\cdot 2^{2n+1}\cdot 2} = 0$$

$\Rightarrow$ Ряд сходится

2586. Признак Коши:

$$\lim_{n\to\infty}\sqrt[n]{a_n} = \begin{cases}<1, & \text{сходится}\\ >1, & \text{расходится}\\ =1, & ?\end{cases}$$

$$\sum_{n=1}^{\infty}\frac{n^2}{\left(2+\frac{1}{n}\right)^n}$$

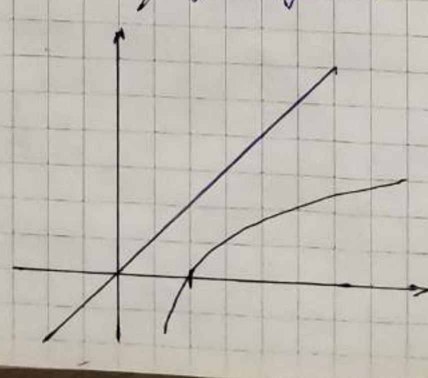
$$\lim_{n\to\infty}\frac{\sqrt[n]{n^2}\to 1}{2+\frac{1}{n}} = \frac{1}{2} < 1 \Rightarrow \text{Ряд сходится}$$

2588 $\sum\limits_{n=2}^{\infty}\frac{1}{\sqrt[n]{\ln n}}$

$n \geq 2 \quad \frac{1}{\sqrt[n]{\ln n}} > 0$

$\ln n < n$

$\sqrt[n]{\ln n} < \sqrt[n]{n}$



$$\frac{1}{\sqrt[4]{\ln n}} > \frac{1}{\sqrt[4]{n}}$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{\sqrt[4]{n}} = 1. \Rightarrow \text{ряд } \sum_{n=1}^{\infty} \frac{1}{\sqrt[4]{n}} \text{ расходится.}$$

По признаку сравнения: исходный ряд также расходится.

Интегральный признак:

Если $f(x)$ $(x>0)$ неотриц. невозр-ая ф-я, то $\sum_{n=1}^{\infty} f(n)$ сходится или расходится одновременно с несобственным интегралом $\int_1^{\infty} f(x)\,dx$

2619 а.

$$a_n = \frac{1}{n \ln^p n} \quad (n \geq 2)$$

$$f(x) = \frac{1}{x \ln^p x} \quad \text{при } x \geq 2 \text{ неотриц.}$$

$$f'(x) = \frac{-(x \ln^p x)'}{x^2 \ln^{2p} x} = -\frac{\left(\ln^p x + x \cdot p \ln^{p-1} x \cdot \frac{1}{x}\right)}{x^2 \ln^{2p} x} = -\frac{1}{(x \ln^p x)^2}\left(\ln^p x + p\ln^{p-1} x\right) =$$

$$= \frac{\ln^{p-1} x}{(x \ln^p x)^2}(\ln x + p)$$

$$\ln x + p > 0 \quad \to \quad x > e^{-p}$$

$$\ln x - 2 > 0$$

$$\int_2^{+\infty} \frac{dx}{x \ln^p x} = \int_2^{+\infty} \frac{d\ln(x)}{\ln^p x} = \left.\frac{\ln^{-p+1} x}{(1-p)}\right|_2^{+\infty} =$$

$$\lim_{A\to+\infty} \left.\frac{\ln^{-p+1} x}{-p+1}\right|_2^A = \lim_{A\to+\infty} \frac{\ln^{-p+1} A}{-p+1} - \frac{(\ln 2)^{1-p}}{1-p}$$

$$-p+1 < 0.$$

$\Rightarrow$ при $p > 1$ ряд сходится.

2584. $\frac{4}{2} + \frac{4}{2} \cdot \frac{7}{6} + \frac{4 \cdot 7 \cdot 10}{2 \cdot 6 \cdot 10}$

$$a_n = \frac{4 \cdot 7 \cdot 10 \cdots (3n+1)}{2 \cdot 6 \cdot 10 \cdots (4n-2)}$$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \left(\frac{4 \cdot 7 \cdot 10 \cdots (3n+1)(3n+4) \cdot 2 \cdot 6 \cdot 10 \cdots (4n-2)}{2 \cdot 6 \cdot 10 \cdots (4n-2)(4n+2) \cdots 4 \cdot 7 \cdot 10 (3n+1)} \right) = \frac{3}{4}$$

$\Rightarrow$ ряд сходится.

Д/З: ~~2578~~, ~~2579~~, ~~2580~~, ~~2581(б)~~, ~~2582~~, ~~2585~~, 2587, 2589(б), 2819(б)

Домашняя работа.

2578. $\frac{1000}{1!}+\frac{1000^2}{2!}+\ldots+\frac{1000^n}{n!}+\ldots$

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to\infty}\frac{1000^{n+1}\cdot n!}{(n+1)!\,1000^n}=\lim_{n\to\infty}\frac{1000}{n+1}=0$$

$\Rightarrow$ ряд сходится.

2579. $\frac{(1!)^2}{2!}+\frac{(2!)^2}{4!}+\ldots+\frac{(n!)^2}{(2n)!}$

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to\infty}\frac{((n+1)!)^2\cdot(2n)!}{(2(n+1))!\,(n!)^2}=\lim_{n\to\infty}\frac{(n+1)!\cdot(n+1)!\cdot(2n)!}{(2n+2)!\,n!\cdot n!}=$$

$$=\lim_{n\to\infty}\frac{(n+1)^2\,(2n)!}{(2n+2)!}=\lim_{n\to\infty}\frac{(n+1)^2\,\cancel{(2n)!}}{(2n+2)(2n+1)\cancel{(2n)!}}=$$

$$=\lim_{n\to\infty}\frac{(n+1)^{\cancel{2}}}{2\cancel{(n+1)}(2n+1)}=\frac{1}{4}$$

$\Rightarrow$ ряд сходится

2580. $\frac{1!}{1}+\frac{2!}{2^2}+\frac{3!}{3^3}+\ldots+\frac{n!}{n^n}$

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to\infty}\frac{(n+1)!\;n^n}{(n+1)^{n+1}\cdot n!}=\lim_{n\to\infty}\frac{(n+1)\cdot n^n}{(n+1)^n\cdot\cancel{(n+1)}}<1.$$

$\Rightarrow$ ряд сходится.

