Метод математической индукции.

(1) $S_n = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \dots + \frac{1}{n(n+1)} = \; + \frac{1}{4\cdot 5} + \frac{1}{5\cdot 6} + \frac{1}{7\cdot 8} + \dots$

$S_1 = \frac{1}{2}$

$S_2 = \frac{1}{2} + \frac{1}{6} = \frac{2}{3}$

$S_3 = \frac{2}{3} + \frac{1}{12} = \frac{3}{4}$

$\dots$

$S_n = \frac{n}{n+1}$

$S_n = \frac{2-1}{1\cdot 2} + \frac{3-2}{2\cdot 3} + \frac{4-3}{3\cdot 4} + \dots + \frac{n+1-n}{n(n+1)} = 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} -$

$- \frac{1}{4} + \dots + \frac{1}{n} - \frac{1}{n+1} = 1 - \frac{1}{n+1} = \frac{n+1-1}{n+1} = \frac{n}{n+1}$

Метод математ. индукции закл. в след: утв-е справедливо для всякого натурального n, если:

1) оно справедливо для $n=1$

2) из справедливости утв-я для какого-либо произвольного натурального $n=k$ следует его справедливость для $n=k+1$

$S_n = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \dots + \frac{1}{n(n+1)} = \frac{n}{n+1}$

1) $n=1$.

$S_1 = \frac{1}{2}$

$\frac{n}{n+1} = \frac{1}{2}$ $\Rightarrow \frac{1}{2} = \frac{1}{2}$ – верно

2) $n=k$

$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \dots + \frac{1}{k(k+1)} = \frac{k}{k+1}$

$\boxed{\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \dots + \frac{1}{k(k+1)}} + \frac{1}{(k+1)(k+2)} = \frac{k}{k+1} + \frac{1}{(k+1)(k+2)} =$

$n = k+1$.

$\frac{n}{n+1} = \frac{k+1}{k+2}$

$= \frac{1}{k+1}\left(k + \frac{1}{k+2}\right) = \frac{1}{k+1}\left(\frac{k^2+2k+1}{k+2}\right) =$

$= \frac{(k+1)^2}{(k+1)(k+2)} = \frac{k+1}{k+2}$

(2) $S_n = 1\cdot 2 + 2\cdot 3 + 3\cdot 4 + \dots + (n-1)n = \frac{(n-1)n(n+1)}{3}$ $(n \geq 2)$

1) $S_2 = (1)\cdot 2 = 2$

$S_2 = \frac{1\cdot 2\cdot 3}{3} = 2$ $\quad \frac{(n-1)n(n+1)}{3} = 2$ – верно

2) $n = k \ (k \geq 2)$

$$1\cdot 2 + 2\cdot 3 + 3\cdot 4 + \ldots + (k-1)k = \frac{(k-1)k(k+1)}{3} \qquad (k \geq 2)$$

3) $n = k+1$

$$\underbrace{\frac{(n-1)n(n+1)}{3} = \frac{(k+1-1)(k+1)(k+2)}{3} = \frac{(k+1)k(k+2)}{3}}\ (1)$$

$$1\cdot 2 + 2\cdot 3 + \ldots + (k-1)k + (k+1-1)(k+1) = \frac{(k-1)k(k+1)}{3} + k(k+1) =$$

$$= \cancel{(k-1)k} \quad (k+1)\cancel{(k(k-1)+} \quad k(k+1)\left(\frac{k-1}{3} + 1\right) = k(k+1)\left(\frac{k-1+3}{3}\right) =$$

$$= \frac{k(k+1)(k+2)}{3} \quad (2)$$

Выражение $(1) = (2)$ – верно.

Сумма всех нечётных чисел:

3) $S_n = 1 + 3 + 5 + \ldots + (2n-1) = n^2$

$S_1 = 2 - 1 = 1$. $\qquad S_3 = 4 + 5 = 9 \qquad 1 + 3 + 5 + 7 + 9 \ldots$

Отлич. на 2.

$S_2 = 1 + 3 = 4 \qquad S_4 = 9 + 7 = 16$

$$S_n = \frac{1 + (2n-1)}{2} \cdot n = \frac{1 + 2n - 1}{2} \cdot n = n^2$$

1) $n = 1$.

$S_1 = 1$.

$n^2 = 1^2 = 1$ $\Rightarrow$ $1 = 1$ – верно.

2) $n = k$.

$$1 + 3 + 5 + \ldots + (2k-1) = k^2$$

3) $n = k+1$.

$$n^2 = (k+1)^2 \quad (1)$$

$$1 + 3 + 5 + \ldots + (2k-1) + (2(k+1)-1) = k^2 + (2(k+1)-1) =$$

$$= k^2 + (2k + 2 - 1) = k^2 + 2k + 1 = (k+1)^2 \quad (2)$$

Выражение $(1) = (2)$ – верно

④ $n(2n^2+7) \vdots 3$

$n + (2n^2+7) \vdots 3$

1) $n=1$

$n(2n^2+7) = 1\cdot(2+7) = 9$ (кратно 3)

2) ~~$n=k$. $k(2k^2+7)$~~

~~$n=k+1$~~
~~$(k+1)(2(k+1)^2+7) = (k+1)(2(k^2+2k+1)+7) =$~~
~~$= (k+1)(2k^2+4k+2+7) = (k+1)(2k^2+4k+9)$~~

~~$k=1$. $2\cdot(2\cdot4+4\cdot2+9) = 2\cdot(8+8+9) =$~~

2) $n=k \Rightarrow k(2k^2+7) \vdots 3$

3) $n=k+1$.

$(k+1)(2(k+1)^2+7) = (k+1)(2(k^2+2k+1)+7) =$
$= (k+1)(2k^2+4k+2+7) = (k+1)(2k^2+4k+9) =$
$= 2k^3+4k^2+9k+2k^2+4k+9 =$
$= 2k^3+6k^2+13k+9 = (2k^3+7k)+6k^2+6k+9 \vdots 3 =$
$=$

⑤ $4^n+15n-1 \vdots 9$ (3^2)

1) $n=1$

$4+15-1=18 \vdots 9$

2) $n=k$.

$\Rightarrow 4^k+15k-1 \vdots 9$

3) $n=k+1$.

$4^{(k+1)}+15(k+1)-1 = 4\cdot4^k+15k+15-1 =$
$= 4\cdot4^k+15k+14 = 4\cdot4^k+16k-k+16-2 =$
$= (4^k+15k-1)+3\cdot4^k+15$ (1)

$3\cdot4^k+15 \vdots 9$

1) $k=1$.

$3\cdot4+15=27 \vdots 9$

2) $k=m \Rightarrow 3\cdot4^m+15 \vdots 9$ (2)

3) $k=m+1$.

$3\cdot4^{m+1}+15 = 3\cdot4\cdot4^m+15 = 12\cdot4^m+15 = 3\cdot4^m+15+9\cdot4^m \vdots 9$ (3)

⑥ $36^n + 10 \cdot 3^n \vdots 11$

1) $n = 1$

$36 + 30 = 66 \vdots 11$

2) $n = k \Rightarrow \underline{36^k + 10 \cdot 3^k} \vdots 11$

3) $n = k+1$

$36^{k+1} + 10 \cdot 3^{k+1} = 36 \cdot 36^k + 10 \cdot 3 \cdot 3^k = \underbrace{36^k + 10 \cdot 3^k} + 35 \cdot 36^k + 20 \cdot 3^k =$

$35 \cdot 36^k + 20 \cdot 3^k \vdots 11$ $\quad = 3(36^k + 10 \cdot 3^k + 11 \cdot 36^k) =$?

1) $k = 1 \Rightarrow 35 \cdot 36 + 60 = 1320 \vdots 11$ $\quad = 3(36^k + 10 \cdot 3^k) + \underline{33 \cdot 36^k} \vdots 11$ преобр

2) $k = m$

$35 \cdot 36^m + 20 \cdot 3^m$

3) $k = m+1$

$35 \cdot 36^{m+1} + 20 \cdot 3^{m+1} = 35 \cdot 36 \cdot 36^m + 60 \cdot 3^m = 35 \cdot 1 \cdot 36^m + 20 \cdot 3^m +$
$+ 40 \cdot 3^m +$

⑦ $n \geq 2$

$S_n = \frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{2n} > \frac{13}{24}$ $\quad$ Расс-ть разность $S_{k+1} - S_k > 0$ по транзитивности.

$\cdot 4^m \vdots 9$ (3)

Домашняя работа.

1) $S_n = 1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$

1) при $n = 1$.

$S_1 = 1$

$\frac{n(n+1)}{2} = \frac{1 \cdot 2}{2} = 1$ $\Rightarrow 1 = 1$ – верно

2) $n = k$

$1 + 2 + 3 + \ldots + k = \frac{k(k+1)}{2}$

3) $n = k + 1$

$\frac{n(n+1)}{2} = \frac{(k+1)(k+2)}{2}$ (1)

$1 + 2 + 3 + \ldots + k + (k+1) = \frac{k(k+1)}{2} + (k+1) = \frac{k^2 + k + 2k + 2}{2} =$

$= \frac{k(k+1) + 2(k+1)}{2} = \frac{(k+1)(k+2)}{2}$ (2)

Выражение (2) совп. с (1), значит всё верно (что и т.д)

2) $S_n = 1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$

1) при $n = 1$.

$S_1 = 1$.

$\frac{n(n+1)(2n+1)}{6} = \frac{1 \cdot 2 \cdot 3}{6} = 1$ $\Rightarrow 1 = 1$ – верно

2) $n = k$

$1^2 + 2^2 + 3^2 + \ldots + k^2 = \frac{k(k+1)(2k+1)}{6}$

3) $n = k + 1$.

$\frac{n(n+1)(2n+1)}{6} = \frac{(k+1)(k+2)(2k+2+1)}{6} = \frac{(k+1)(k+2)(2k+3)}{6}$ (1)

$1^2 + 2^2 + 3^2 + \ldots + k^2 + (k+1)^2 = \frac{k(k+1)(2k+1)}{6} + (k+1)^2 =$

$= \frac{k(k+1)(2k+1) + 6(k+1)^2}{6} = \frac{(k+1)(k(2k+1) + 6(k+1))}{6} =$

$= \frac{(k+1)(2k^2 + k + 6k + 6)}{6} = \frac{(k+1)(2k^2 + 7k + 6)}{6} = \frac{(k+1)(k+2)(2k+3)}{6}$ (2)

$2k^2 + 7k + 6 = 0$

$D = 1$.

$k_1 = \frac{-7+1}{4} = -\frac{3}{2}$; $k_2 = -2$

$2k^2 + 7k + 6 = 2(k + \frac{3}{2})(k+2) = (2k+3)(k+2)$

Выражение (2) совп. с (1), значит всё верно (что и т.д.)

③ $S_n = 1^3 + 2^3 + 3^3 + \ldots + n^3 = \left[\frac{n(n+1)}{2}\right]^2$

1) при $n = 1$

$S_1 = 1$

$\left[\frac{n(n+1)}{2}\right]^2 = \left(\frac{1 \cdot 2}{2}\right)^2 = 1$ $\Rightarrow$ $1 = 1$ - верно

2) $n = k$

$1^3 + 2^3 + 3^3 + \ldots + k^3 = \left[\frac{k(k+1)}{2}\right]^2$

3) $n = k+1$

(1) $\left[\frac{(k+1)(k+2)}{2}\right]^2 = \left[\frac{n(n+1)}{2}\right]^2$

$1^3 + 2^3 + 3^3 + \ldots + k^3 + (k+1)^3 = \left[\frac{k(k+1)}{2}\right]^2 + (k+1)^3 =$

$= \frac{(k^2+k)^2}{4} + (k+1)^3 = \frac{k^2(k+1)^2}{4} + (k+1)^3 = \frac{k^2(k+1)^2 + 4(k+1)^3}{4} =$

$= \frac{(k+1)^2(k^2+4k+4)}{4} = \frac{(k+1)^2(k+2)^2}{4} = \left[\frac{(k+1)(k+2)}{2}\right]^2$ (2)

Выражение (2) совп. с (1), значит всё верно (что и т.д.)

④ $S_n = 1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + 3 \cdot 4 \cdot 5 + \ldots + n(n+1)(n+2) = \frac{n(n+1)(n+2)(n+3)}{4}$

1) при $n = 1$

$S_1 = 1 \cdot 2 \cdot 3 = 6$

$\frac{n(n+1)(n+2)(n+3)}{4} = \frac{1 \cdot 2 \cdot 3 \cdot 4}{4} = 6$ $\Rightarrow$ $6 = 6$ - верно

2) при $n = k$.

$1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + \ldots + k(k+1)(k+2) = \frac{k(k+1)(k+2)(k+3)}{4}$

3) $n = k+1$

$\frac{n(n+1)(n+2)(n+3)}{4} = \frac{(k+1)(k+2)(k+3)(k+4)}{4}$ (1)

$1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + \ldots + k(k+1)(k+2) + (k+1)(k+2)(k+3) = \frac{k(k+1)(k+2)(k+3)}{4} +$

$+ (k+1)(k+2)(k+3) = \frac{k(k+1)(k+2)(k+3) + 4(k+1)(k+2)(k+3)}{4} =$

$= \frac{(k+1)(k+2)(k+3)(k+4)}{4}$ (2)

Выражение (2) совп. с (1), значит всё верно (что и т.д.)

⑤ $(n^4+2n^3+2n^2+n) \vdots 6$

1) при $n=1$.

$n^4+2n^3+2n^2+n = 1+2+2+1 = 6 \quad (\vdots 6)$

2) $n=k$.

$(k^4+2k^3+2k^2+k) \vdots 6$

3) $n=k+1$

$(k+1)^4+2(k+1)^3+2(k+1)^2+(k+1) = (k+1)((k+1)^3+2(k+1)^2+2(k+1)+$
$+1) = (k+1)((k+1)^2(k+1)+2(k^2+2k+1)+2k+1+1) =$
$= (k+1)(k^3+3k^2+3k+1+2k^2+4k+2+2k+2+1) =$
$= (k+1)(k^3+5k^2+9k+6) = k^4+5k^3+9k^2+6k+k^3+5k^2+9k+6 =$
$= k^4+6k^3+14k^2+15k+6 = \underbrace{k^4+2k^3+2k^2+k}_{\vdots 6} + \underbrace{12k^2+6}_{\vdots 6} + \underbrace{4k^3+14k}$

Рас-м $(4k^3+14k)$ и докажем, что оно тоже $\vdots 6$

$(4k^3+14k) \vdots 6$

1) $k=1$.

$4+14 = 18 \vdots 6$

2) $k=m$

$\Rightarrow 4m^3+14m \vdots 6$

3) $k=m+1$.

$4(m+1)^3+14(m+1) = (m+1)((m^2+2m+1)\cdot 4+14) = (m+1)(4m^2+8m+18) =$
$= 4m^3+8m^2+18m+4m^2+8m+18 = \underbrace{4m^3+14m}_{\vdots 6} + \underbrace{12m^2+12m+18}_{\vdots 6}$

$\Rightarrow (n^4+2n^3+2n^2+n) \vdots 6$, что и требовалось док-ть.

⑥ $S_n = \frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{2n} > \frac{13}{24} \quad (n \geq 2)$

II

I сп. $S_n = \frac{1}{2} + \frac{1}{4} + \ldots + \frac{1}{2n} > \frac{13}{24} \quad (n \geq 2)$

1) $n=2 \Rightarrow \frac{1}{2} + \frac{1}{4} = \frac{3^{6}}{4} = \frac{18}{24} > \frac{13}{24}$ – верно

2) $n=k$; $\frac{1}{2} + \frac{1}{4} + \ldots + \frac{1}{2k} > \frac{13}{24}$

3) $n=k+1$

$$\frac{1}{2}+\frac{1}{4}+\ldots+\frac{1}{2k}+\frac{1}{2k+1} \overset{?}{>} \frac{13}{24}$$

Из пункта 2: $\frac{1}{2}+\frac{1}{4}+\ldots+\frac{1}{2k} > \frac{13}{24}$, тогда:

$$\frac{1}{2}+\frac{1}{4}+\ldots+\frac{1}{2k}+\frac{1}{2k+1} > \frac{13}{24}+\frac{1}{2k+1} > \frac{13}{24}$$

$$\Updownarrow$$

$$\frac{1}{2k+1} > 0$$ при $k \geq 2$ - выполняется всегда.

=> неравенство доказано.

II способ.

Т.к. справа стоит константа а слева ряд чисел (при $k \geq 2$), то попробуем усилить неравенство:

$$S_n = \frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\ldots+\frac{1}{2n} > \frac{13}{24}+\frac{1}{2n} \quad (n \geq 2)$$

1) $n = 2 \Rightarrow \frac{1}{2}+\frac{1}{4} = \frac{18}{24}$; $\frac{18}{24} < \frac{19}{24}$ - не подходит.

$\frac{13}{24}+\frac{1}{4} = \frac{19}{24}$

$$S_n = \frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\ldots+\frac{1}{2n} > \frac{13}{24}+\frac{1}{4n} \quad (n \geq 2)$$

1) $n=2 \Rightarrow \frac{1}{2}+\frac{1}{4} = \frac{18}{24}$; $\frac{18}{24} > \frac{16}{24}$ - верно

$\frac{13}{24}+\frac{1}{8}^{\,3} = \frac{16}{24}$

$\frac{13}{24}+\frac{1}{4k}+\frac{1}{2(k+1)} > \frac{13}{24}+\frac{1}{4(k+1)}$

$\frac{1}{4k}+\frac{1}{2k+2}-\frac{1}{4k+4} > 0$

2) $n = k \Rightarrow \frac{1}{2}+\frac{1}{4}+\ldots+\frac{1}{2k} > \frac{13}{24}+\frac{1}{4k}$

3) $n = k+1$.

$$\frac{1}{2}+\frac{1}{4}+\ldots+\frac{1}{2k}+\frac{1}{2(k+1)} > \underbrace{\frac{13}{24}+\frac{1}{4k}+\frac{1}{2(k+1)} > \frac{13}{24}+\frac{1}{4(k+1)}}$$

$$\Updownarrow$$

$$\frac{13}{24}+\frac{1}{4k}+\cancel{\frac{1}{2(k+1)}} \cancel{>} \cancel{\frac{13}{24}}+\cancel{\frac{1}{2(k+1)}}$$

$$\Updownarrow$$

$$\frac{1}{4k} > 0$$ - выполняется при $k \geq 2$ всегда.