2581(б)

$\frac{3\cdot1!}{1}+\frac{3^2\cdot2!}{2^2}+\frac{3^3\cdot3!}{3^3}+\ldots+\frac{3^n\,n!}{n^n}$

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to\infty}\frac{3^{n+1}\,(n+1)!\;n^n}{(n+1)^{n+1}\cdot3^n\cdot n!}=\lim_{n\to\infty}\frac{3\,(n+1)\;n^n}{(n+1)^n\cdot(n+1)}$$

$$=3\lim_{n\to\infty}\frac{n^n}{(n+1)^n}=3\lim_{n\to\infty}\left(\frac{n+1-1}{n+1}\right)^n=3\lim_{n\to\infty}\left(1-\frac{1}{n+1}\right)^n=$$

$$=3\lim_{n\to\infty}\left(\left(1+\frac{1}{-n-1}\right)^{-n-1}\right)^{\frac{n}{-n-1}}=3\cdot e^{\lim\limits_{n\to\infty}\frac{n}{-n-1}}=\frac{3}{e}>1.$$

$\Rightarrow$ ряд расходится.

2582

$\frac{(1!)^2}{2}+\frac{(2!)^2}{2^4}+\frac{(3!)^2}{2^9}+\ldots+\frac{(n!)^2}{2^{n^2}}+\ldots$

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to\infty}\frac{(n+1)!\,(n+1)!\;2^{n^2}}{2^{(n+1)^2}\;n!\;n!}=\lim_{n\to\infty}\frac{(n+1)^2\cdot 2^{n^2}}{2^{n^2+2n+1}}=\lim_{n\to\infty}\frac{(n+1)^2\cdot \cancel{2^{n^2}}}{\cancel{2^{n^2}}\cdot 2^{2n}\cdot 2}=$$

$$=\frac{1}{2}\lim_{n\to\infty}\frac{(n+1)^2}{2^{2n}}=\left\{\frac{\infty}{\infty}\right\}=\frac{1}{2}\lim_{n\to\infty}\frac{\cancel{2}(n+1)}{2^{2n}\cdot\cancel{2}\ln 2}=\left\{\frac{\infty}{\infty}\right\}=$$

$$=\frac{1}{2}\lim_{n\to\infty}\frac{1}{\ln^2 2\cdot 2\cdot 2^{2n}}=0$$

$\Rightarrow$ ряд сходится.

2583.

$$\frac{1000}{1}+\frac{1000\cdot 1001}{1\cdot 3}+\frac{1000\cdot 1001\cdot 1002}{1\cdot 3\cdot 5}+\dots$$

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to\infty}\frac{1000+n}{2n+1}=\frac{1}{2}<1$$

$\Rightarrow$ ряд сходится.

2587. $\sum\limits_{n=1}^{\infty}\frac{n^{n+\frac{1}{n}}}{\left(n+\frac{1}{n}\right)^n}$

$$\lim_{n\to\infty}\sqrt[n]{\frac{n^n\cdot n^{\frac{1}{n}}}{\left(n+\frac{1}{n}\right)^n}}=\lim_{n\to\infty}\frac{n\cdot\sqrt[n]{n^{\frac{1}{n}}}}{n+\frac{1}{n}}=\lim_{n\to\infty}\frac{n\cdot\sqrt[n^2]{n}}{n+\frac{1}{n}}=$$

$$=\lim_{n\to\infty}\frac{\sqrt[n]{n^{\frac{1}{n}}}}{\left(1+\frac{1}{n^2}\right)}=\lim_{n\to\infty}\frac{\sqrt[nn]{n}\to 1}{\left(1+\frac{1}{n^2}\right)}=1\ ?$$

$$\frac{n^{n+\frac{1}{n}}}{\left(n+\frac{1}{n}\right)^n}=\frac{n^n\cdot n^{\frac{1}{n}}\,n^n}{(n^2+1)^n}=\frac{n^{2n}\cdot n^{\frac{1}{n}}}{(n^2+1)^n}$$

2589 (δ) $\sum_{n=1}^{\infty} \frac{n^5}{2^n+3^n}$

2619. δ $a_n = \frac{1}{n(\ln n)^p(\ln\ln n)^q}$ $(n>2)$

$$\int_2^{+\infty} \frac{dx}{n(\ln n)^p(\ln\ln n)^q} = \int_2^{+\infty} \frac{d(\ln n)}{(\ln n)^p(\ln(\ln n))^q}$$

18.05.21. Знакопеременные ряды.

2668(а) $\sum_{n=1}^{\infty} (-1)^n \left(\frac{2n+100}{3n+1}\right)^n$

Ряд из модулей: $\sum_{n=1}^{\infty} \left(\frac{2n+100}{3n+1}\right)^n$

$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\left(\frac{2n+100}{3n+1}\right)^n} = \lim_{n\to\infty} \left(\frac{2n+100}{3n+1}\right) = \frac{2}{3} < 1 \Rightarrow$

ряд $\sum_{n=1}^{\infty} \left(\frac{2n+100}{3n+1}\right)^n$ сходится, а значит $\sum_{n=1}^{\infty} (-1)^n \left(\frac{2n+100}{3n+1}\right)^n$ сходится абсолютно.

2669. $\sum_{n=1}^{\infty} (-1)^n \frac{\sqrt{n}}{n+100}$

Ряд из модулей: $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n+100}$

$\lim_{n\to\infty} \frac{\sqrt{n}}{n+100} = 0$

$\lim_{n\to\infty} \frac{\sqrt{n} \cdot \sqrt{n}}{n+100} = 1$

$\Rightarrow \sum_{n=1}^{\infty} \frac{\sqrt{n}}{n+100}$ и $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ ведут себя одинаково

$p = \frac{1}{2}$

$\Rightarrow \sum_{n=1}^{\infty} \frac{\sqrt{n}}{n+100}$ - расходится.

По признаку Лейбница: $\frac{\sqrt{n}}{n+100}$ убыв.?

$f(x) = \frac{\sqrt{x}}{x+100}$

~~$f'(x) = \frac{\frac{1}{2\sqrt{x}}(x+100) - \sqrt{x}}{(x+100)^2} = \frac{(x+100-2x)}{2\sqrt{x}(x+100)^2} = \frac{100}{2\sqrt{x}(x+100)^2}$~~

$f'(x) = \frac{100 - x}{2\sqrt{x}(x+100)^2} < 0$

$x > 100$

$n \geq 101$

$\;\; n_0$

2) $\lim_{n\to\infty} \frac{\sqrt{n}}{n+100} = 0$

Ряд $\sum_{n=1}^{\infty} (-1)^n \frac{\sqrt{n}}{n+100}$ сходится по призн. Лейбница.

Ряд $\sum_{n=1}^{\infty} (-1)^n \frac{\sqrt{n}}{n+100}$ сх-ся условно

2673(б.) $\sum_{n=2}^{\infty} \frac{1}{\ln^2 n} \cdot \cos \frac{\pi n^2}{n+1}$

$$\boxed{\cos\frac{\pi n^2}{n+1} = (-1)^n \cos\left(\pi\frac{n^2}{n+1} - \pi n\right)} \qquad \cos\alpha = (-1)^n\cos(\alpha - \pi n)$$

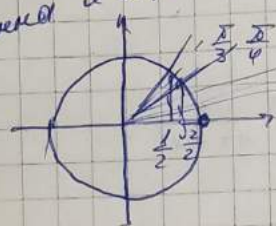
$n = 2k \qquad \frac{\pi\cdot 4k^2}{2k+1}$

$n = 2k+1$

$$\cos\frac{\pi n^2}{n+1} = (-1)^n\cos\left(\frac{\pi n^2}{n+1} - \pi n\right) = (-1)^n\cos\left(\frac{\pi n^2 - \pi n(n+1)}{n+1}\right) = (-1)^n\cos\left(-\frac{\pi n}{n+1}\right) =$$

$$= (-1)^n\cos\left(\frac{-\pi(n+1)+\pi}{n+1}\right) = (-1)^n\cos\left(-\pi + \frac{\pi}{n+1}\right) = (-1)^{n+1}\cos\frac{\pi}{n+1}$$

$\Rightarrow \sum\limits_{n=2}^{\infty}\frac{1}{\ln^2 n}\cdot(-1)^{n+1}\cos\frac{\pi}{n+1}$ — монотонна и огр.



~~Признак Абеля~~:

$\cos\frac{\pi}{n+1}$ монотонна и огр.

$\sum\limits_{n=2}^{\infty}\frac{1}{\ln^2 n}(-1)^{n+1}$

Признак Лейбница $\lim\limits_{n\to\infty}\frac{1}{\ln^2 n} = 0$

$f(x) = \frac{1}{\ln^2 x}$

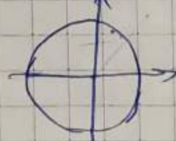
$f'(x) = -\frac{1}{\ln^4 x}\cdot 2\ln x\cdot\frac{1}{x} = -\frac{2}{x\ln^3 x}$

при $x \geq 2$ $f(x)$ убыв.

Ряд $\sum\limits_{n=2}^{\infty}\frac{1}{\ln^2 n}(-1)^{n+1}$ — сходится по признаку Лейбница,

а ряд $\sum\limits_{n=2}^{\infty}\frac{1}{\ln^2 n}(-1)^n\cos\frac{\pi}{n+1}$ сходится по признаку Абеля.

2667. $\sum\limits_{n=1}^{\infty}\frac{\ln^{100} n}{n}\sin\frac{n\pi}{4}$



$\{b_n\} = \left\{\frac{\ln^{100} n}{n}\right\}$

$f(x) = \frac{\ln^{100} x}{x}$

$$f'(x) = \frac{100\ln^{99}x\cdot\frac{1}{x}\cdot x - \ln^{100}x}{x^2} = \frac{\ln^{99}x\left(\frac{100}{x}\cdot x - \ln x\right)}{x^2} =$$

$$= \frac{\ln^{99}x\,(100 - \ln x)}{x^2} = \frac{\ln^{99}x\,(100-\ln x)}{x^2} < 0$$

$\ln x > 100$

$x > e^{100}$

$\lim\limits_{n\to\infty}\frac{\ln^{100} n}{n} = 0$ (по правилу ~~Абел~~ Лопиталя)

$\{b_n\}$ монотонно убывает до нуля.

$$\left|\sum_{k=1}^{n} \sin\frac{k\pi}{4}\right| \le \dots$$

Суммы вида $\sin\alpha + \sin 2\alpha + \sin 3\alpha + \ldots + \sin n\alpha$ вычисляются домножением и делением на $\frac{\alpha}{2}$.

$$\frac{\sin\frac{\alpha}{2}}{\sin\frac{\alpha}{2}}\left(\sin\alpha + \sin 2\alpha + \ldots + \sin n\alpha\right) = \frac{1}{\sin\frac{\alpha}{2}}\left(\sin\frac{\alpha}{2}\sin\alpha + \sin\frac{\alpha}{2}\cdot\sin 2\alpha + \ldots + \sin\frac{\alpha}{2}\sin n\alpha\right) =$$

$$\sin\alpha\sin\beta = \frac{1}{2}\left(\cos(\alpha-\beta) - \cos(\alpha+\beta)\right)$$

$$= \frac{1}{2\sin\frac{\alpha}{2}}\left(\cos\frac{\alpha}{2} - \cancel{\cos\frac{3\alpha}{2}} + \cancel{\cos\frac{3\alpha}{2}} - \cos\frac{5\alpha}{2} + \cos\frac{(2n-1)\alpha}{2} - \cos\frac{(2n+1)\alpha}{2}\right) =$$

$$= \frac{1}{2\sin\frac{\alpha}{2}}\left(\cos\frac{\alpha}{2} - \cos\frac{(2n+1)\alpha}{2}\right) = \frac{\sin\frac{(n+1)\alpha}{2}\sin\frac{n\alpha}{2}}{\sin\frac{\alpha}{2}}$$

$$\cos\alpha - \cos\beta = -2\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}$$

$$\left|\sum_{k=1}^{n}\sin\frac{n\pi}{4}\right| = \left|\frac{\sin\frac{(n+1)\pi}{8}\sin\frac{n\pi}{8}}{\sin\frac{\pi}{8}}\right| \le \frac{1}{\sin\frac{\pi}{8}}$$

$\Rightarrow$ исходный ряд сходится по Дирихле.

2671. $\sum_{n=1}^{\infty}\sin\left(\pi\sqrt{n^2+k^2}\right)$

$$\sin x = (-1)^n\sin(x+\pi n)$$

Д/З: 2671, 2665(б), 2668, 2672, 2679(а), 2683 | 2659, 2661*

20.05.21.

Степенные ряды. Нахождение радиуса и интервала сходимости.

$$\sum_{n=0}^{\infty} a_n (x-x_0)^n$$

$\{a_n\}$ числовая последовательность.

a_n - коэффициенты степенного ряда.