=> По принципу транзитивности доказывается и исходное неравенство.

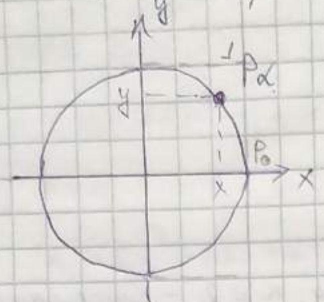
$$\frac{1}{4k}+\frac{1}{2(k+1)}-\frac{1}{4(k+1)} = \frac{k+1+2k-k}{4k(k+1)} > 0$$

$$\frac{2k+1}{4k(k+1)} > 0$$

$-\quad -1 \quad + \quad -\frac{1}{2} \quad - \quad 0 \quad +$

при $k \geq 2$ - выполняется всегда.

Формулы тригонометрии и обратные тригонометрические функции.



$P_\alpha \to$ „x" $\cos\alpha$

$P_\alpha \to$ „y" $\sin\alpha$

$\cup P_0P_1 = 1$

$\angle P_0OP_1 = 1$ рад.

$x^2 + y^2 = 1$

$\cos^2\alpha + \sin^2\alpha = 1$

$\sin(-\alpha) = -\sin\alpha$
$\cos(-\alpha) = \cos\alpha$

$tg\alpha = \frac{\sin\alpha}{\cos\alpha} \Rightarrow tg(-\alpha) = -tg\alpha$
$ctg(-\alpha) = -ctg\alpha$

α	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\sin\alpha$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\cos\alpha$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
$tg\alpha$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	–
$ctg\alpha$	–	$\sqrt{3}$	1	$\frac{\sqrt{3}}{3}$	0

Формулы приведения.

$\sin(\frac{\pi}{2} - \alpha) = \cos\alpha$

$\sin(\frac{\pi}{2} + \alpha) = \cos\alpha$

$\sin(\pi - \alpha) = \sin\alpha$

$\sin(\pi + \alpha) = -\sin\alpha$

$\sin(\frac{3}{2}\pi - \alpha) = -\cos\alpha$

$\sin(\frac{3}{2}\pi + \alpha) = -\cos\alpha$

$\cos(\frac{\pi}{2} - \alpha) = \sin\alpha$

$\cos(\frac{\pi}{2} + \alpha) = -\sin\alpha$

$\cos(\pi - \alpha) = -\cos\alpha$

$\cos(\pi + \alpha) = -\cos\alpha$

$\cos(\frac{3}{2}\pi - \alpha) = -\sin\alpha$

$\cos(\frac{3}{2}\pi + \alpha) = \sin\alpha$

I. Соотношение между тригонометрическими функциями одного и того же аргумента.

1) $\sin^2\alpha + \cos^2\alpha = 1$

2) $tg\alpha = \frac{\sin\alpha}{\cos\alpha}$

3) $ctg\alpha = \frac{\cos\alpha}{\sin\alpha}$

4) $tg\alpha \cdot ctg\alpha = 1$

5) $1 + tg^2\alpha = \frac{1}{\cos^2\alpha}$

6) $1+ctg^2\alpha = \dfrac{1}{\sin^2\alpha}$

7) $\sec\alpha = \dfrac{1}{\cos\alpha}$

8) $\operatorname{cosec}\alpha = \dfrac{1}{\sin\alpha}$

II. Формулы сложения:

1) $\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$

2) $\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \sin\beta\cos\alpha$

3) $tg(\alpha \pm \beta) = \dfrac{tg\alpha \pm tg\beta}{1 \mp tg\alpha\, tg\beta}$

4) $ctg(\alpha \pm \beta) = \dfrac{ctg\alpha \cdot ctg\beta \mp 1}{ctg\alpha \pm ctg\beta}$

III. Формулы двойных аргументов:

1) $\sin 2\alpha = 2\sin\alpha\cos\alpha$

2) $\cos 2\alpha = \cos^2\alpha - \sin^2\alpha = 1 - 2\sin^2\alpha = 2\cos^2\alpha - 1$

3) $tg 2\alpha = \dfrac{2tg\alpha}{1-tg^2\alpha}$ 4) $ctg 2\alpha = \dfrac{ctg^2\alpha - 1}{2ctg\alpha}$

IV. Формула понижения степени:

1) $\sin^2\alpha = \dfrac{1-\cos 2\alpha}{2}$

2) $\cos^2\alpha = \dfrac{1+\cos 2\alpha}{2}$

3) $tg^2\alpha = \dfrac{1-\cos 2\alpha}{1+\cos 2\alpha}$

V. Формулы перехода от произведения тригонометр. ф-й к сумме.

1) $\sin\alpha\cos\beta = \dfrac{1}{2}\left(\sin(\alpha+\beta) + \sin(\alpha-\beta)\right)$

2) $\cos\alpha\cos\beta = \dfrac{1}{2}\left(\cos(\alpha-\beta) + \cos(\alpha+\beta)\right)$

3) $\sin\alpha\sin\beta = \dfrac{1}{2}\left(\cos(\alpha-\beta) - \cos(\alpha+\beta)\right)$

VI. Формулы перехода от суммы к произведению.

1) $\sin\alpha + \sin\beta = 2\sin\dfrac{\alpha+\beta}{2}\cos\dfrac{\alpha-\beta}{2}$

2) $\sin\alpha - \sin\beta = 2\sin\frac{\alpha-\beta}{2}\cos\frac{\alpha+\beta}{2}$

3) $\cos\alpha + \cos\beta = 2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$

4) $\cos\alpha - \cos\beta = -2\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}$

VII. Формулы тройных аргументов:

1) $\sin 3\alpha = 3\sin\alpha - 4\sin^3\alpha$ $\qquad tg\,3x = \frac{3tg\,x - tg^3 x}{1 - 3tg^2 x}$

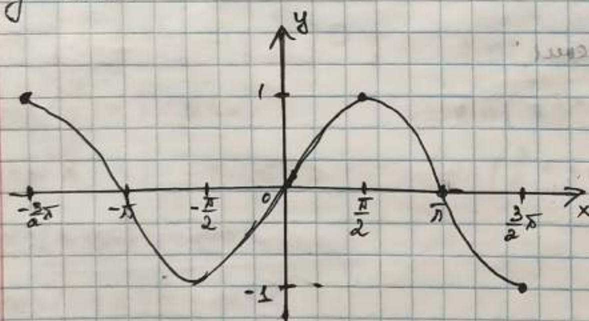
? 2) $\cos 3\alpha = 4\cos^3\alpha - 3\cos\alpha$ $\qquad ctg\,3x = \frac{ctg^3 x - 3tg\,x}{3tg^2 x - 1}$

VIII. Выражение тригонометр. функций через тангенс половинного аргумента:

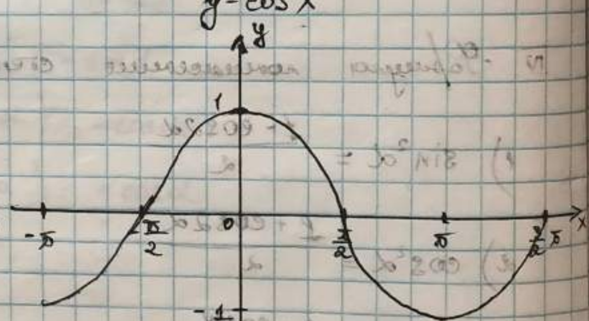
1) $\sin\alpha = \frac{2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}}{\cos^2\frac{\alpha}{2} + \sin^2\frac{\alpha}{2}} = \frac{2tg\frac{\alpha}{2}}{1 + tg^2\frac{\alpha}{2}}$, $\cos\frac{\alpha}{2} \neq 0$

2) $\cos\alpha = \frac{1 - tg^2\frac{\alpha}{2}}{1 + tg^2\frac{\alpha}{2}}$

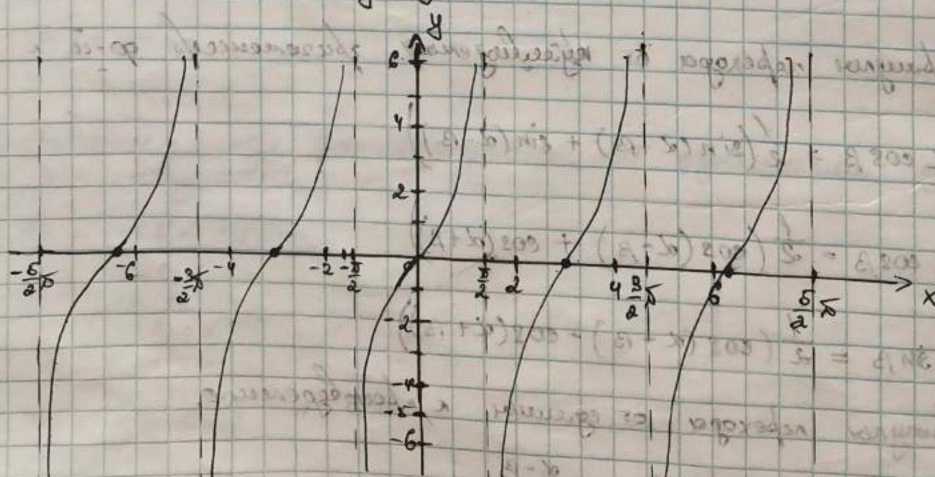
$y = \sin x$



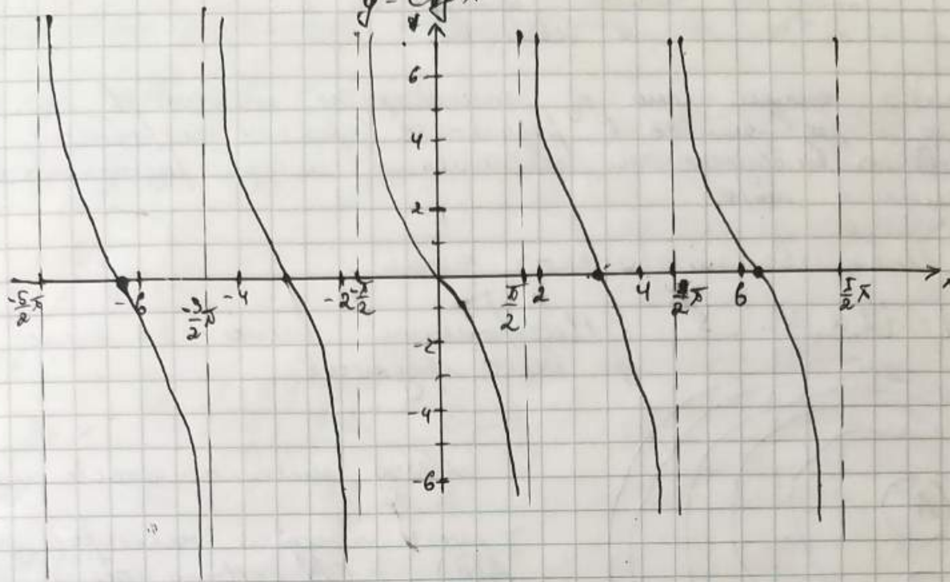
$y = \cos x$



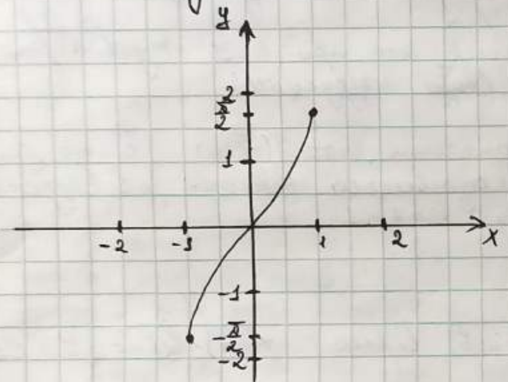
$y = tg\,x$



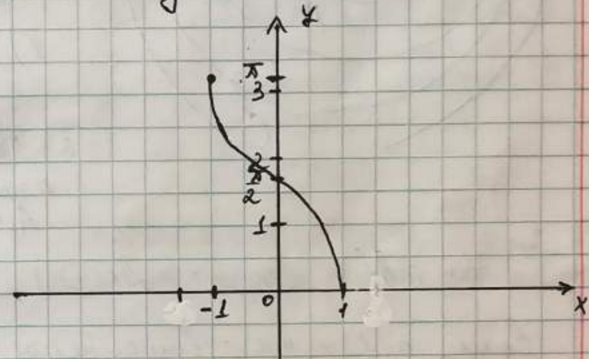
$y = ctg\,x$



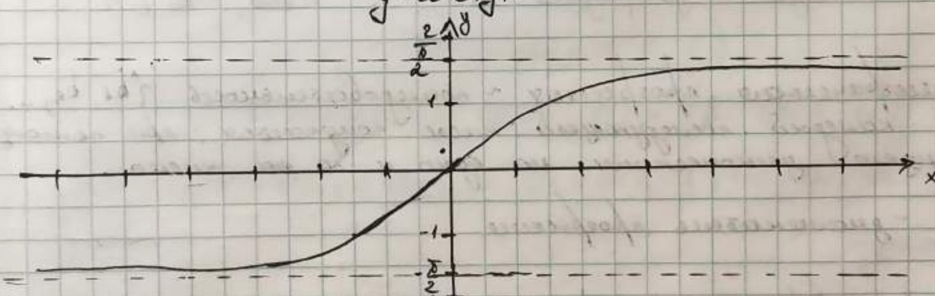
$y = arcsin\,x$



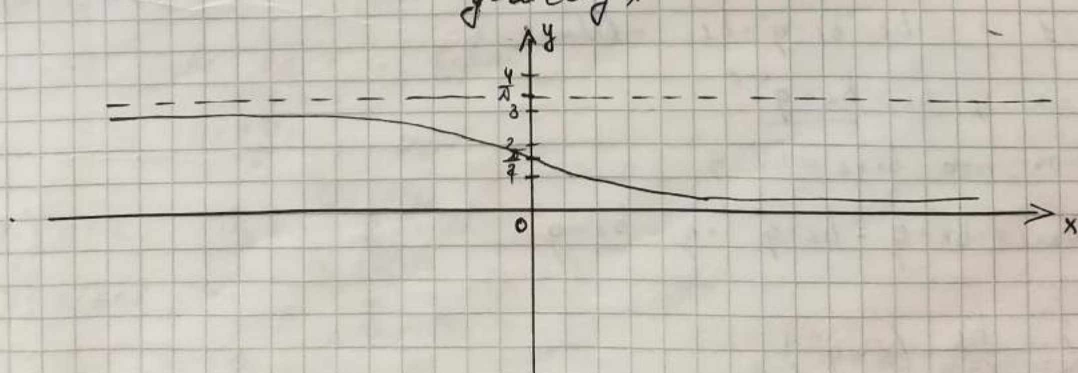
$y = arccos\,x$



$y = arctg\,x$



$y = arcctg\,x$



5) $\left(\left(1+\cos 44^\circ \cos 1^\circ - \sin 44^\circ \sin 1^\circ\right)^2 - 1{,}5\right)^2 = \left(\left(1+\cos 45^\circ\right) - \frac{3}{2}\right)^2 =$

$= \left(1 + 2\cos 45^\circ + \cos^2 45^\circ - \frac{3}{2}\right)^2 = \left(1 + \sqrt{2} + \frac{2}{4} - \frac{6}{4}\right)^2 =$

$= \left(\sqrt{2}\right)^2 = 2$

6) $tg(45^\circ + \alpha) = \frac{tg45 + tg\alpha}{1 - tg45\, tg\alpha} = \frac{1+6}{1-6} = -\frac{7}{5}$

7) $\left(\frac{\cos 11^\circ \cos 19^\circ - \sin 11^\circ \sin 19^\circ}{\cos 11^\circ \sin 34^\circ + \sin 11^\circ \cos 34^\circ}\right)^4 = \left(\frac{\cos 30^\circ}{\sin 45^\circ}\right)^4 = \left(\frac{\sqrt{3}\,\cancel{2}}{2 \cdot \sqrt{2}}\right)^4 =$

$= \frac{3 \cdot 3}{2 \cdot 2} = \frac{9}{4}$

8) $21 ctg^2 x - 4 tg^2 x =$ $\qquad (\cos x = 0{,}4.)$

$1 + tg^2 x = \frac{1}{\cos^2 x} \Rightarrow tg^2 x = \frac{1}{\cos^2 x} - 1 = \frac{1}{0{,}16} - 1 =$

$= \frac{100 - 16}{16} = \frac{84}{16} = \frac{42}{8} = \frac{21}{4}$

$\sin^2 x = 1 - \cos^2 x = 1 - 0{,}16 = 0{,}84$

$1 + ctg^2 x = \frac{1}{\sin^2 x} \Rightarrow ctg^2 x = \frac{1}{0{,}84} - 1 = \frac{100 - 84}{84} = \frac{16}{84} =$

$= \frac{4}{21}$

$\Rightarrow 21 ctg^2 x - 4 tg^2 x = \frac{\cancel{21} \cdot 4}{\cancel{21}} - \frac{\cancel{4} \cdot 21}{\cancel{4}} = 4 - 21 = -17$

9) $\frac{(\sin 40^\circ + \sin 50^\circ)^2}{\sin^2 80^\circ} = \frac{(2 \sin 60^\circ \cos 10^\circ)^2}{\sin^2 80^\circ} = \frac{4 \sin^2 60^\circ \cdot \cos^2(90-80)}{\sin^2 80} =$

$= \frac{4 \cdot \left(\frac{\sqrt{3}}{2}\right)^2 \cdot (\cos 90 \cdot \cos 80 + \sin 90 \sin 80)^2}{\sin^2 80} = \frac{4 \cdot 3}{4} \cdot \frac{\sin^2 80}{\sin^2 80} = 3$

10) c

a

a

a

si

a

a

si

10) $\arccos(\sin 45^\circ + \cos 135^\circ) = \arccos\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\right) = \arccos(0) = 1$

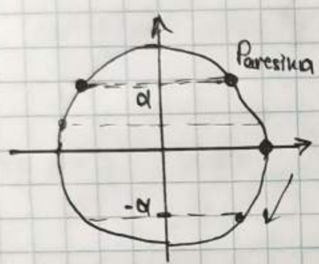
$\arcsin\left(\cos \frac{7}{3}\pi\right) = \arcsin\left(\cos\left(2\pi + \frac{\pi}{3}\right)\right) = \arcsin\left(\cos \frac{\pi}{3}\right) =$

$= \arcsin\left(\frac{1}{2}\right) = \frac{\pi}{6}$

$$\frac{\arccos(\sin 45^\circ + \cos 135^\circ)}{\arcsin(\cos \frac{7}{3}\pi)} = \frac{1 \cdot 6}{\pi} = \frac{6}{\pi} \approx \frac{6}{3{,}14}$$

$\Rightarrow \approx$ на 91 %

Обратные тригонометрические функции.