$t = x - x_0$ ~ замена переводит степенной ряд с центром в нуле.

$$\sum_{n=0}^{\infty} a_n t^n$$

(1) $\sum_{n=0}^{\infty} a_n x^n$

Первая теорема Абеля:

~ Если ряд (1) сходится в некоторой точке $c \neq 0$, то он сходится абсолютно на интервале $(-|c|, |c|)$

Следствие: каждому степенному ряду $\sum_{n=0}^{\infty} a_n (x-x_0)^n$ соответствует действительное число $R \geq 0$ или символ $+\infty$ так, что при $|x-x_0| < R$ ряд сходится абсолютно, а при $|x-x_0| > R$ расходится.

R - радиус сходимости

$R > 0$ $(-R, R)$ - интервал сходимости.

Формула Коши-Адамара.

$$\frac{1}{R} = \overline{\lim_{n\to\infty}} \sqrt[n]{a_n}$$

$$R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right|$$

2812. $\sum_{n=1}^{\infty} \frac{x^n}{n^p}$

$$\cancel{R = \lim_{n\to\infty} \left| \frac{x^n}{n^p} \cdot \frac{(n+1)^p}{x^{n+1}} \right|}$$

$$R = \lim_{n\to\infty} \left| \frac{(n+1)^p}{n^p} \right| = \lim_{n\to\infty} \left| \left(1+\frac{1}{n}\right)^p \right| = \lim_{n\to\infty} \underbrace{\left(1+\frac{1}{n}\right)^{n}}_{e}{}^{\cdot\frac{p}{n}} = e^{\lim_{n\to\infty}\frac{p}{n}} = e^0 = 1$$

$|x| < 1$ - ряд сх-ся абсолютно $(-1 < x < 1)$

$x = 1$

$\sum_{n=1}^{\infty} \frac{1}{n^p}$

$p > 1$ сх-ся ; $p \leq 1$ расх.

$x = -1$

$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^p}$

$\lim_{n\to\infty} \frac{1}{n^p} = 0$; $p > 0$

$a_n > a_{n+1}.$

$\frac{1}{n^p} \geq \frac{1}{(n+1)^p} \quad p>0$

При $x=1$ сх. абс. при $p>1$.

При $x=-1$ сх. абс. при $p>1$, сх. условно при $0<p\leq 1$.

2813. $\sum_{n=1}^{\infty} \frac{3^n+(-2)^n}{n}(x+1)^n$

$\frac{1}{R} = \overline{\lim_{k\to\infty}} \sqrt[n]{\left|\frac{3^n+(-2)^n}{n}\right|} = \lim_{k\to\infty} \sqrt[2k]{\frac{3^{2k}+2^{2k}}{2k}} = \lim_{k\to\infty} \frac{\left(3^{2k}+2^{2k}\right)^{\frac{1}{2k}}}{\underbrace{\sqrt[2k]{2k}}_{\to 1}} =$

$= \lim_{k\to\infty} \left(3^{2k}+2^{2k}\right)^{\frac{1}{2k}} = \lim_{k\to\infty} \left|3^{2k}\left(1+\left(\frac{2}{3}\right)^{2k}\right)\right|^{\frac{1}{2k}} =$

$= \lim_{k\to\infty} 3\cdot\left(1+\left(\frac{2}{3}\right)^{2k}\right)^{\frac{1}{2k}} = 3.$ $\to 1$

$R = \frac{1}{3}.$

$|x+1| < \frac{1}{3} \quad \to -\frac{1}{3} < x+1 < \frac{1}{3}$

$-\frac{4}{3} < x < -\frac{2}{3}$ – при таких x ряд сходится абсолютно.

$x = -\frac{4}{3} \quad \sum_{n=1}^{\infty} \frac{3^n+(-2)^n}{n}\cdot\frac{(-1)^n}{3^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} + \frac{1}{n}\left(\frac{2}{3}\right)^n$

$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ – сходится по Лейбницу условно.

$\sum_{n=1}^{\infty} \frac{1}{n}\cdot\left(\frac{2}{3}\right)^n$

$\lim_{n\to\infty} \frac{2}{3\sqrt[n]{n}} = \frac{2}{3} < 1. \to$ сходится по пр. Коши.

при $x=-\frac{4}{3}$ ряд сходится условно.

$x = -\frac{2}{3} \quad \sum_{n=1}^{\infty} \frac{3^n+(-2)^n}{3^n\cdot n}$

$\frac{3^n+(-2)^n}{3^n\cdot n} > \frac{1}{4n}$ расходится.

$n=1 \to \frac{1}{3}, \ldots \frac{1}{9}, \frac{19}{27}$

$\sum \frac{1}{4n}$ расходится $\Rightarrow$ при $x=-\frac{2}{3}$ $\sum_{n=1}^{\infty} \frac{3^n+(-2)^n}{3^n\cdot 4}$ расходится.

$\left(-\frac{4}{3}; -\frac{2}{3}\right)$ ряд сх. абс

$x = -\frac{4}{3}$ сх. условно.

2814. $\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!} x^n$

$$R = \lim_{n\to\infty} \left| \frac{(n!)^2 \cdot (2(n+1))!}{(2n)! \cdot ((n+1)!)^2} \right| = \lim_{n\to\infty} \left| \frac{n!\, n! \cdot (2n+2)!}{(2n)!\; n!\, (n+1)^2\, n!} \right| =$$

$$= \lim_{n\to\infty} \left| \frac{(2n)!\,(2n+1)(2n+2)}{(2n)!\,(n+1)^2} \right| = \lim_{n\to\infty} \frac{(2n+1)(2n+2)}{(n+1)^2} = 4.$$

$|x| < 4 \longrightarrow -4 < x < 4$ – сходится абсолютно

при $x = -4$.

$$\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!} (-1\cdot 4)^n = \sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!} 4^n \cdot (-1)^n$$

при $x = 4$ $\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!} 4^n$

$$\frac{a_n}{a_{n+1}} = \frac{(n!)^2\, 4^n\, (2n+2)!}{(2n)!\,((n+1)!)^2\, 4^{n+1}} = \frac{(2n+1)(2n+2)}{(n+1)^2 \cdot 4} =$$

$$= \frac{4n^2 + 4n + 2n + 2}{4n^2 + 8n + 4} = \frac{4n^2 + 6n + 2}{4n^2 + 8n + 4} = \frac{(2n+1)\,\cancel{2}(n+1)}{(n+1)^2 \cdot \cancel{4}\,2} = \frac{2n+1}{2n+2} =$$

$$= \frac{2n+2-1}{2n+2} = 1 - \frac{1}{2n+2} < 1.$$ посл-ть возрастающая

$\lim_{n\to\infty} a_n \neq 0$ → ряд расходится

при $x = -4$. будет расходится.