$\sin x = a, \quad |a| \le 1$

$\boxed{\begin{array}{l} \arcsin a = x \\ 1)\ x \in [-\frac{\pi}{2}; \frac{\pi}{2}] \\ 2)\ |a| \le 1 \\ 3)\ \sin x = a \end{array}}$

$x^2 = 4$

$x = \pm 2$

$x^2 = 5$

$\sqrt{a} = b$

$\boxed{\begin{array}{l} a \ge 0 \\ b \ge 0 \\ a = b^2 \end{array}}$

$\arcsin(-a) = -\arcsin a$

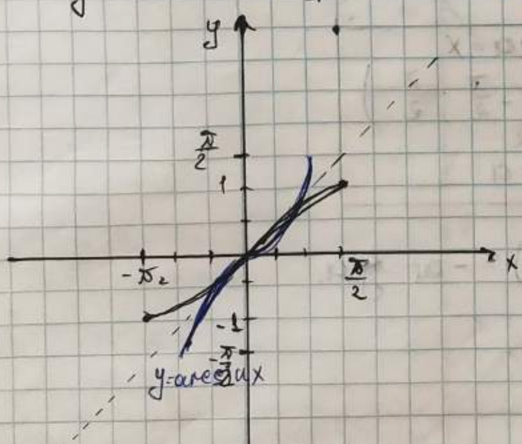
$\sin(\arcsin a) = a, \quad |a| \le 1$

$\arcsin(\sin x) = x, \quad x \in [-\frac{\pi}{2}; \frac{\pi}{2}]$

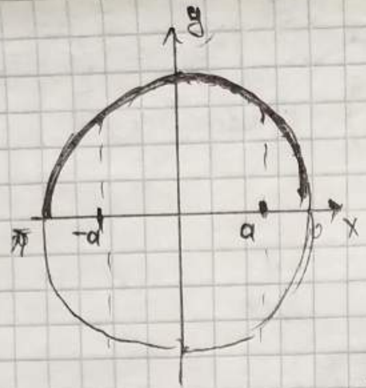
$\arcsin(\sin 3) = \pi - 3$, т.к. $\sin\alpha = \sin(\pi - \alpha)$

$\sin(\arcsin(\frac{\pi}{3})) \ne \frac{\pi}{3}$ – не существует

$y = \arcsin x$ – возрастающая



$\arcsin\frac{1}{3} > \arcsin\frac{1}{4}$



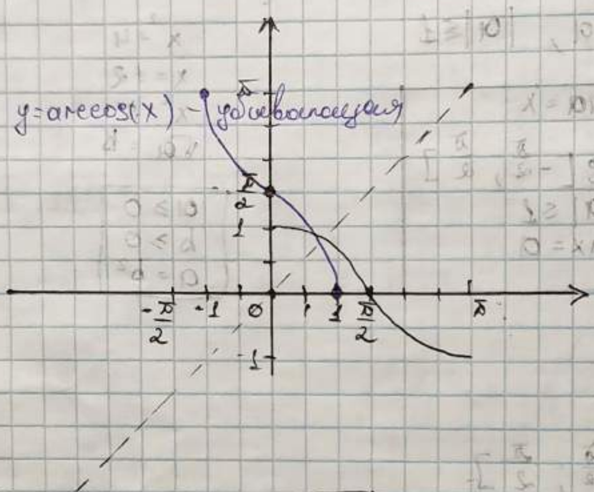
$\cos x = a$, $|a| \le 1$

$$\boxed{\begin{aligned} & y = \arccos(a) = x \\ & 1)\ x \in [0; \pi] \\ & 2)\ |a| \le 1 \\ & 3)\ \cos x = a \end{aligned}}$$

$\arccos(-a) = \pi - \arccos(a)$

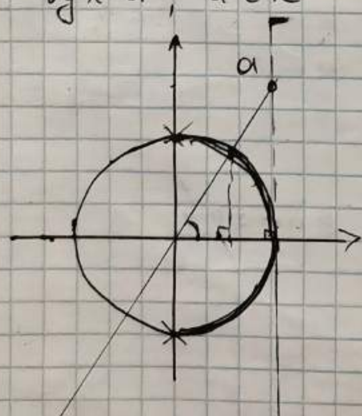
$\arccos(\cos x) = x \quad (x \in [0; \pi])$

$\cos(\arccos a) = a$, $|a| \le 1$.



$$\boxed{\arcsin a + \arccos a = \frac{\pi}{2}}$$

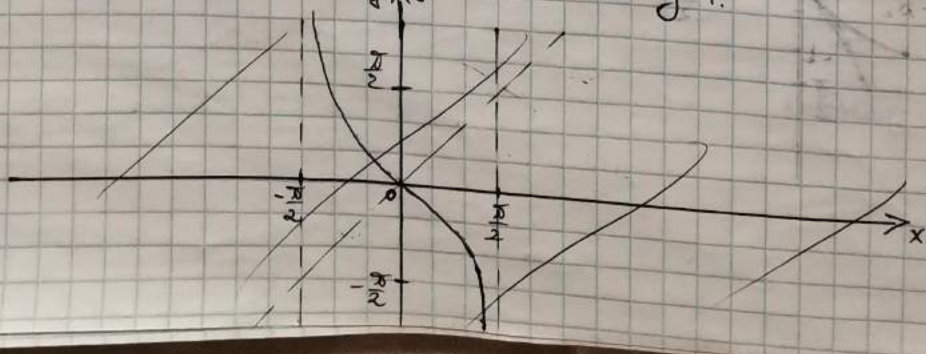
$\operatorname{tg} x = a$, $a \in R$

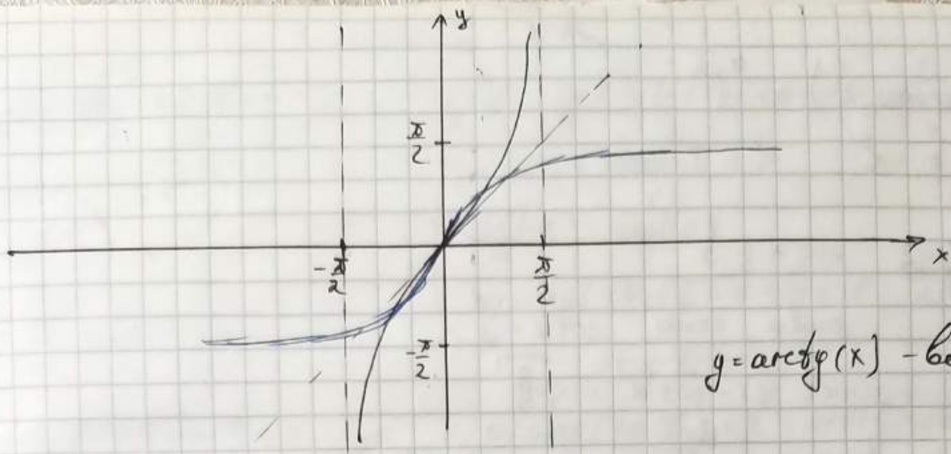


$\operatorname{tg} x = \frac{\sin x}{\cos x} = a$

$$\boxed{\begin{aligned} & \operatorname{arctg} a = x \\ & 1)\ x \in \left(-\frac{\pi}{2}; \frac{\pi}{2}\right) \\ & 2)\ a \in R \\ & 3)\ \operatorname{tg} x = a \end{aligned}}$$

$\operatorname{arctg}(-a) = -\operatorname{arctg} a.$

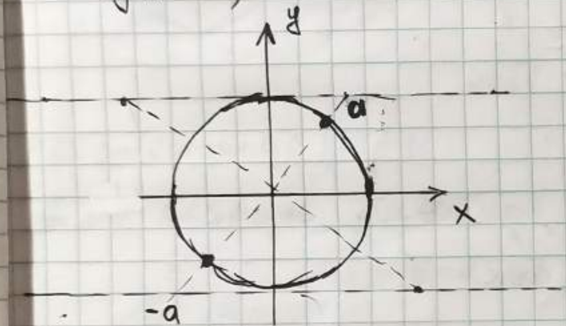




$y = arctg(x)$ - возвр.

$arctg\,a + arcctg\,a = \frac{\pi}{2}$

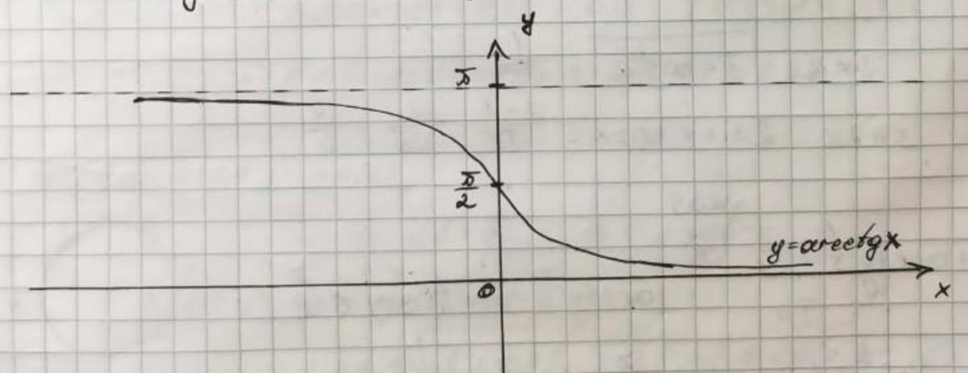
$ctg\,x = a, \quad a \in R$



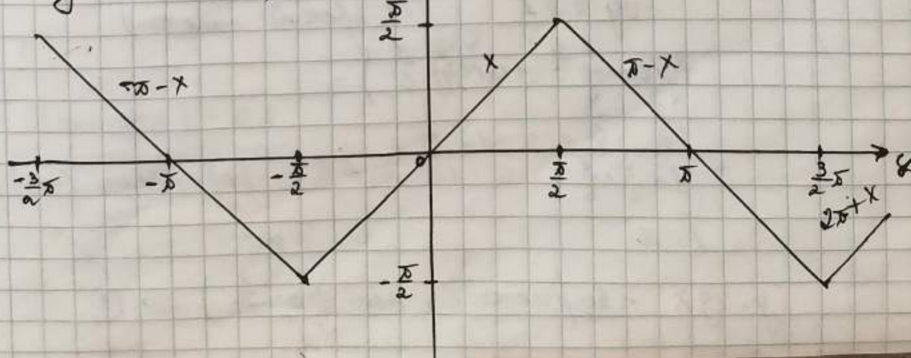
$ctg\,x = \frac{\cos x}{\sin x} = a$

$$\boxed{\begin{array}{l} arcctg(a) = x \\ 1)\ x \in (0; \pi) \\ 2)\ a \in R \\ 3)\ ctg\,x = a \end{array}}$$

$arcctg(-a) = \pi - arcctg\,a$



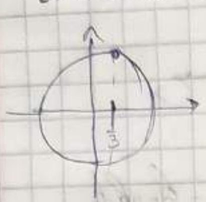
$y = arcsin(sinx)$



№226

1) $\sin\left(2\arccos\frac{1}{3}\right)=\frac{4\sqrt{2}}{9}$

$\arccos\frac{1}{3}=\alpha$, $\alpha\in$ I четв.



$\cos\alpha=\frac{1}{3}$

$\sin 2\alpha$, если $\cos\alpha=\frac{1}{3}$.

$\sin\alpha=\sqrt{1-\cos^2\alpha}=\sqrt{\frac{8}{9}}$

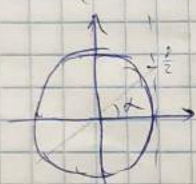
$\sin\alpha=\pm\frac{2\sqrt{2}}{3}$, т.к. $\alpha\in$ I ч., то $\sin\alpha>0$

$\sin\alpha=\frac{2\sqrt{2}}{3}$

$\sin 2\alpha=2\sin\alpha\cos\alpha=\frac{2\cdot2\sqrt{2}}{3}\cdot\frac{1}{3}=\frac{4\sqrt{2}}{9}$

№227.

1) $\sin\left(2\operatorname{arctg}\frac{1}{2}\right)=\frac{4}{5}$



$\operatorname{arctg}\frac{1}{2}=\alpha$, $\alpha\in$ I ч.

$\operatorname{tg}\alpha=\frac{1}{2}$.

$\sin 2\alpha$, если $\operatorname{tg}\alpha=\frac{1}{2}$; т.к. $\alpha\in$ I ч., то $\sin\alpha>0$ $\cos\alpha>0$

$1+\operatorname{tg}^2\alpha=\frac{1}{\cos^2\alpha}$ $1+\frac{1}{4}=\frac{1}{\cos^2\alpha}$

$\frac{5}{4}=\frac{1}{\cos^2\alpha}$ $\Rightarrow$ $\cos^2\alpha=\frac{4}{5}$ $\Rightarrow$ $\cos\alpha=\frac{2}{\sqrt{5}}$

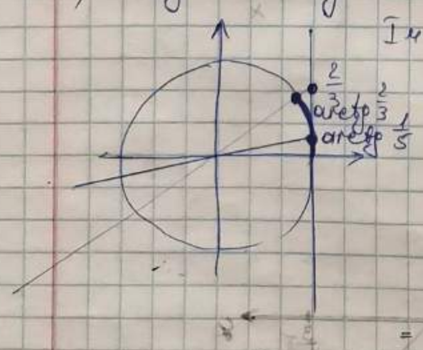
$\sin\alpha=\sqrt{1-\cos^2\alpha}=\frac{1}{\sqrt{5}}$

$\sin 2\alpha=2\sin\alpha\cos\alpha=\frac{2\cdot1}{\sqrt{5}}\cdot\frac{2}{\sqrt{5}}=\frac{4}{5}$

№235

1) $\operatorname{arctg}\frac{2}{3}+\operatorname{arctg}\frac{1}{5}=\frac{\pi}{4}$ (*)

I ч.



$\left(\operatorname{arctg}\frac{2}{3}+\operatorname{arctg}\frac{1}{5}\right)\in[0;\frac{\pi}{2}]$

$\frac{\pi}{4}\in(0;\frac{\pi}{2})$

$\operatorname{tg}\frac{\pi}{4}=1$ (от правой части.)

$\operatorname{tg}\left(\operatorname{arctg}\frac{2}{3}+\operatorname{arctg}\frac{1}{5}\right)=$

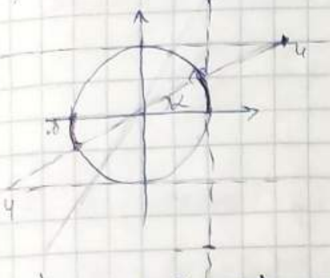
$=\frac{\frac{2}{3}+\frac{1}{5}}{1-\frac{2}{3}\cdot\frac{1}{5}}=\frac{13\cdot15}{15\cdot13}=1$. (от левой части.)

$1=1$.

$y=\operatorname{tg}x$ - возрастает на $(0;\frac{\pi}{2})\Rightarrow$ (*)

№225.

1) $\operatorname{arcctg}(\operatorname{ctg} 4) = 4 - \pi$



2) $\operatorname{arcctg}(\operatorname{tg}(-6)) = \operatorname{arcctg}\left(\frac{1}{\operatorname{ctg}(-6)}\right)$

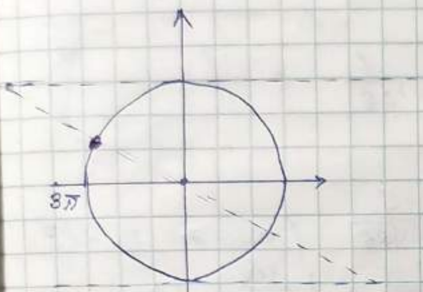
$\operatorname{tg}\alpha \cdot \operatorname{ctg}\alpha = 1$

$\operatorname{tg}(-6) = \frac{1}{\operatorname{ctg}(-6)}$

$\operatorname{arcctg}(\operatorname{ctg}(6 - \frac{3}{2}\pi)) = 6 - \frac{3}{2}\pi$

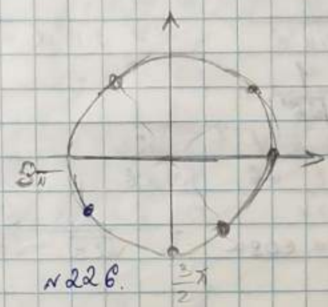
по другому.

3) $\operatorname{arcctg}(\operatorname{ctg} 9) = 9 - 2\pi$



4) $\operatorname{arcctg}(\operatorname{tg} 10) = \frac{\pi}{2} - \operatorname{arctg}(\operatorname{tg} 10) = \frac{\pi}{2} - (10 - 3\pi) =$

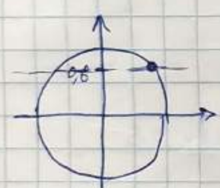
$= 3{,}5\pi - 10$



№226.

2) $\sin(2\arcsin 0{,}6) = 0{,}96$

$\arcsin 0{,}6 = \alpha$, $\alpha \in$ I четв.