=> ряд сходится только при $x \in (-4; 4)$

2826. $\sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \left(\frac{n}{e}\right)^n x^n$

$$R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| = \lim_{n\to\infty} \left| \frac{(-1)^n}{n!} \left(\frac{n}{e}\right)^n \frac{e^{n+1}\,(n+1)!}{(-1)^{n+1}\,(n+1)^{n+1}} \right| =$$

$$= \lim_{n\to\infty} \cancel{\left(-\frac{e(n+1)}{n}\right)} = \cancel{-e}$$

$$= \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n e = \lim_{n\to\infty} \left(1 - \frac{1}{n+1}\right)^n e = 1.$$

$\downarrow \frac{1}{e}$

$-1 < x < 1.$

$$\frac{a_n}{a_{n+1}} = \frac{e^1}{\left(1+\frac{1}{n}\right)^n} > 1.$$

$\forall n \in \mathbb{N}$

$$\lim_{n\to\infty} \left(\frac{n}{e}\right)^n \cdot \frac{1}{n!} = \lim_{n\to\infty} \left(\frac{n}{e}\right)^n \cdot \left(\frac{e}{n}\right)^n \frac{1}{\sqrt{2\pi n}} = 0 \; -$$

Формула Стирлинга: $n! \approx \left(\frac{n}{e}\right)^n \sqrt{2\pi n}$

$\Rightarrow$ ряд сходится по пр. Лейбница.

$$\sum_{n=1}^{\infty} \frac{1}{n!}\left(\frac{n}{e}\right)^n \quad \to \text{ряд расходится.}$$

$\dots \quad n\to\infty \quad \frac{1}{n!}\left(\frac{n}{e}\right)^n \sim \frac{1}{\sqrt{2\pi n}}$

$(-1; 1)$

$x = -1$ сх. усл.

Д/з: ~~2815~~, ~~2822~~, ~~2823~~, 2824, 2825, 2828.

Функциональные ряды.

2717. $\sum\limits_{n=1}^{\infty} \frac{(-1)^n}{2n-1}\left(\frac{1-x}{1+x}\right)^n$

Ряд из модулей: $\sum\limits_{n=1}^{\infty} \frac{1}{2n-1}\left(\left|\frac{1-x}{1+x}\right|\right)^n$

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\frac{|1-x|^{n+1}}{|1+x|^{n+1}}\cdot\frac{|1+x|^n}{2(n+1)-1}\cdot\frac{2n-1}{|1-x|^n} = \lim_{n\to\infty}\frac{(2n-1)\cdot|1-x|}{(2n+1)\cdot|1+x|} = \left|\frac{1-x}{1+x}\right|$$

Чтобы он сходился:

$$\left|\frac{1-x}{1+x}\right| < 1 \qquad |f(x)| < |g(x)| \ \uparrow^2$$

$$f^2(x) < g^2(x)$$

$$\left|\frac{-(x-1)}{x+1}\right| < 1$$

$$\left(\frac{x-1}{x+1}\right)^2 < 1$$

$$\left(\frac{x-1}{x+1}\right)^2 - 1 < 0 \rightarrow \left(\frac{x-1}{x+1} - 1\right)\left(\frac{x-1}{x+1}+1\right) < 0$$

$$\left(\frac{-2}{x+1}\right)\left(\frac{2x}{x+1}\right) < 0$$

$$\frac{-4x}{(x+1)^2} < 0 \rightarrow \underline{\underline{x > 0}}$$

При $x>0$ сходится абсолютно.

$x=0$: $\sum\limits_{n=1}^{\infty}\frac{(-1)^n}{2n-1}$

$$b_n = \frac{1}{2n-1}$$

$$\lim_{n\to\infty}\frac{1}{2n-1} = 0 \quad ; \quad b_{n+1} < b_n - ?$$

$$\frac{1}{2(n+1)-1} < \frac{1}{2n-1}$$

$$\frac{b_n}{b_{n+1}} = \frac{2n+1}{2n-1} = 1 + \frac{2}{2n-1} \Rightarrow b_n > b_{n+1}.$$

При $x>0$, ряд сходится абсолютно

При $x=0$, сходится условно.

2719. $\sum\limits_{n=1}^{\infty}\frac{1\cdot3\ldots(2n-1)}{2\cdot4\ldots(2n)}\cdot\left(\frac{2x}{1+x^2}\right)^n = \sum\limits_{n=1}^{\infty}\frac{(2n-1)!!}{(2n)!!}\cdot\left(\frac{2x}{1+x^2}\right)^n$

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\frac{(2n+1)!!}{(2n+2)!!}\cdot\frac{|2x|^{n+1}}{|1+x^2|^{n+1}}\cdot\frac{(2n)!!}{(2n-1)!!}\cdot\frac{|1+x^2|^n}{|2x|^n} = \lim_{n\to\infty}\frac{(2n+1)}{2n+2}\left|\frac{2x}{1+x^2}\right|$$

$x=1$:

$x=-1$

$$\left|\frac{2x}{1+x^2}\right| < 1$$

$$\frac{|2x|}{1+x^2} < 1 \qquad |x| = t \rightarrow x^2 = t^2$$

$$\frac{2t}{1+t^2} - 1 < 0$$

$$\frac{2t-1-t^2}{1+t^2} < 0$$

$$\frac{-(t^2-2t+1)}{1+t^2} < 0 \Rightarrow \frac{(t-1)^2}{1+t^2} > 0$$

$$t \neq 1.$$

$$|x| \neq 1 \rightarrow x \neq \pm 1$$

$x=1$: $\sum\limits_{n=1}^{\infty} \frac{(2n-1)!!}{(2n)!!}$

$$\sum_{n=1}^{\infty} \frac{(2n-1)!!\,(2n)!!}{(2n)!!\,(2n)!!} = \frac{(2n)!}{2^{2n}\cdot(n!)^2}$$

$$(n!) \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$$

$$(2n)! \approx \sqrt{4\pi n}\left(\frac{2n}{e}\right)^{2n}$$

$$\sum_{n=1}^{\infty} \frac{(2n)!}{2^{2n}(n!)^2} \approx \sum_{n=1}^{\infty} \frac{\sqrt{4\pi n}\left(\frac{2n}{e}\right)^{2n}}{2^{2n}\cdot 2\pi n\left(\frac{n}{e}\right)^{2n}} \approx \sum_{n=1}^{\infty} \frac{2^{2n}\left(\frac{n}{e}\right)^{2n}}{2^{2n}\cdot\left(\frac{n}{e}\right)^{2n}}\cdot\sqrt{\frac{4\pi n}{4\pi^2 n^2}}$$

$$\sum_{n=1}^{\infty} \sqrt{\frac{1}{\pi n}}$$

$$\frac{1}{\sqrt{\pi}}\cdot\frac{1}{n^{\frac{1}{2}}} \leq \frac{1}{n^{\frac{1}{2}}}$$ – расходится