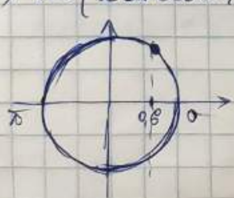
$\Rightarrow \sin\alpha = 0{,}6$

$\sin 2\alpha = 2\sin\alpha\cos\alpha =$

$\cos\alpha = \sqrt{1 - 0{,}36} = 0{,}8$ ($\cos\alpha > 0$, т.к. $\alpha \in$ I четв.)

$\Rightarrow \sin(2\arcsin 0{,}6) = 2 \cdot 0{,}6 \cdot 0{,}8 = 0{,}96$

3) $\cos(2\arccos 0{,}6) = -0{,}28$



$\arccos 0{,}6 = \alpha$, $\alpha \in$ I четв.

$\Rightarrow \cos\alpha = 0{,}6$

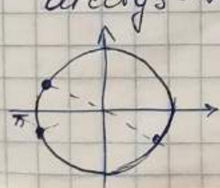
$\sin\alpha = \sqrt{1 - 0{,}36} = 0{,}8$ ($\sin\alpha > 0$, т.к. $\alpha \in$ I четв.)

$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha = 0{,}36 - 0{,}64 = -0{,}28$

№227.

2) $\cos(2\operatorname{arcctg} 3) = 0{,}8.$

$\operatorname{arcctg} 3 = \alpha$, $\alpha \in$ II четв.



$\operatorname{ctg}\alpha = 3$

$1 + \operatorname{ctg}^2\alpha = \frac{1}{\sin^2\alpha}$

$1+9=\frac{1}{\sin^2\alpha} \Rightarrow \sin^2\alpha=\frac{1}{10}$

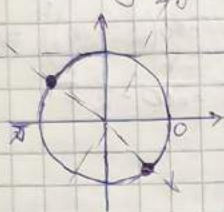
$\cos^2\alpha = 1-\frac{1}{10}=\frac{9}{10}$

$\cos2\alpha=\cos^2\alpha-\sin^2\alpha=\frac{9}{10}-\frac{1}{10}=\frac{8}{10}=\frac{4}{5}=0{,}8$

3) $\sin(2\,\mathrm{arcctg}\,5) = \frac{5}{13}$?

$\mathrm{arcctg}\,5=\alpha$; $\alpha \in \mathrm{I}$ четв.

$+5$



$\mathrm{ctg}\,\alpha=5$

$1+\mathrm{ctg}^2\alpha=\frac{1}{\sin^2\alpha}$

$1+25=\frac{1}{\sin^2\alpha} \Rightarrow \sin^2\alpha=\frac{1}{26}$

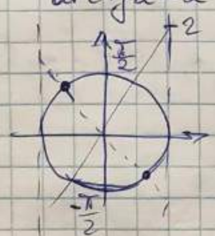
$\sin\alpha=\frac{1}{\sqrt{26}}$

$\cos\alpha=\sqrt{1-\frac{1}{26}}=\sqrt{\frac{25}{26}}=-\frac{5}{\sqrt{26}}$ ($\cos\alpha>0$, т.к. $\alpha\in\mathrm{I}$ ч.

$\sin2\alpha=2\sin\alpha\cos\alpha=+\frac{2\cdot1}{\sqrt{26}}\cdot\frac{5}{\sqrt{26}}=+\frac{10}{26}=+\frac{5}{13}$

4) $\cos(2\,\mathrm{arctg}\,2)=-\frac{1}{3}$

$\mathrm{arctg}\,2=\alpha$; $\alpha\in\mathrm{I}$ ч.



$\mathrm{tg}\,\alpha=2$

$1+\mathrm{tg}^2\alpha=\frac{1}{\cos^2\alpha}$

$1+4=\frac{1}{\cos^2\alpha} \Rightarrow \cos^2\alpha=\frac{1}{5}$

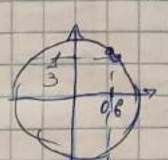
$\sin^2\alpha=1-\frac{1}{5}=\frac{4}{5}$

$\cos2\alpha=\cos^2\alpha-\sin^2\alpha=\frac{1}{5}-\frac{4}{5}=-\frac{1}{3}$

№ 228.

1) $\sin(\arccos 0{,}6+\arcsin\frac{1}{3})=\frac{8\sqrt{2}+3}{15}$

$\arccos 0{,}6=\alpha$, $\alpha\in\mathrm{I}$ ч.



$\cos\alpha=0{,}6$

$\sin\alpha=\sqrt{1-\cos^2\alpha}=0{,}8$ ($\sin\alpha>0$, т.к. $\alpha\in\mathrm{I}$ ч)

$\arcsin\frac{1}{3}=\beta$, $\beta\in\mathrm{I}$ ч.

$\sin\beta=\frac{1}{3}$

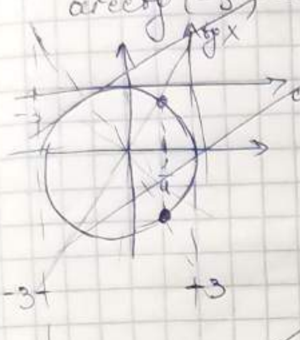
$\cos\beta = \sqrt{1-\frac{1}{9}} = \frac{2\sqrt{2}}{3}$ ($\cos\beta > 0$, т.к. $\beta \in I$ ч.)

$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \sin\beta\cos\alpha = 0,8\cdot\frac{2\sqrt{2}}{3} + \frac{1}{3}\cdot\frac{6}{10} =$

$= \frac{16\sqrt{2}+6}{30} = \frac{8\sqrt{2}+3}{15}$

2) $\cos\left(\operatorname{arcctg}(-3) + \arccos\frac{1}{4}\right) =$

$\operatorname{arcctg}(-3) = \alpha$, $\alpha \in IV$ четв.



$\operatorname{ctg}\alpha = -3$

$1 + \operatorname{ctg}^2\alpha = \frac{1}{\sin^2\alpha}$

$1 + 9 = \frac{1}{\sin^2\alpha} \Rightarrow \sin^2\alpha = \frac{1}{10}$

$\sin\alpha = \frac{-1}{\sqrt{10}}$ (т.к. $\alpha \in IV$ ч.)

$\cos\alpha = \sqrt{1-\frac{1}{10}} = \frac{3}{\sqrt{10}}$ (т.к. $\alpha \in IV$ ч.)

?

$\arccos\frac{1}{4} = \beta$, $\beta \in I$ четв.

$\cos\beta = \frac{1}{4}$

$\sin\beta = \sqrt{1-\frac{1}{16}} = \frac{\sqrt{15}}{4}$ (т.к. $\beta \in I$ четв)

$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta = \frac{3\cdot 1}{\sqrt{10}\cdot 4} + \frac{1\cdot\sqrt{15}}{\sqrt{10}\cdot 4} =$

$= \frac{3+\sqrt{15}}{4\sqrt{10}}$

№229

2) $\operatorname{ctg}\left(\arccos(-0,6) - \arccos\frac{\sqrt{3}}{2}\right) =$

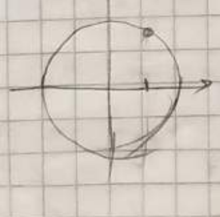
$\arccos(-0,6) = \alpha$, $\alpha \in II$ ч.

$\cos\alpha = -0,6$

$\sin\alpha = \sqrt{1-0,36} = 0,8$ ($\sin\alpha > 0$, $\alpha \in II$ ч.)

$\operatorname{ctg}\alpha = -\frac{6}{8} = -\frac{3}{4}$

$\arccos\frac{\sqrt{3}}{2} = \beta$, $\beta \in I$ ч.



$\cos\beta = \frac{\sqrt{3}}{2}$; $\sin\beta = \sqrt{1-\frac{3}{4}} = \frac{1}{2}$ ($\sin\beta > 0$, $\alpha \in I$ ч.)

$\operatorname{ctg}\beta = \frac{\sqrt{3}\cdot 2}{2\cdot 1} = \sqrt{3}$.

$\operatorname{ctg}(\alpha-\beta) = \frac{\operatorname{ctg}\alpha\operatorname{ctg}\beta + 1}{\operatorname{ctg} - \operatorname{ctg}} = \frac{-\frac{3}{4}\cdot\sqrt{3}+1}{-\frac{3}{4}-\sqrt{3}} = \frac{\frac{3\sqrt{3}}{4}-1}{\frac{3}{4}+\sqrt{3}}$

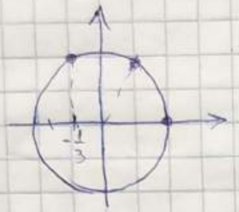
№230

3) $\cos\left(\frac{1}{2}\arccos\left(-\frac{1}{3}\right)\right) = \frac{1}{\sqrt{3}}$

$\arccos\left(-\frac{1}{3}\right) = \alpha$, $\alpha \in$ II ч.

$\cos\alpha = -\frac{1}{3}$

$\cos^2\frac{\alpha}{2} = \frac{1+\cos\alpha}{2} \Rightarrow \cos\frac{\alpha}{2} = \sqrt{\frac{1+\cos\alpha}{2}} = \sqrt{\frac{1-\frac{1}{3}}{2}} =$



$= \sqrt{\frac{2}{6}} = \sqrt{\frac{1}{3}} = \frac{1}{\sqrt{3}}$

№231.

1) $tg\left(\frac{1}{2}\arcsin\frac{4}{5}\right) = \frac{1}{4}$

$\arcsin\frac{4}{5} = 2\alpha$, $2\alpha \in$ I ч.

$\sin 2\alpha = \frac{4}{5}$

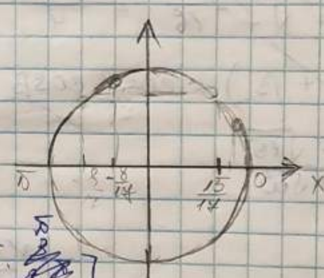
$\cos 2\alpha = \sqrt{1-\frac{16}{25}} = \frac{3}{5}$ ($\cos 2\alpha > 0$, т.к. $2\alpha \in$ I ч.)

$tg\alpha = \frac{1-\cos 2\alpha}{1+\cos 2\alpha} \Rightarrow tg\alpha = \frac{1-\frac{3}{5}}{1+\frac{3}{5}} = \frac{2\cdot 5}{5\cdot 8} = \frac{1}{4}$

№232.

2) $\pi - \arcsin\frac{15}{17} = \arccos\left(-\frac{8}{17}\right)$

$\pi = \arccos\left(-\frac{8}{17}\right) + \arcsin\frac{15}{17}$



$\pi \in [0;\pi]$, $\pi \in \left[\frac{\pi}{2};\frac{3}{2}\pi\right]$

$\arccos\left(-\frac{8}{17}\right) + \arcsin\left(\frac{15}{17}\right) \in \left[\frac{\pi}{2}; \pi\right]$

$\cos\pi = -1$

$\cos\left(\arccos\left(-\frac{8}{17}\right) + \arcsin\left(\frac{15}{17}\right)\right) =$

$\arccos\left(-\frac{8}{17}\right) = \alpha$, $\alpha \in$ II ч.

$\cos\alpha = -\frac{8}{17}$

$\sin\alpha = \sqrt{1-\frac{64}{289}} = \frac{15}{17}$ ($\sin\alpha > 0$, т.к. $\alpha \in$ II ч.)

$\arcsin\left(\frac{15}{17}\right) = \beta$, $\beta \in$ I ч.

$\sin\beta = \frac{15}{17}$

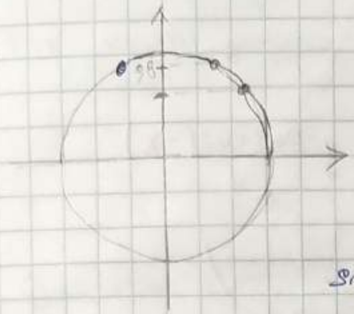
$\cos\beta = \sqrt{1-\frac{225}{289}} = \frac{8}{17}$ ($\cos\beta > 0$, т.к. $\beta \in$ I ч.)

$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta = -\frac{8}{17}\cdot\frac{8}{17} - \frac{15}{17}\cdot\frac{15}{17} = -1.$

Т.к. $y = \cos x$ - убывает на $\left[\frac{\pi}{2}, \pi\right] \Rightarrow$ равенство верно

№233.

1) $\arcsin 0{,}6 + \arcsin 0{,}8 = \arcsin\left(\frac{3}{5}\right) + \arcsin\left(\frac{4}{5}\right) = \frac{\pi}{2}$



$\arcsin\frac{3}{5} = \alpha$, $\alpha \in$ I ч.

$\arcsin\frac{4}{5} = \beta$, $\beta \in$ I ч.

$\sin\alpha = \frac{3}{5}$; $\sin\beta = \frac{4}{5}$

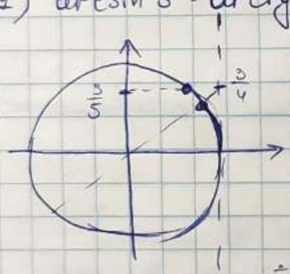
$\cos\alpha = \frac{4}{5}$; $\cos\beta = \frac{3}{5}$

$\sin(\alpha+\beta) = \frac{3}{5}\cdot\frac{3}{5} + \frac{4}{5}\cdot\frac{4}{5} = \frac{9+16}{25} = 1.$

$\Rightarrow \arcsin\left(\frac{3}{5}\right) + \arcsin\left(\frac{4}{5}\right) = \frac{\pi}{2}$

№234.

1) $\arcsin\frac{3}{5} = \operatorname{arctg}\frac{3}{4}$



$\arcsin\frac{3}{5} = \alpha \in [0; \frac{\pi}{2}]$

$\operatorname{arctg}\frac{3}{4} = \beta \in [0; \frac{\pi}{2}]$

$y = \sin x$ – возрастает на $[0; \frac{\pi}{2}]$

$\Rightarrow \sin(\arcsin\frac{3}{5}) = \sin(\operatorname{arctg}\frac{3}{4})$

$\sin\alpha = \frac{3}{5}$; $\operatorname{tg}\beta = \frac{3}{4}$

$1 + \operatorname{tg}^2\beta = \frac{1}{\cos^2\beta}$

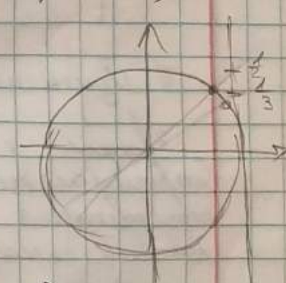
$1 + \frac{9}{16} = \frac{1}{\cos^2\beta} \Rightarrow \cos^2\beta = \frac{16}{25}$

$\sin^2\beta = 1 - \frac{16}{25} = \frac{9}{25}$

$\sin\beta = \frac{3}{5}$ ($\sin\beta > 0$, т.к. $\beta \in$ I ч.)

$\frac{3}{5} = \frac{3}{5}$ – верно.

№236.



1) $\operatorname{arctg}\frac{1}{2} + \operatorname{arctg}\frac{1}{3} = \frac{\pi}{4}$

$\operatorname{arctg}\frac{1}{2} = \alpha$ $\qquad$ $\operatorname{arctg}\frac{1}{3} = \beta$

$\operatorname{tg}\alpha = \frac{1}{2}$ $\qquad$ $\operatorname{tg}\beta = \frac{1}{3}$

$\operatorname{tg}(\operatorname{arctg}\frac{1}{2} + \operatorname{arctg}\frac{1}{3}) = \operatorname{tg}(\alpha+\beta) = \frac{\operatorname{tg}\alpha + \operatorname{tg}\beta}{1 - \operatorname{tg}\alpha\operatorname{tg}\beta} = \frac{\frac{1}{2}+\frac{1}{3}}{1-\frac{1}{6}} = \frac{5\cdot 6}{6\cdot 5} = 1$

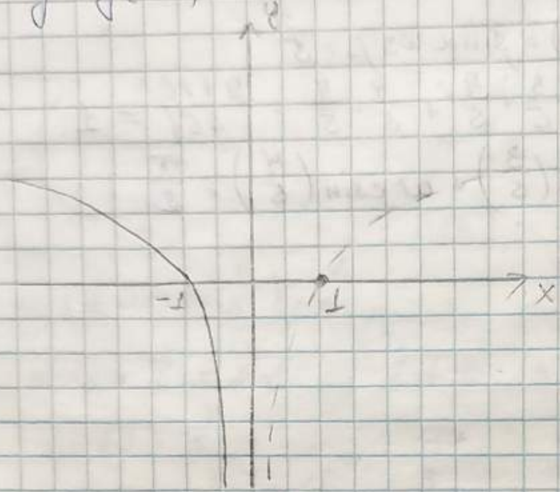
$\Rightarrow \operatorname{arctg}\frac{1}{2} + \operatorname{arctg}\frac{1}{3} = \frac{\pi}{4}$

Построение графиков функций.
Сложение и умножение графиков.

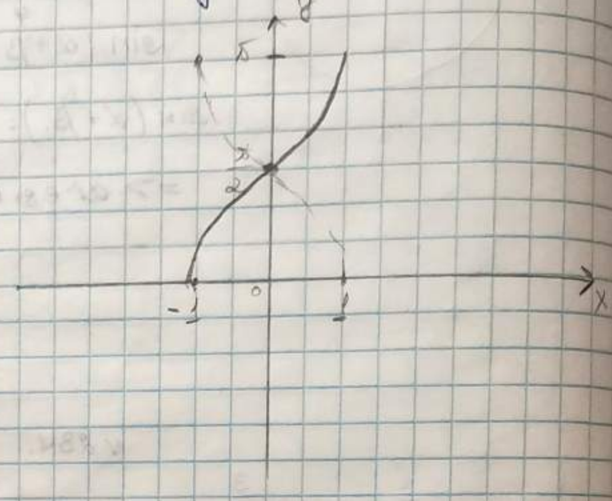
I. Смена знака у аргумента.

График $y = f(-x)$ получается симметричным отображением $y = f(x)$ о[тн.] оси y.

$y = \lg(-x)$



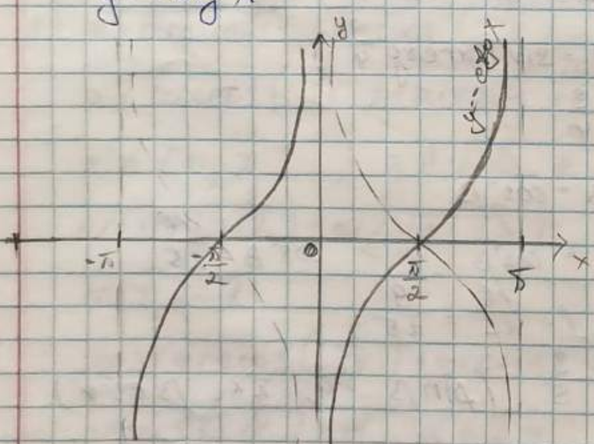
$y = \arccos(-x)$



II. Смена знака у функции.

График $y = -f(x)$ получается симметричным отображение $y = f(x)$ отн. оси x.

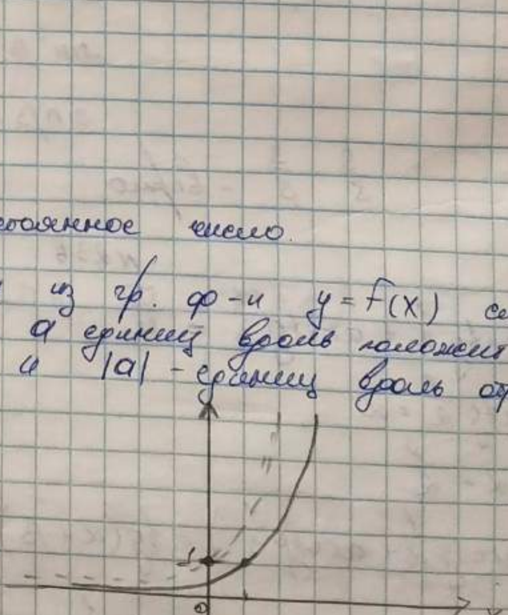
$y = -\operatorname{ctg} x$



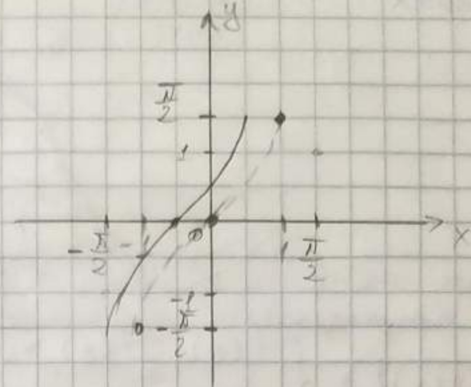
III. Изменение аргумента на постоянное число.

График $y = f(x-a)$, получается из гр. ф-и $y = f(x)$ смещением (параллельным переносом) на a единиц вдоль положительного направления Ox, если $a > 0$ и $|a|$ - единиц вдоль отриц. направления Ox, если $a < 0$

$y = \frac{1}{2} \cdot 2^x = 2^{x-1}$



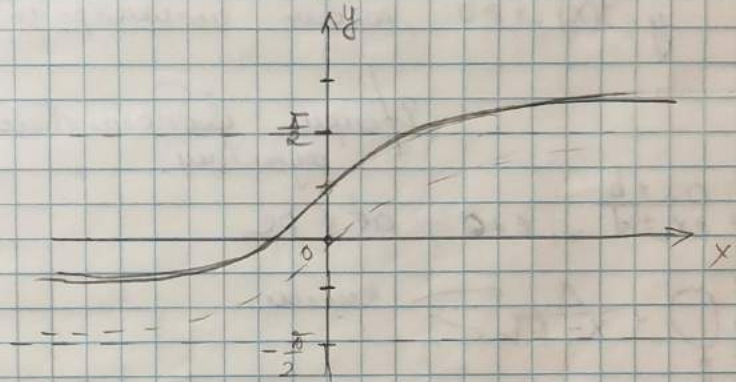
$y = \arcsin(x+1)$



IV. Изменение значения функ-и на постоянное число.

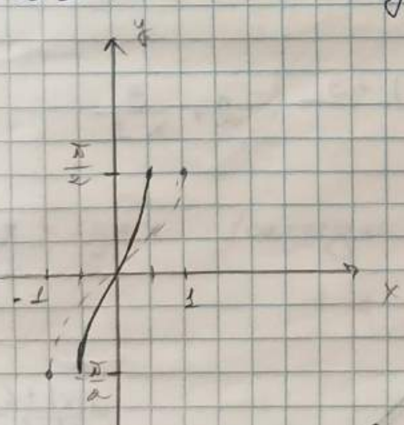
Гр. ф-и $y=f(x)+b$ получается графиком ф-и $y=f(x)$ следующим на b единиц вдоль полож. напр оси y если $b>0$ или $|b|$ - единиц вдоль отриц., если $b<0$

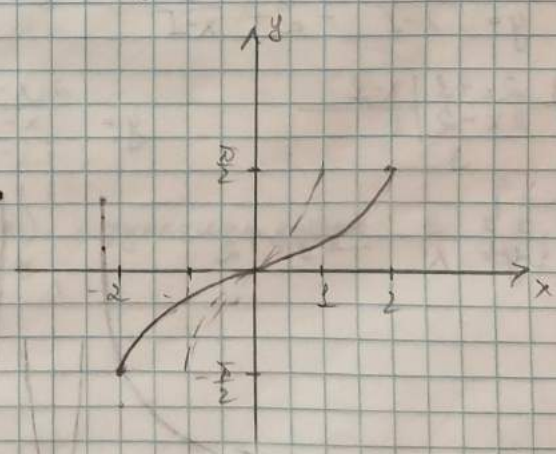
$y = \operatorname{arctg} x + \frac{\pi}{4}$



V. График ф-и $y=f(ax)$, $a>0$, $a\neq 1$ получается из графика ф-и $y=f(x)$ сжатием графика вдоль оси Ox к оси Oy в a раз, если $a>1$ и растяжением графика функции вдоль оси Ox в $\frac{1}{a}$ раз, если $\boxed{0<a<1}$

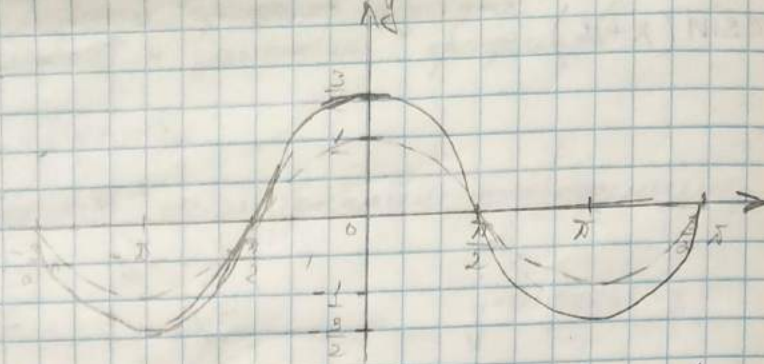
$y=\arcsin 2x$ и $y=\arcsin\frac{x}{2}$





VI. График ф-и $y=Af(x)$, где $A>0$, $A\neq 1$ получается из гр ф-и $y=f(x)$ растяжением графика вдоль оси Oy от оси Ox в A раз, если $A>1$ и сжатием графика вдоль Oy в $\frac{1}{A}$ раз, если $0<A<1$.

$y = \frac{3}{2}\cos x$



VII. $y = |f(x)| = \begin{cases} f(x), & f(x) > 0 \\ -f(x), & f(x) < 0 \end{cases}$ Строим график $y = f(x)$, выделяем ту часть графика которая выше Ox, а та часть, которая ниже отображается относительно Ox.

VIII. $y = f(|x|)$

$y = f(x),\ x \geq 0$, потом симметрично отображают отн. Oy.

График дробнолинейной функции.

$y = \frac{ax+b}{cx+d}$, $c \neq 0$; $ad \neq bc$

$y = \bigcirc + \frac{\triangle}{x - \square}$ ← числа.

целая часть

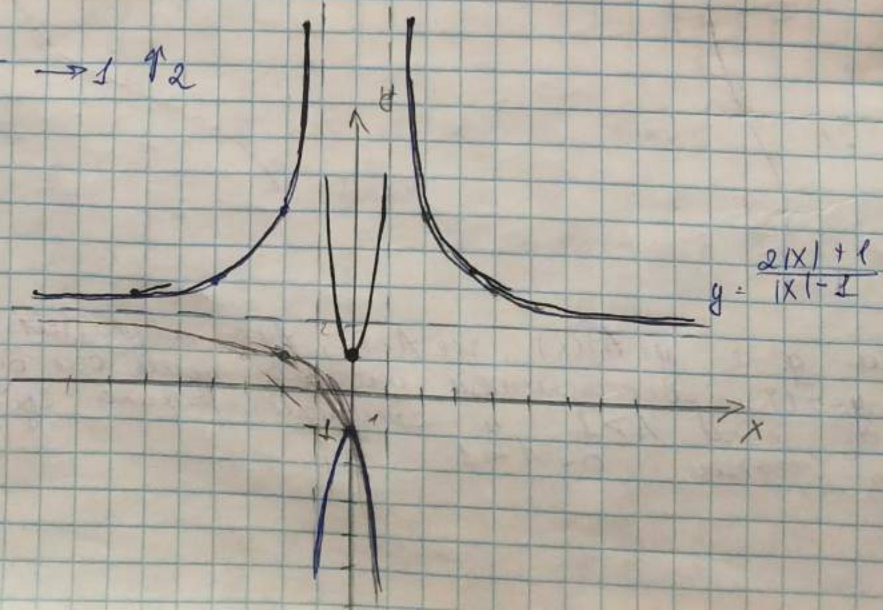
$y = y_0 + \frac{m}{x - x_0} \Rightarrow \begin{cases} y = y_0 \\ x = x_0 \end{cases}$ – асимптоты

$y = \frac{2x+1}{x-1} = 2 + \frac{3}{x-1}$

1) $\begin{array}{r|l} 2x+1 & x-1 \\ \cline{2-2} 2x-2 & 2 \\ \hline 3 & \end{array}$

$y = \frac{2(x-1)+3}{x-1} = 2 + \frac{3}{x-1}$.

$y = \frac{3}{x} \rightarrow 1\ \uparrow 2$



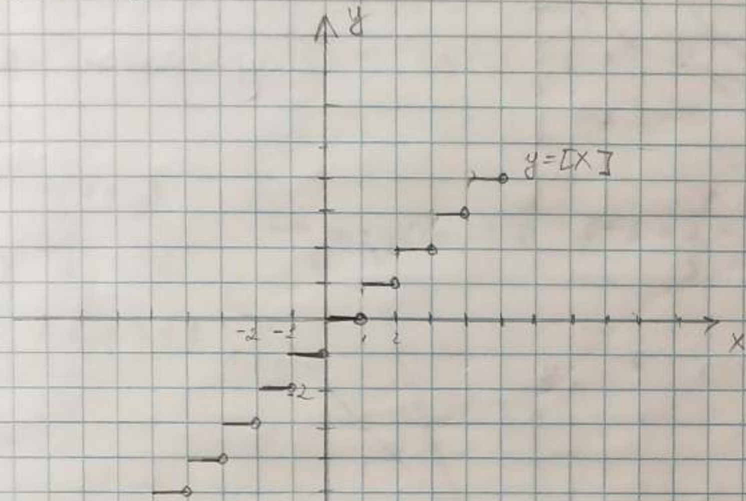
Если $y = \frac{2|x|+1}{|x|-1}$, если $y = \left|\frac{2|x|+1}{|x|-1}\right|$

$\left|\frac{2|x|+1}{|x|-1}\right| = a$

$y = [x]$ ([...] - целая часть числа) - это наибольшее целое число, не превосходящая x.

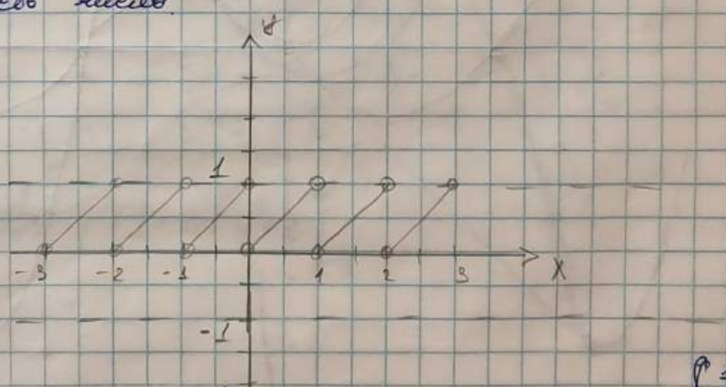
$[5,6] = 5$ (антье)

$[-5,6] = -6$

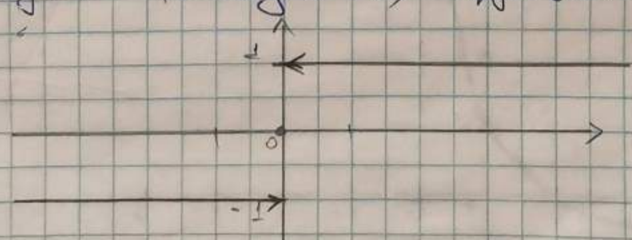


$\{x\}$ - дробная часть числа

$\{x\} = x - [x]$



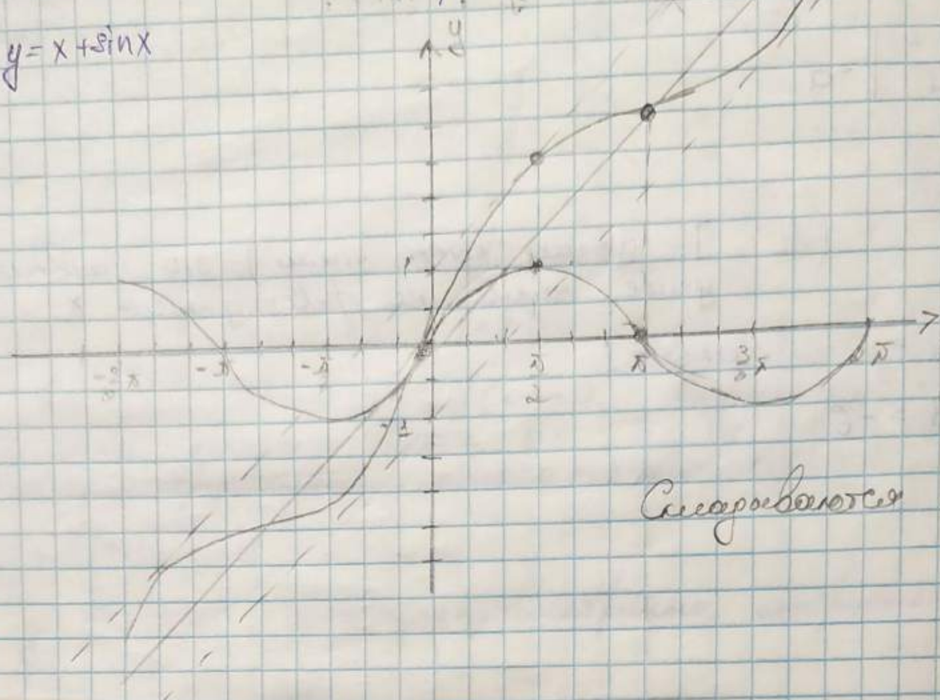
$y = \mathrm{sign}\, x = \mathrm{sgn}\, x$ (сигнум x) - функция знака. $= \begin{cases} 1, & x > 0 \\ 0, & x = 0 \\ -1, & x < 0 \end{cases}$



$|x| = x \cdot \mathrm{sgn}\, x$

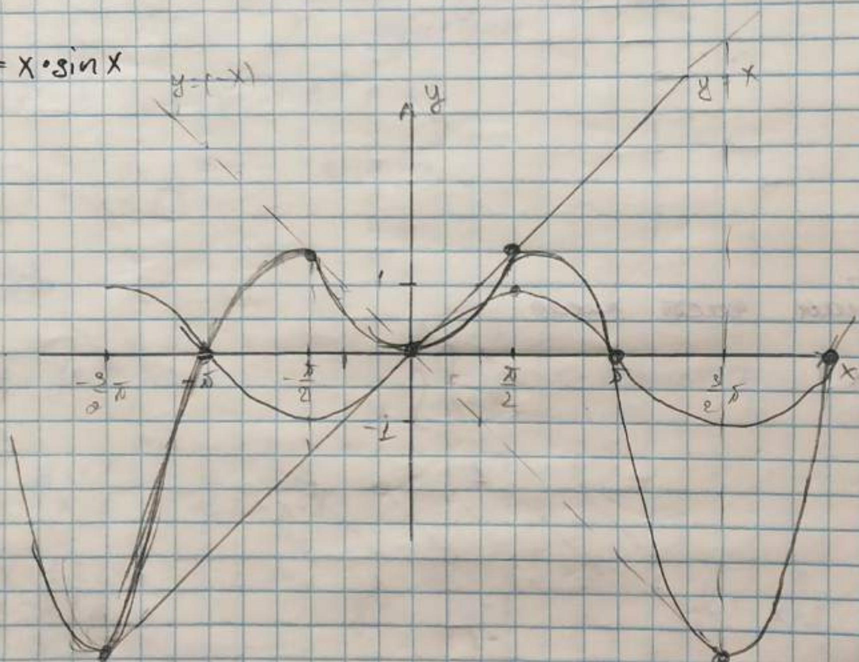
Сложение графиков

383. $y = x + \sin x$



Складываются координаты

$y = x \cdot \sin x$



Домашняя работа.