2 способ:

$$\sum_{n=1}^{\infty} \frac{1\cdot 3\ldots(2n-1)}{2\cdot 4\ldots 2n}\left(\frac{2x}{1+x^2}\right)^n$$

$x=1$: $\lim\limits_{n\to\infty} n\cdot\left(\frac{a_n}{a_{n-1}} - 1\right) = ?$ $\begin{cases} >1 & \text{сходится} \\ <1 & \text{расходится} \\ =1, ? \end{cases}$

$$\lim_{n\to\infty} n\cdot\left(\frac{1\cdot 3\ldots(2n-1)\cdot\ \cdot 2\cdot 4\ldots(2n+2)}{2\cdot 4\ldots 2n\cdot\ \cdot 1\cdot 3\ldots(2n+1)} - 1\right) =$$

$$= \cancel{\lim_{n\to\infty} n\left(\frac{(2n-1)}{2n}(\ldots) - 1\right)} = \lim_{n\to\infty}$$

$$\lim_{n\to\infty} n\left(\frac{\cancel{1\cdot 3\ldots(2n-1)}\cdot(2n+2)}{\cancel{2\cdot 4\ldots(2n)}\,(2n+1)} - 1\right) = \lim_{n\to\infty} n\left(\frac{2n+2}{2n+1} - 1\right) =$$

$= \lim\limits_{n\to\infty} n\left(\frac{1}{2n+1}\right) = \frac{1}{2} < 1$ — расходится.

$\frac{b_n}{b_{n+1}} = \frac{(2n-1)}{2n} \cdot \frac{(2n+2)\,2n}{(2n-1)(2n+1)} = \frac{2n+2}{2n+1} = 1 + \frac{1}{2n+1} > 1$ — убыв.

$\lim\limits_{n\to\infty} b_n = \lim\limits_{n\to\infty} \frac{1}{\sqrt{\pi n}} = \lim\limits_{n\to\infty} \frac{(2n)!}{2^{2n}(n!)^2} = 0$ — по признаку Лейбница условно сходится при $x=-1$

2431. $\sum\limits_{n=1}^{\infty} \frac{(n+x)^n}{n^{n+x}} = \sum\limits_{n=1}^{\infty} \frac{(n+x)^n}{n^n \cdot n^x}$ $\quad \frac{1}{n^x}$

$\lim\limits_{n\to\infty} \frac{(n+x)^n \cdot n^x}{n^{n+x}} = \lim\limits_{n\to\infty} \frac{(n+x)^n}{n^n} = \lim\limits_{n\to\infty}\left(1+\frac{x}{n}\right)^n = e^x, \quad x \in R$

При $x \le 1$ ряд расходится, а при $x > 1$ ряд сходится.

2427. $\sum\limits_{n=1}^{\infty} \frac{x^n}{(1+x)(1+x^2)\ldots(1+x^n)}$

$\lim\limits_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim\limits_{n\to\infty} \left| \frac{x^{n+1}\,(1+x)\ldots(1+x^n)}{(1+x)(1+x^2)\ldots(1+x^n)(1+x^{n+1})\cdot x^n} \right| = \lim\limits_{n\to\infty} \left| \frac{x}{1+x^{n+1}} \right| =$

$= |x| \lim\limits_{n\to\infty} \frac{1}{|1+x^{n+1}|} = |x|$, $\quad |x| < 1$ ~ ряд сходится.

$|x| > 1$ ~ ряд расходится ?

Если $|x| = 1$

27.05.21. Равномерная сходимость по Вейерштрассу.

2744 (б.) $\sum_{n=1}^{\infty} \frac{(-1)^n}{x+2^n} \quad (x>-2)$

$$\left|\frac{(-1)^n}{x+2^n}\right| = \frac{1}{|x+2^n|} = \frac{1}{x+2^n} < \frac{1}{2^n-2} \leq \frac{1}{2^{n-1}}$$, при $n \geq 2$

$2^n \geq 4$

$2^n - 2 \overset{?}{\geq} 2^{n-1}$

$2^n - 2 \geq 4 - 2$

$\sum_{n=2}^{\infty} \frac{1}{2^{n-1}}$ сходится.

$\Rightarrow \sum_{n=1}^{\infty} \frac{(-1)^n}{x+2^n}$ сходится равномерно.

$2^n - 2 \geq 2^{n-1}$

$2^n - 2 \geq \frac{1}{2}\cdot 2^n$

$2\cdot 2^n - 4 \geq 2^n$

$2^n \geq 4$

г) $\sum_{n=1}^{\infty} \frac{nx}{1+n^5x^2} \quad |x|<+\infty$

$\underbrace{1+n^5x^2}_{(n^{\frac{5}{2}}x)^2} \geq 2\cdot n^{\frac{5}{2}}x \qquad a^2+b^2 \geq 2ab$

$$\left|\frac{nx}{1+n^5x^2}\right| \leq \left|\frac{nx}{2\,x\cdot n^{\frac{5}{2}}}\right| = \frac{1}{2n^{\frac{3}{2}}}$$

$\sum_{n=1}^{\infty} \frac{1}{2n^{\frac{3}{2}}}$ сходится

$\Rightarrow \sum_{n=1}^{\infty} \frac{nx}{1+n^5x^2}$ сходится равномерно.