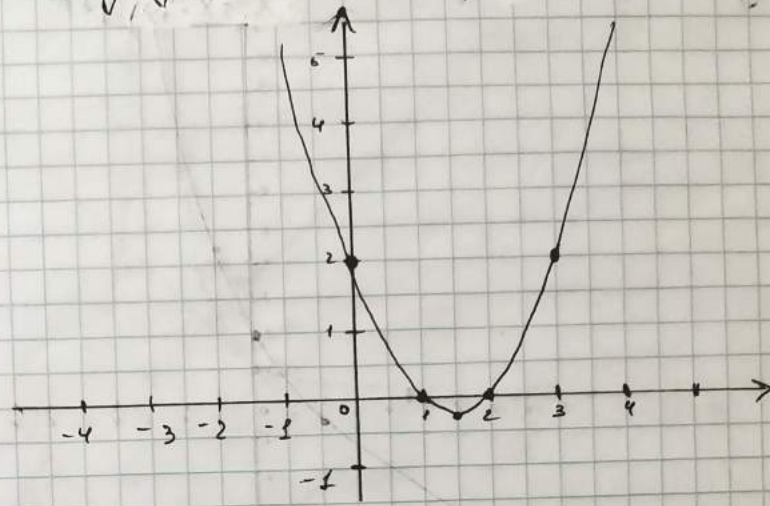
243. б) $y = x^2 - 3x + 2$ $\longrightarrow$ $y_0 = y(x_0) = \frac{9}{4} - \frac{9}{2} + 2 = -\frac{9}{4} + \frac{8}{4} = -\frac{1}{4}$

$x_0 = \frac{3}{2} = 1{,}5$

$y = -\frac{1}{4} + \left(x - \frac{3}{2}\right)^2$

$x_1 = 1$

$x_2 = 2$

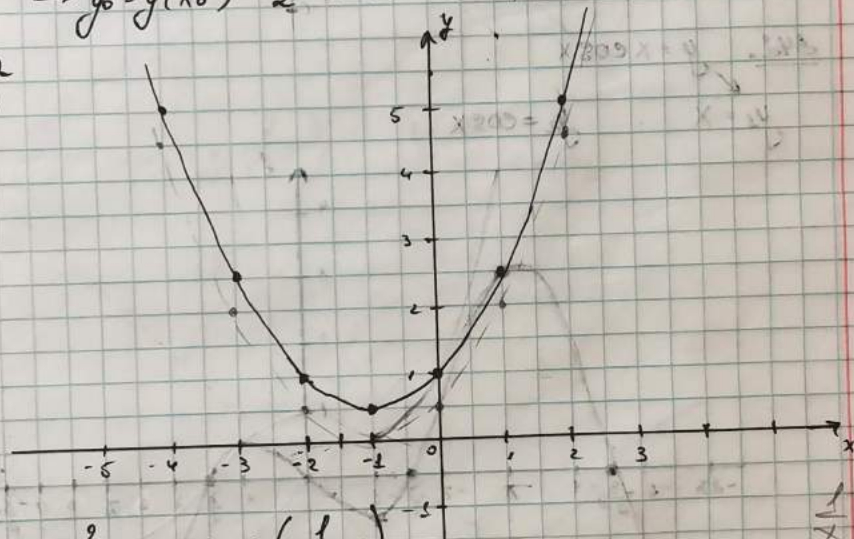


г) $y = \frac{1}{2}x^2 + x + 1.$ $\longrightarrow$ $y_0 = y(x_0) = \frac{1}{2} - 1 + 1 = \frac{1}{2} = 0{,}5$

$x_0 = \frac{-1}{1} = -1$

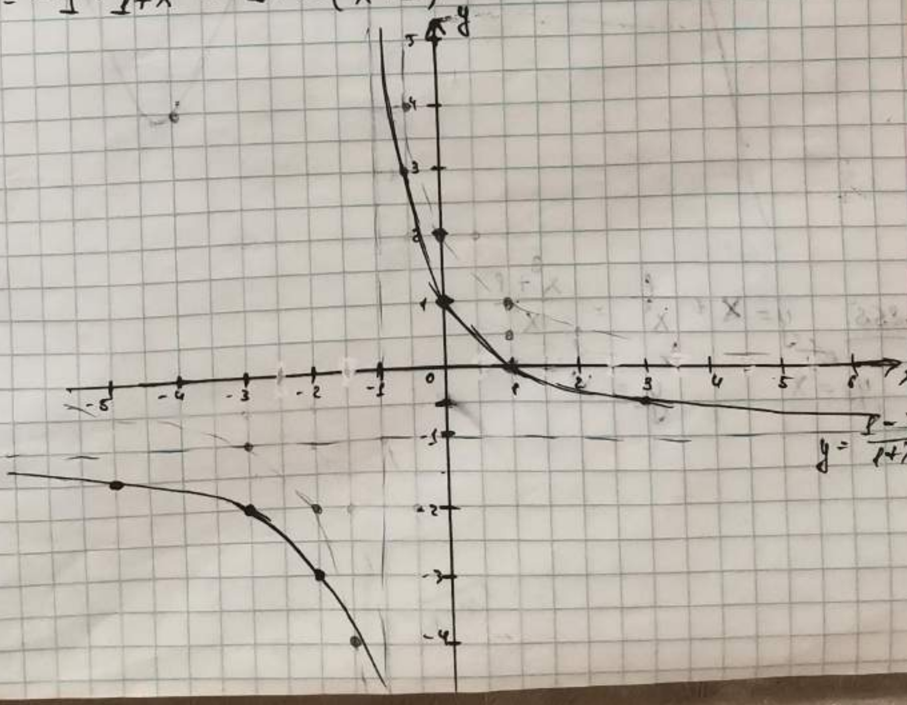
$y = \frac{1}{2} + \frac{1}{2}(x+1)^2$

$y = y_0 + a(x - x_0)^2$



$\frac{1}{x}$

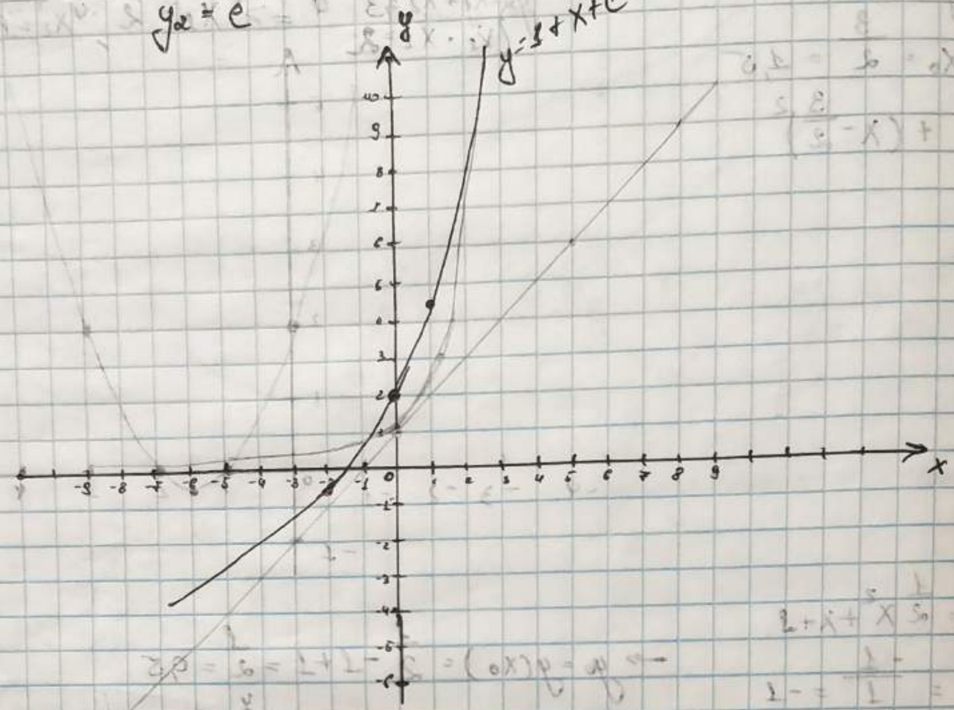
250. $y = \frac{1-x}{1+x} = -1 + \frac{2}{1+x} = -1 + 2\left(\frac{1}{x+1}\right)$



$y = \frac{1-x}{1+x}$

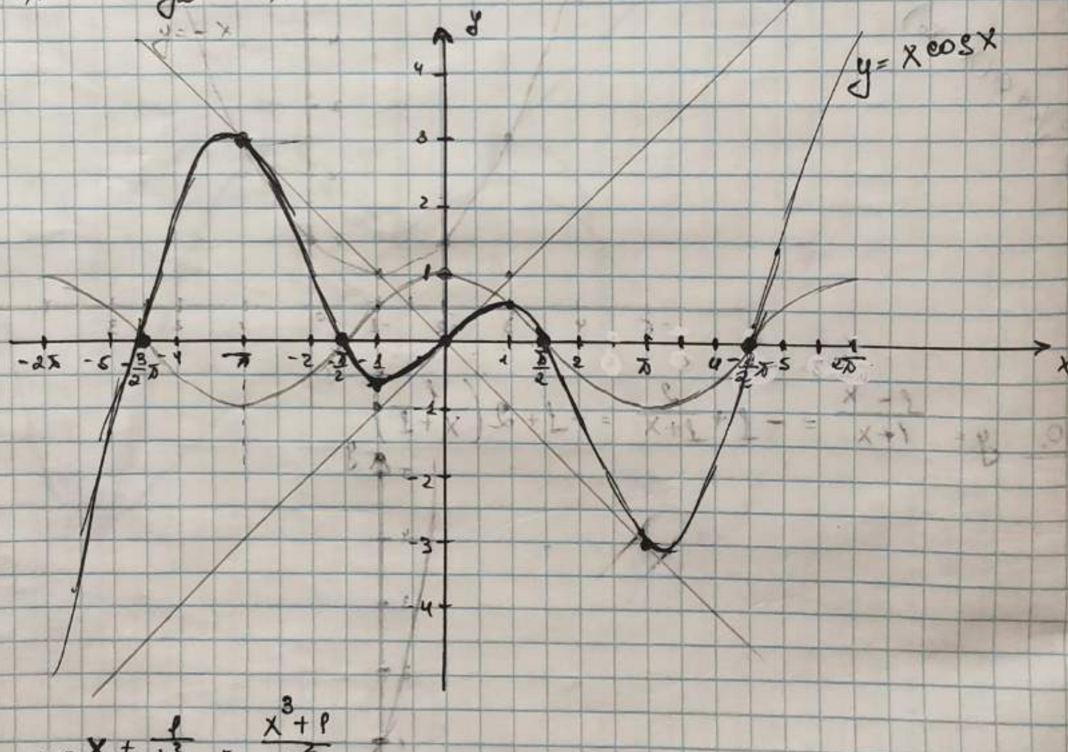
381. $y = 1 + x + e^x$

$y_1 = 1 + x$ $\qquad$ $y_2 = e^x$



342. $y = x\cos x$

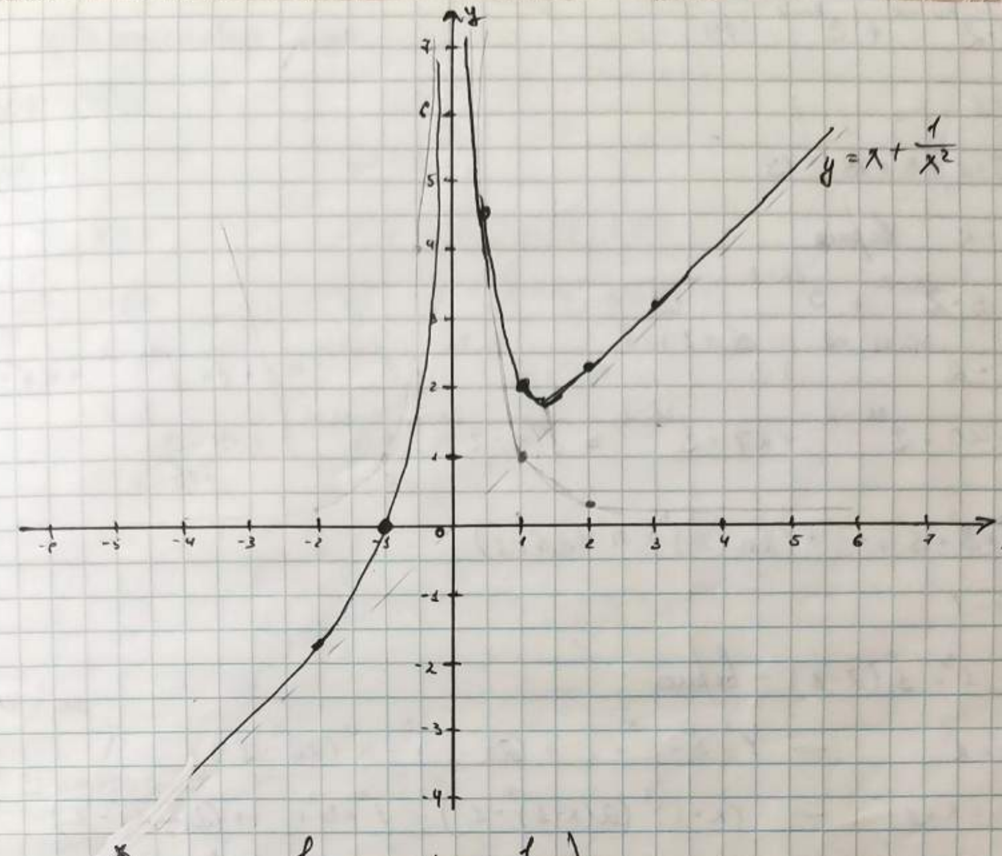
$y_1 = x$ $\qquad$ $y_2 = \cos x$



255. $y = x + \frac{1}{x^2} = \frac{x^3 + 1}{x^2}$

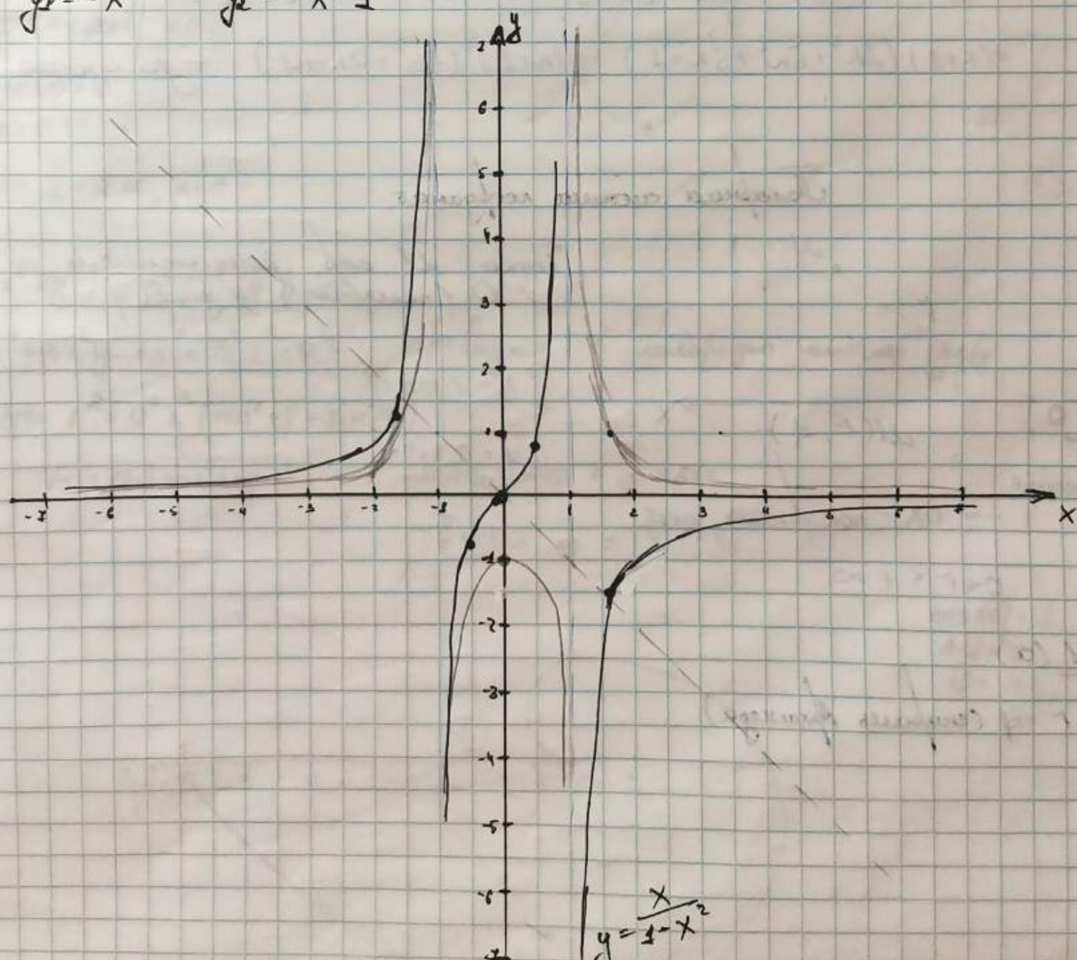
$y_1 = x$ $\qquad$ $y_2 = \frac{1}{x^2}$

259.



259. $y = \frac{x}{1 - x^2} = x \cdot \frac{1}{1 - x^2} = -\left(x \cdot \frac{1}{x^2 - 1}\right)$

$y_1 = -x$ $\qquad$ $y_2 = \frac{1}{x^2 - 1}$



① $5 \cdot 2^{3n-2} + 3^{3n-1} \,\vdots\, 19$

1) $n=1$

$5 \cdot 2 + 3^2 = 19 \,\vdots\, 19$

2) $n=k$ верно

$5 \cdot 2^{3k-2} + 3^{3k-1} \,\vdots\, 19$

$5 \cdot 2^{3(k+1)-2} + 3^{3(k+1)-1} = 5 \cdot 2^{3k+1} + 3^{3k+2} = 5 \cdot 8 \cdot 2^{3k-2} + 27 \cdot 3^{3k-1} =$

$= 40 \cdot 2^{3k-2} + 27 \cdot 3^{3k-1} = 8\underbrace{(5 \cdot 2^{3k-2} + 3^{3k-1})}_{\vdots\, 19} + \underbrace{19 \cdot 3^{3k-1}}_{\vdots\, 19}$

② $1^3 + 3^3 + 5^3 + \ldots + (2n-1)^3 = n^2(2n^2-1)$

1) $n=1$

$1^3 = 1^2(2-1)$ – верно

2) $n=k \quad \to \quad 1^3 + 3^3 + 5^3 + \ldots + (2k-1)^3 = k^2(2k^2-1)$ $\quad 2k^2+4k+1$

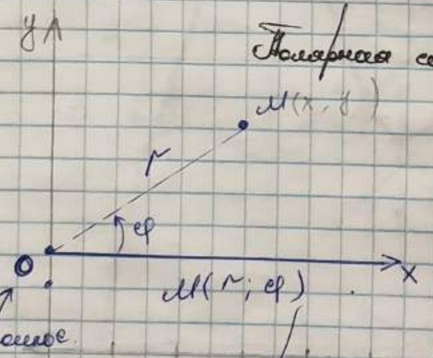
3) $n=k+1 \quad \to \quad (k+1)^2(2(k+1)^2-1) = 1^3 + 3^3 + \ldots + (2(k+1)-1)^3$

$1^3 + 3^3 + \ldots + (2k-1)^3 + (2k+1)^3 = k^2(2k^2-1) + (2k+1)^3 =$

$= 2k^4 - k^2 + 8k^3 + 12k^2 + 6k + 1 = 2k^4 + 8k^3 + 11k^2 + 6k + 1 =$

$= (k+1)(2k^3 + 6k^2 + 5k + 1) = (k+1)^2(2k^2+4k+1)$ – доказано.

Полярная система координат.



Точка M опр. расстоянием до т. O (полюса) и углом φ.

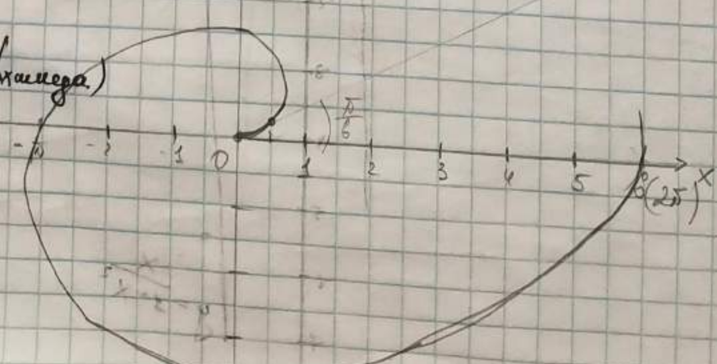
$$\begin{cases} x = r\cos\varphi \\ y = r\sin\varphi \end{cases}$$

→ OX – полярная ось.

$0 < r < +\infty$

371. (a.)

$r = \varphi$ (спираль Архимеда)



371. (г.)

$r = \ldots$

$\sin 3\ldots$

$3\varphi \ldots$

$\varphi = \ldots$

371. (д.)

$r = 2(1 \ldots$

$1 + \cos\ldots$

$\varphi = \pi + \ldots$

$1 + \cos\varphi = \ldots$

$\cos\varphi = 0\ldots$

$\varphi = \frac{\pi}{2} + \ldots$

$(x^2+y^2)\ldots$

$\cancel{x^4 + 2x^2y\ldots}$

$r^2\ldots$

$r = \ldots$

371 (г.) – трёхлепестковая роза.

$r = 10\sin 3\varphi$

$\sin 3\varphi = 0$

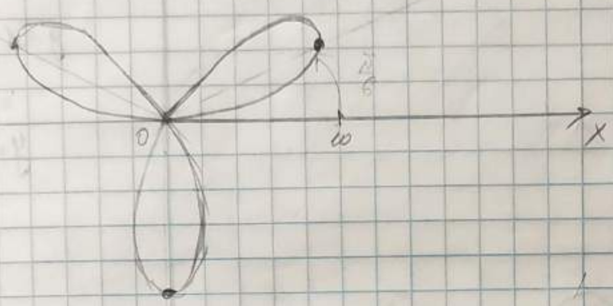
$3\varphi = \pi n,\ n \in Z$

$\varphi = \frac{\pi}{3} n,\ n \in Z$

$\sin 3\varphi = 1$

$3\varphi = \frac{\pi}{2} + 2\pi k,\ k \in Z$

$\varphi = \frac{\pi}{6} + \frac{2}{3}\pi k,\ k \in Z$



371 (д) – кардиоида.

$r = 2(1+\cos\varphi)$

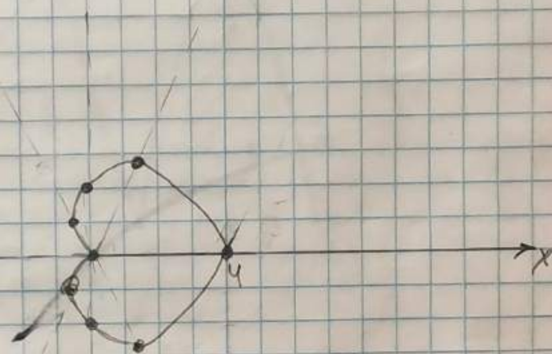
$1+\cos\varphi = 0$

$\varphi = \pi + 2\pi k,\ k \in Z$

$1+\cos\varphi = 1$

$\cos\varphi = 0$

$\varphi = \frac{\pi}{2} + \pi m,\ m \in Z$



$(x^2+y^2)^2 = a^2(x^2-y^2)$

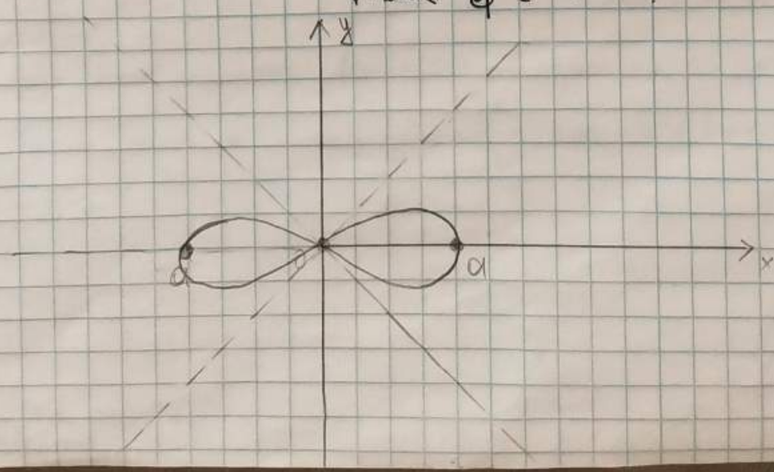
$\cancel{x^4+2x^2y^2+y^4 = a^2x^2-a^2y^2}$ переводим в полярную систему коорд.

$r^2 = a^2x^2(\cos^2\varphi - \sin^2\varphi)$

$r = a\sqrt{\cos 2\varphi}$

$-\frac{\pi}{2}+2\pi k \le 2\varphi \le \frac{\pi}{2} + 2\pi k$

$-\frac{\pi}{4}+\pi k \le \varphi \le \frac{\pi}{4} + \pi k$



$\cos 2\varphi = 1$

$2\varphi = 2\pi k$

$\varphi = \pi k$

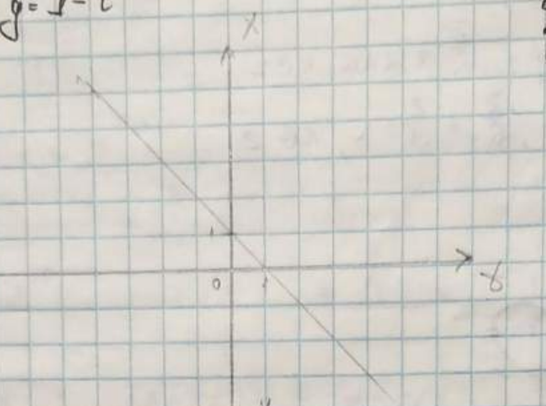
Построение параметрически заданных функций.

$x^2 + y^2 = a^2$

369. (а.) $\begin{cases} x = 1 - t \\ y = 1 - t^2 \end{cases}$

$\begin{cases} x = a\cos t \\ y = a\sin t \end{cases}$

$t \in [0; 2\pi]$

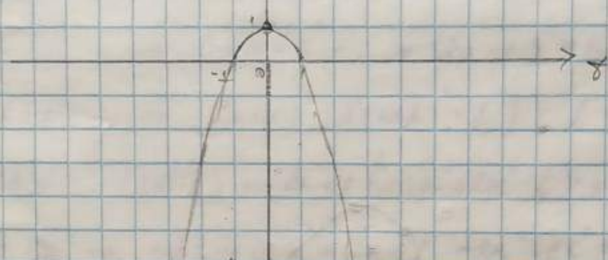


$t = 1 - x$

$y = 1 - 1 + 2x - x^2 = 2x - \ldots$

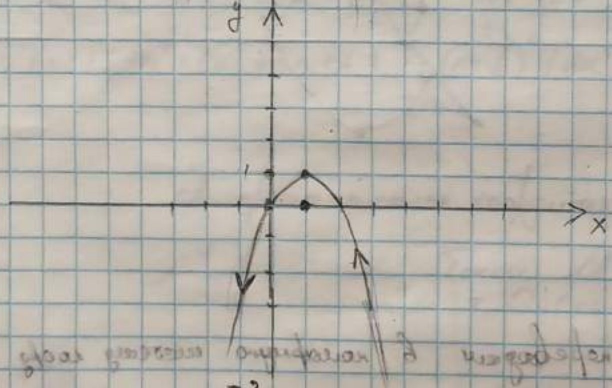
$x_0 = \frac{-2}{-2} = 1$

$y_0 = 1.$



$t \in (-\infty; 0]$ – правая ветка.

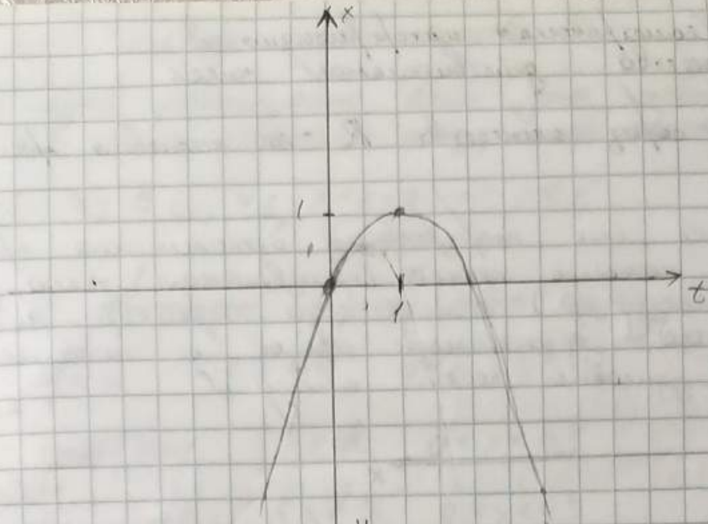
$t \in [0; \infty)$ – левая ветка

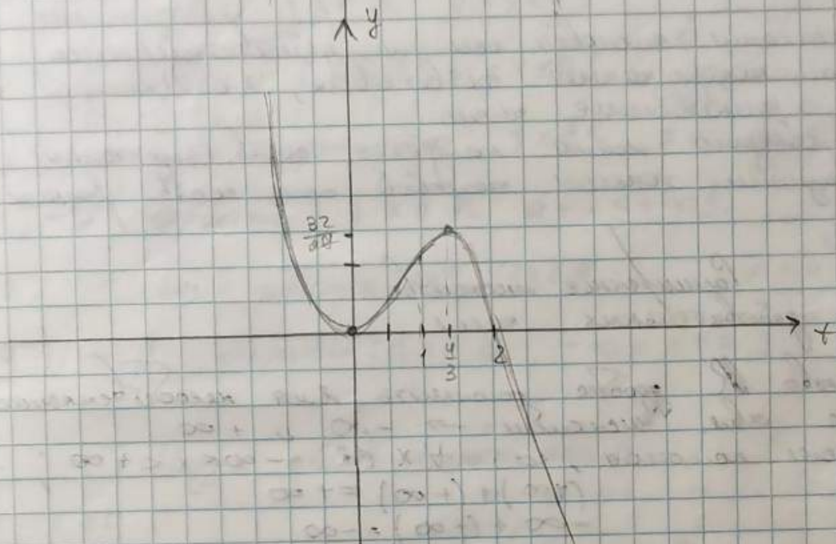


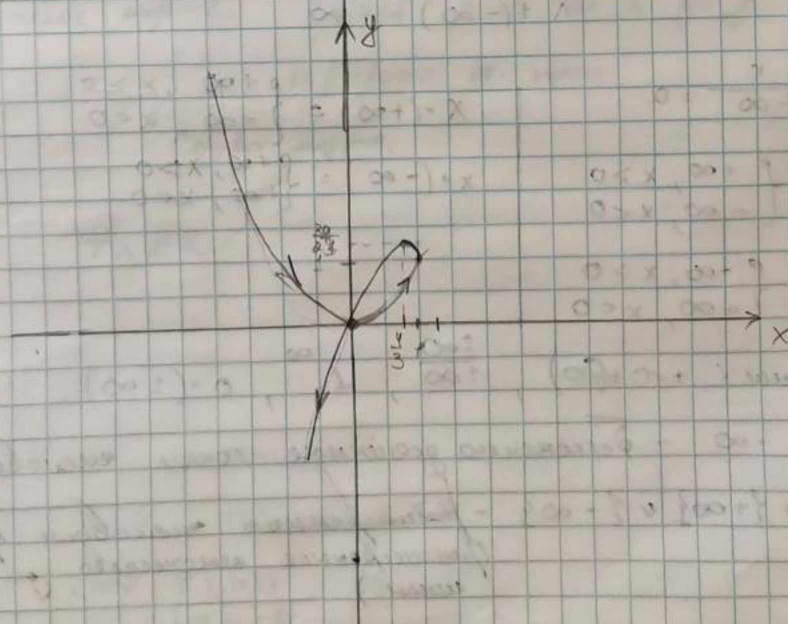
$\begin{cases} x = 2t - t^2 \\ y = 2t^2 - t^3 \end{cases}$

$x_0 = \frac{-2}{-2} = 1$ $\quad x_0 = 1$

$t \in (-\infty; 0]$


x
t
1
1


y
t
1
2


y
x

$[0]=0$

$\{x\}$ - дробная часть действ. числа. - разность между числом и его целой частью

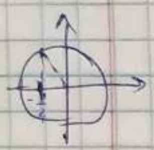
$\{x\}=x-[x]$

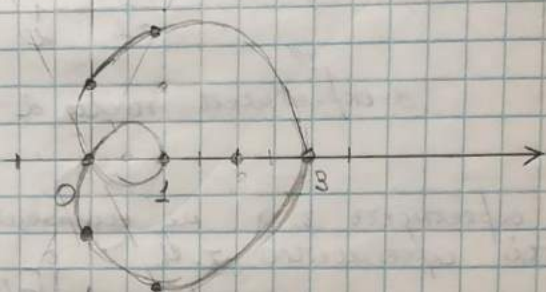
$\{1,7\}=1,7-1=0,7$

$\{-0,3\}=-0,3+1=0,7$

Домашняя работа

③ 1) $r=1+2\cos\varphi$





$1+2\cos\varphi=0$

$\cos\varphi=-\frac{1}{2}$

$\varphi=\frac{\pi}{2}+\frac{\pi}{6}+2\pi k=\frac{2}{3}\pi+2\pi k$

$\varphi=-\frac{2}{3}\pi+2\pi k$

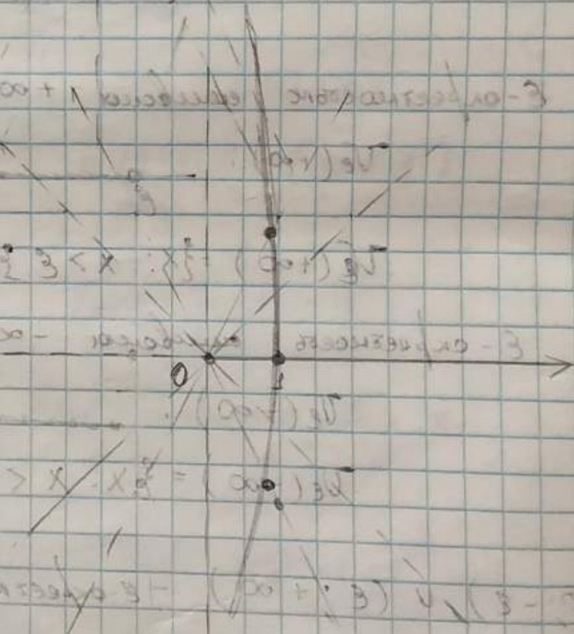
$1=1+2\cos\varphi$

$2\cos\varphi=0$

$\cos\varphi=0$

$\varphi=\frac{\pi}{2}+\pi k,\ k\in Z$

2) $r=\frac{1}{\cos\varphi}$



$\cos\varphi\neq0$

$\varphi\neq\frac{\pi}{2}+\pi n,\ n\in Z$

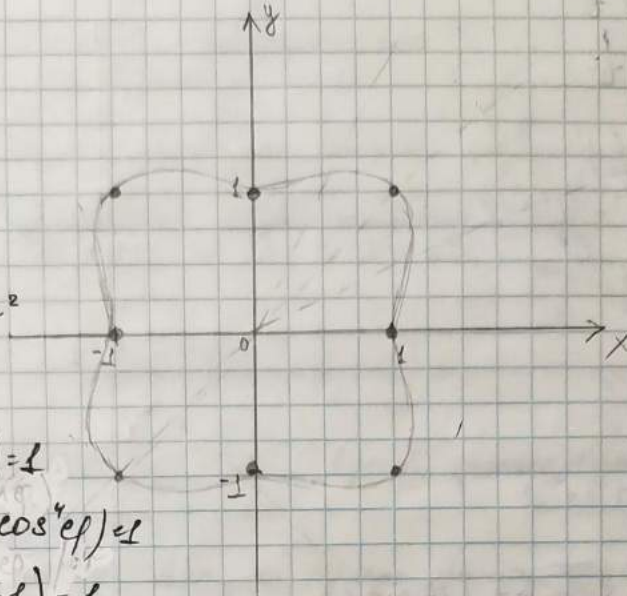
② $x^4+y^4=x^2+y^2$

$r^2=x^2+y^2$

$(r\cos\varphi)^4+(r\sin\varphi)^4=r^2$

$r^4\cos^4\varphi+r^4\sin^4\varphi=r^2$

$r^4(\cos^4\varphi+\sin^4\varphi)=r^2 \quad /:r^2$



$r^2(\cos^4\varphi+\sin^4\varphi)=1$

$r^2(\cos^4\varphi+(1-\cos^2\varphi)^2)=1$

$r^2(\cos^4\varphi+1-2\cos^2\varphi+\cos^4\varphi)=1$

$r^2(2\cos^4\varphi-2\cos^2\varphi+1)=1$

$$r^2=\frac{1}{2\cos^4\varphi-2\cos^2\varphi+1}$$

$\varphi=\frac{\pi}{4} \rightarrow \frac{1}{2\frac{4}{16}-2\cdot\frac{2}{4}+1}=\frac{1}{\frac{1}{2}-1+1}=2 \Rightarrow r=\pm\sqrt{2}$ $\qquad \varphi=0$