д) $\sum_{n=1}^{\infty} \frac{n^2}{\sqrt{n!}}(x^n+x^{-n}) \qquad \frac{1}{2} \leq |x| \leq 2$

$0<b<c$ числа одного знака $\frac{1}{c} < \frac{1}{b} < \frac{1}{a}$

$\left|\frac{n^2(x^n+x^{-n})}{\sqrt{n!}}\right| \qquad \frac{1}{2^n} \leq |x^n| \leq 2^n$

$\frac{1}{2} \leq \frac{1}{|x|} \leq 2$

$\frac{1}{2^n} \leq \frac{1}{|x|^n} \leq 2^n$

$\frac{2}{2^n} \leq |x|^n + \frac{1}{|x|^n} \leq 2\cdot 2^n$

$$\left|\frac{n^2(x^n+\frac{1}{x^n})}{\sqrt{n!}}\right| \leq \frac{n^2}{\sqrt{n!}}\left(|x|^n + \frac{1}{|x|^n}\right) \leq \frac{n^2}{\sqrt{n!}}2^{n+1}$$

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\frac{(n+1)^2\cdot 2^{(n+1)+1}}{\sqrt{(n+1)!}}\cdot\frac{\sqrt{n!}}{n^2\,2^{n+1}} = \lim_{n\to\infty}\frac{(n^2+2n+1)\cdot 2\,\cancel{\sqrt{n!}}}{\cancel{\sqrt{n!}}\cdot\sqrt{n+1}\cdot n^2} = \lim_{n\to\infty}\frac{\cancel{n^2}(1+\frac{2}{n}+\frac{1}{n^2})\cdot 2}{\cancel{n^2}\sqrt{n+1}} =$$

$= 0$

$$\sum_{n=1}^{\infty} \frac{n^2}{\sqrt{n!}} \cdot 2^{n+1}$$ сходится.

$$\Rightarrow \sum_{n=1}^{\infty} \frac{n^2}{\sqrt{n!}} \left(x^n + x^{-n}\right)$$ сходится равномерно.

ж) $$\sum_{n=1}^{\infty} \frac{\sin nx}{\sqrt[3]{n^4+x^4}} \qquad |x| < +\infty$$

$$\left|\frac{\sin nx}{\sqrt[3]{n^4+x^4}}\right| \le \frac{1}{\sqrt[3]{n^4+x^4}} \le \frac{1}{\sqrt[3]{n^4}}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^{\frac{4}{3}}}$$ сходится.

$$\Rightarrow \sum_{n=1}^{\infty} \frac{\sin nx}{\sqrt[3]{n^4+x^4}}$$ сходится равномерно.

к) $$\sum_{n=2}^{\infty} \ln\left(1+\frac{x^2}{n\ln^2 n}\right) \qquad |x| < a$$

$$\left|\ln\left(1+\frac{x^2}{n\ln^2 n}\right)\right| \le \frac{x^2}{n\ln^2 n} \le \frac{a^2}{n\ln^2 n}$$

$$\sum_{n=1}^{\infty} \frac{a^2}{n\ln^2 n}$$

Интегральный признак Коши:

$$\int_2^{\infty} \frac{dx}{x\ln^2 x} = \int_2^{\infty} \frac{d(\ln x)}{\ln^2 x} = \left(-\frac{1}{\ln x}\Big|_2^{\infty}\right) =$$

$$= \lim_{A\to\infty} \int_2^{A} \frac{d(\ln x)}{\ln^2 x} = \lim_{A\to\infty}\left(-\frac{1}{\ln x}\Big|_2^{A}\right) = \frac{1}{\ln 2}$$ — равен числу.

$$\boxed{\text{Д/з}: 2774\ (а, б, з, и, л, м)}$$

$$\left(\frac{1}{x\ln^2 x}\right)' = \frac{-(\ln^2 x + 2\ln x)}{x^2\ln^4 x} = \frac{-\ln x(\ln x + 2)}{x^2\ln^4 x} = \frac{-(\ln x+2)}{x^2\ln^3 x}$$

убывает при больших x

Разложение функций в степенные ряды.

I. $e^x = 1 + x + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!} + \ldots \qquad |x| < +\infty$

II. $\sin x = x - \frac{x^3}{3!} + \ldots + (-1)^{n-1}\frac{x^{2n-1}}{(2n-1)!} + \ldots \qquad |x| < +\infty$

III. $\cos x = 1 - \frac{x^2}{2!} + \ldots + (-1)^n \frac{x^{2n}}{(2n)!} + \ldots \qquad |x| < +\infty$

IV. $(1+x)^m = 1 + mx + \frac{m(m-1)}{2!}x^2 + \ldots + \frac{m(m-1)\ldots(m-n+1)}{n!}x^2 + \ldots \qquad -1 < x < 1$

V. $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots + (-1)^{n-1}\frac{x^n}{n} + \ldots \qquad -1 < x \leq 1$

2851. $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad -\infty < x + \infty$

$$e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!}$$

2853. $\sin^3 x$ -?

$\sin 3x = 3\sin x - 4\sin^3 x \quad \Rightarrow \sin^3 x = \frac{3}{4}\sin x - \frac{1}{4}\sin 3x$

$$\sin^3 x = \frac{3}{4}\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!} - \frac{1}{4}\sum_{n=0}^{\infty}(-1)^n \frac{(3x)^{2n+1}}{(2n+1)!} =$$

$$= \frac{3}{4}\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!} - \frac{1}{4}\sum_{n=0}^{\infty}(-1)^n \frac{3^{2n+1}\cdot x^{2n+1}}{(2n+1)!} =$$

$$= \frac{3}{4}\sum_{n=0}^{\infty}\frac{(-1)^n(1-3^{2n})x^{2n+1}}{(2n+1)!} \qquad |x| < +\infty \quad (x \in \mathbb{R})$$

2855. $\frac{1}{(1-x)^2} = (1-x)^{-2}$

$$(1+x)^m = 1 + \sum_{n=1}^{\infty}\frac{m(m-1)(m-2)\ldots(m-(n-1))}{n!}x^m$$

$$\frac{1}{(1-x)^2} = 1 + \sum_{n=1}^{\infty}\frac{-2(-2-1)(-2-2)\ldots(-2-n+1)}{n!}(-x)^n =$$

$$= 1 + \sum_{n=1}^{\infty}\frac{(-1)^n(-1)^n\cdot 2\cdot 3\cdot 4\ldots(n+1)}{n!}x^n =$$

$$= 1 + \sum_{n=1}^{\infty}\frac{(n+1)!\,x^n}{n!} = 1 + \sum_{n=1}^{\infty}(n+1)x^n = \sum_{n=0}^{\infty}(n+1)x^2$$

2 способ: $\left(\frac{1}{1-x}\right)' = \frac{1}{(1-x)^2}$

$$(1-x)^{-1} = (1+(-x))^{-1} = 1 + (-x)(-1) + \frac{(-1)(-2)}{1\cdot 2}(-x)^2 + \frac{(-1)(-2)(-3)}{1\cdot 2\cdot 3}(-x)^3 + \ldots$$

$$+ \frac{(-1)(-2)(-3)(-4)}{1\cdot 2\cdot 3\cdot 4}(-x)^4 + \ldots =$$

$$= 1 + x + x^2 + x^3 + x^4 + \ldots$$

Внутри области сходимости, степенной ряд можно почленно дифференцировать и интегрировать, при этом интервал сходимости полученного таким образом ряда совпадает с интервалом сходимости исходного ряда.

$$\left(\frac{1}{1-x}\right)' = 1+2x+3x^2+4x^2+\ldots = \sum_{n=0}^{\infty}(n+1)x^n, \quad |x|<1.$$

2857.

$$\ln\sqrt{\frac{1+x}{1-x}} = \frac{1}{2}\ln\frac{1+x}{1-x} =$$

$$\frac{1+x}{1-x}>0$$

$$-1<x<1$$

$$= \frac{1}{2}\left(\ln|1+x| - \ln|1-x|\right) =$$

$$= \frac{1}{2}\left(\ln(1+x) - \ln(1-x)\right) =$$

$$= \frac{1}{2}\left(\left(x - \cancel{\frac{x^2}{2}} + \frac{x^3}{3} - \cancel{\frac{x^4}{4}} + \frac{x^5}{5} + \ldots\right) - \left(-x - \cancel{\frac{x^2}{2}} - \frac{x^3}{3} - \cancel{\frac{x^4}{4}} - \frac{x^5}{5} - \ldots\right)\right) =$$

$$= \frac{1}{2}\left(2x + 2\cdot\frac{x^3}{3} + 2\cdot\frac{x^5}{5} + \ldots\right) = x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \ldots =$$

$$= \sum_{n=0}^{\infty}\frac{x^{2n+1}}{2n+1}$$

область сходимости:

$\ln(1-x)$: $-1<-x\le 1 \Rightarrow -1\le x<1$

$\ln(1+x)$: $-1<x\le 1$ $\quad -1<x\le 1.$

$$-1<x<1$$

$$\ln\sqrt{\frac{1+x}{1-x}} = \sum_{n=0}^{\infty}\frac{x^{2n+1}}{2n+1}, \quad |x|<1$$

2860.

$$\frac{x}{(1-x)(1-x^2)}$$

$$\frac{x}{(1-x)^2(1+x)} = -\frac{1}{4(1+x)} - \frac{1}{4(1-x)} + \frac{1}{2(1-x)^2} =$$

$$= -\frac{1}{4}\sum_{n=0}^{\infty}(-1)^n x^n - \frac{1}{4}\sum_{n=0}^{\infty}x^n + \frac{1}{2}\sum_{n=0}^{\infty}(n+1)x^n =$$

$$= \frac{1}{4}\sum_{n=0}^{\infty}\left((-1)^{n+1} - 1 + 2n + 2\right)x^n = \frac{1}{4}\sum_{n=0}^{\infty}\left(2n+1+(-1)^{n+1}\right)x^n =$$

$$= \frac{1}{2}\sum_{n=1}^{\infty}\left(n + \frac{1-(-1)^n}{2}\right)x^n$$

Если $n=0 \rightarrow$

$$\frac{1}{R} = \lim_{k\to\infty}\sqrt[2k+1]{2k+1+\frac{2}{2}} = \lim_{k\to\infty}\sqrt[2k+1]{2k+2} = \lim_{k\to\infty}(2k+2)^{\frac{1}{2k+1}} =$$

$$= \lim_{k\to\infty}(2k+1+1)^{\frac{1}{2k+1}} = \lim_{k\to\infty}\underbrace{(2k+1)^{\frac{1}{2k+1}}}_{\to 1}\underbrace{\left(1+\frac{1}{2k+1}\right)^{\frac{1}{2k+1}}}_{\to 1} = 1.$$

$R = 1$

При $|x|<1$ $\frac{x}{(1-x)(1-x^2)} = \frac{1}{2}\sum_{n=1}^{\infty}\left(n + \frac{1-(-1)^n}{2}\right)x^n$

2869

$f(x) = \operatorname{arctg} x$ $\qquad \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{2n-1}$

$$\operatorname{arctg} x = \int_0^x \frac{dt}{1+t^2} = \{1+t^2 = 1-(-t^2)\} =$$

$$= \int_0^x (1-t^2+t^4-t^6+\ldots+(-1)^n t^{2n}+\ldots)\,dt =$$

$$= t - \frac{t^3}{3} + \frac{t^5}{5} - \frac{t^7}{7} + \ldots + \frac{(-1)^n t^{2n+1}}{2n+1} + \ldots \Big|_0^x =$$

$$= \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2n+1} \quad (*) \qquad |x|<1.$$

При $x = \pm 1$ $(*)$ сх-ся.

$y = \operatorname{arctg} x$ – непрерывная ф-я из второй теоремы Абеля:

Пусть R – радиус сходимости степенного ряда и ряд сходится в точке $x = R$ ($x=-R$), тогда сумма ряда непрерывна слева (справа) т.е.

$$\lim_{x\to R-0}\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n R^n$$

$$\left(\lim_{x\to -R+0}\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n (-R)^n\right)$$

Из этой теоремы следует, что разложение функции $\ln(1+x)$ в степенной ряд с центром в нуле верно и при $x=1$.

$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n}$ сходится при $x=1$

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{2n-1} = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2n+1} = \qquad \text{при } x=1.$$

$$= \operatorname{arctg} 1 = \frac{\pi}{4}$$

$$\pi = 4\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{2n-1}$$

Применение рядов к приближенным вычислениям –

$\ln 1{,}1$ с точностью до 10^{-4}

$\ln(1+\underset{0{,}1}{x}) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots (-1)^{n-1}\frac{x^n}{n} + \ldots$ сх-ся $(-1; 1]$

$$\ln 1{,}1 = 0{,}1 - \frac{0{,}1^2}{2} + \frac{0{,}1^3}{3} - \frac{0{,}1^4}{4} + \ldots$$