$\varphi=\frac{\pi}{6} \rightarrow \frac{1}{2\cdot\frac{9}{16}-2\cdot\frac{3}{4}+1}=\frac{1}{\frac{9}{8}-\frac{3}{2}+1}=\frac{1}{\frac{1}{8}+\cancel{1}+\cancel{1}-\cancel{1}-\frac{1}{2}}=\frac{1}{\frac{1}{8}+\frac{1}{2}}$

$=\frac{8}{5}=1{,}6 \qquad \Rightarrow r=\pm\sqrt{1{,}6}$

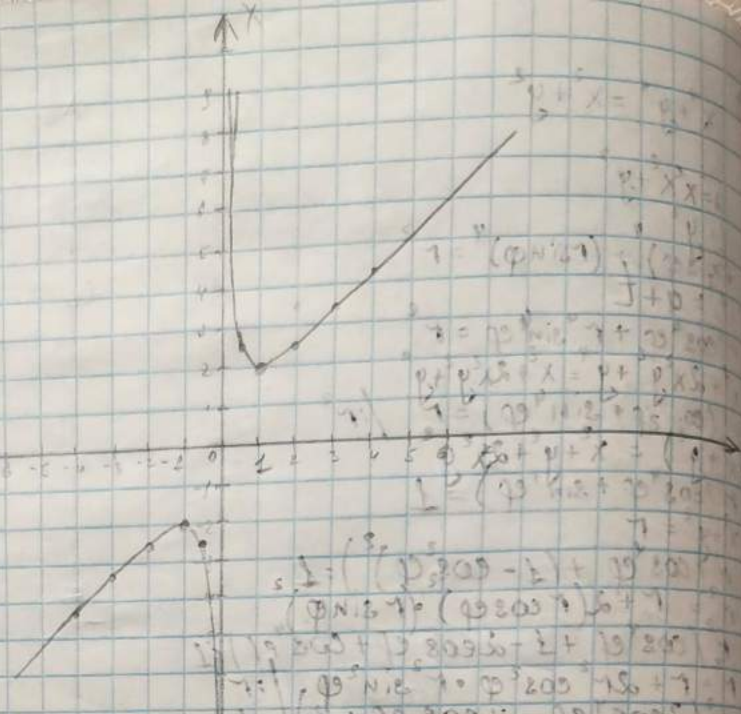
$\varphi=\frac{\pi}{3} \rightarrow \frac{1}{2\cdot\frac{1}{16}+1-2\cdot\frac{1}{4}}=\frac{1}{\frac{1}{8}-\frac{1}{2}+1}=1{,}6 \Rightarrow r=\pm\sqrt{1{,}6}$

① $x = t + \frac{1}{t}$

$y = t + \frac{1}{t^2}$

$$y' = t' + \left(\frac{1}{t^2}\right)' = 1 + \frac{1' t^2 - (t^2)' \cdot 1}{t^4} = 1 + \frac{-2t}{t^4} = \cancel{1 - \frac{2t}{t^4}} = \cancel{\frac{t^4 - 2}{t^4}} = \frac{(t^2-\sqrt{2})(t^2+\sqrt{2})}{t^4}$$

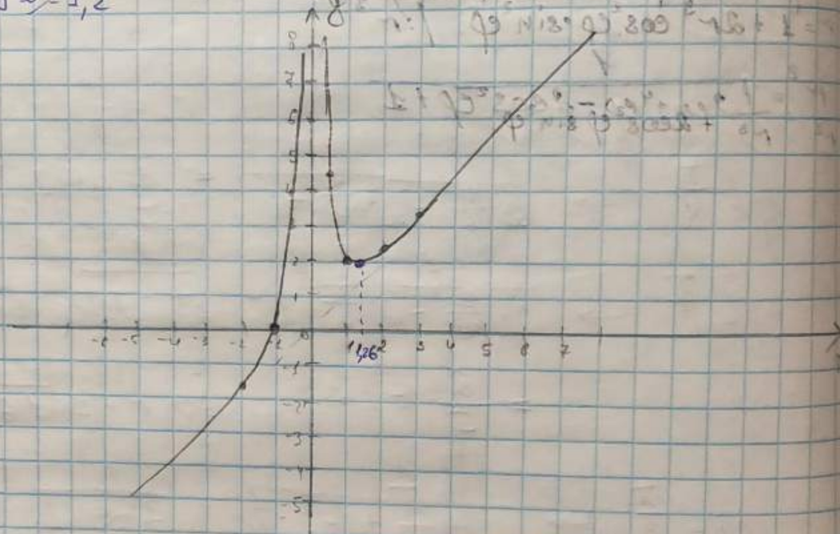
$\Rightarrow \cancel{t_{1,2} = \pm\sqrt{\sqrt{2}} \approx \pm 1,19 \approx \pm 1,2}$



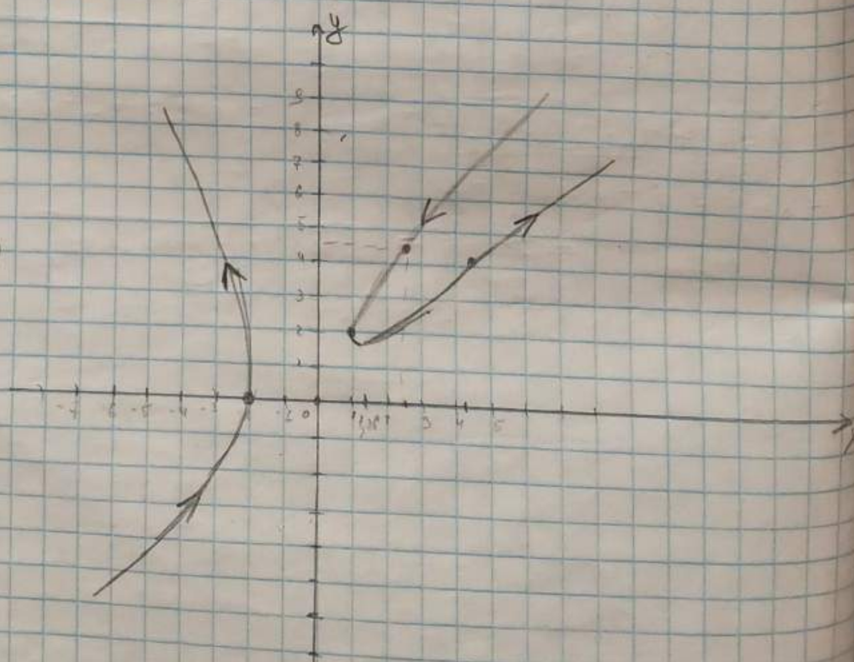
$$= 1 - \frac{2}{t^3} = \frac{t^3 - 2}{t^3}$$

$t_1 = 0$

$t_2 = \sqrt[3]{2} \approx 1,26$



1) $t \in (-\infty, -1]$
2) $t \in [-1; 0]$
3) $t \in [0; 1]$
4) $t \in [1; 1,26...]$
5) $t \in [1,26...; +\infty]$



378.

$y = \cos x \cdot \mathrm{sgn}(\sin x)$

$y_1 = \cos x \qquad y_2 = \mathrm{sgn}(\sin x)$

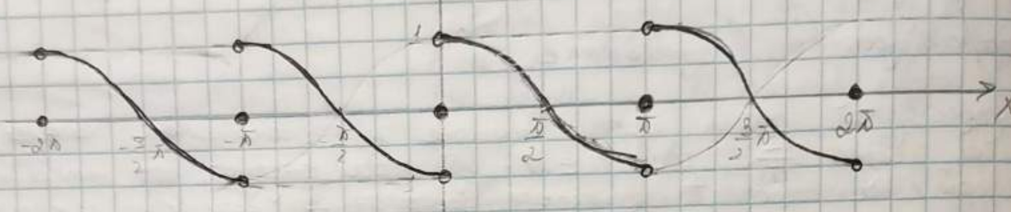
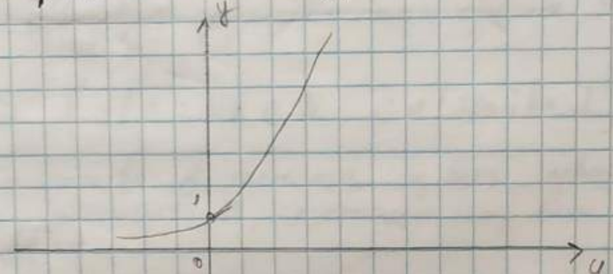


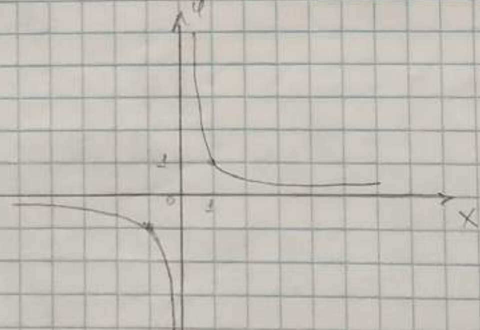
График сложной функции.

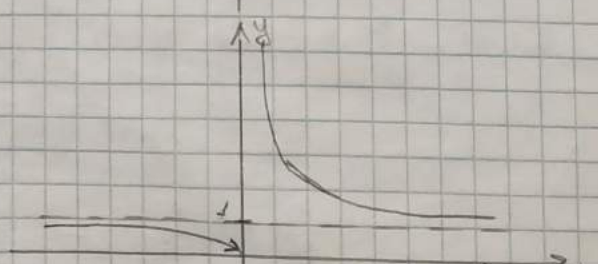
$y = e^{\frac{1}{x}}$

$u(x) = \frac{1}{x}$

$y = e^{u}$







282. (б.)

$y = \ln(x-$

$y = \ln u$

$u = (x-1)$

$y \cdot 3 = 12$

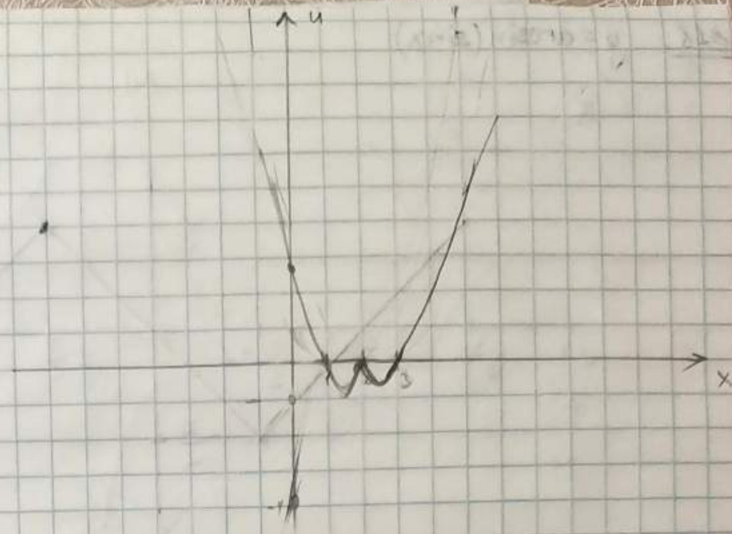
$y = \ln u$

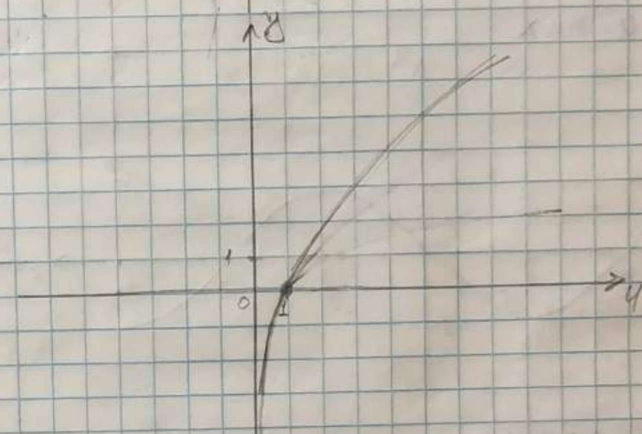
$(x-3)^3$

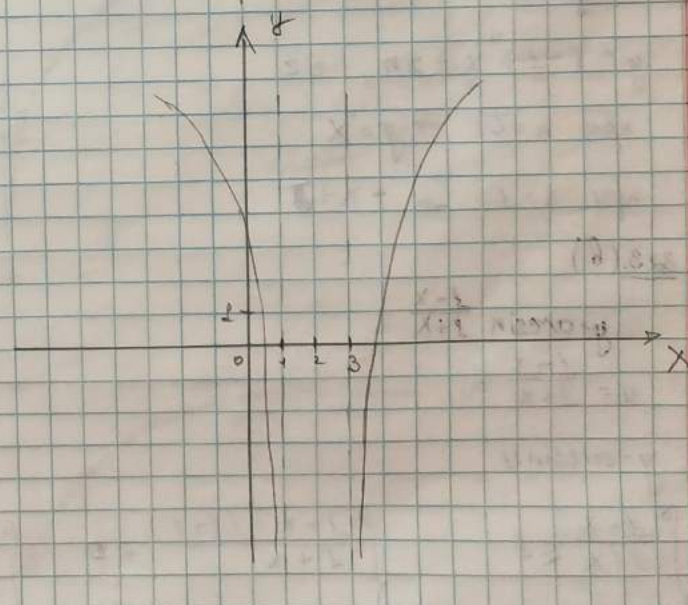
x^3

282. (б.)

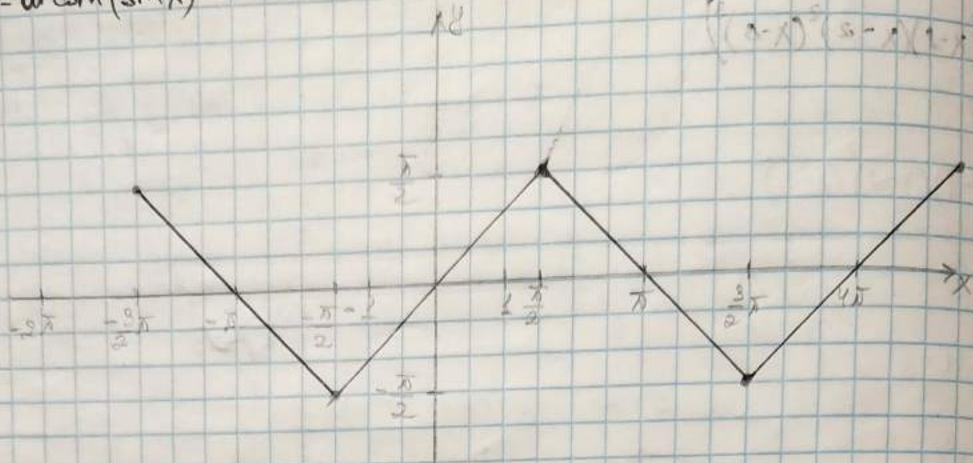
$y = \ln\left((x-1)(x-2)^2(x-3)^3\right)$





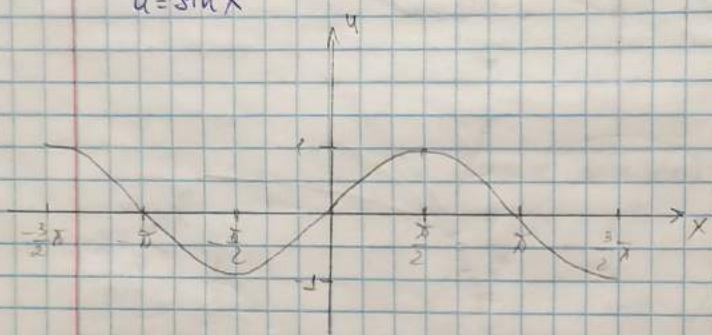


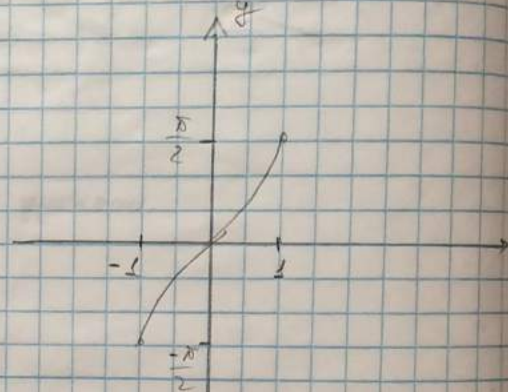
318. $y = \arcsin(\sin x)$



$y = \arcsin u$

$u = \sin x$





324(б)

$y = \operatorname{arct}$

$y = \operatorname{arct}$

$u = \ln$

$\sin y = \sin x$

$y = (-1)^n x + \pi n, n \in z$

при $n = 0 \to y = x$

при $n = 1 \Rightarrow -x + \pi$

$$\begin{array}{r|l} 1-x & 1+x \\ 1+x & 1 \\ \hline -2x & \end{array}$$

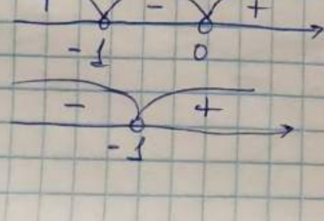
323.(б)

$y = \arcsin \frac{1-x}{1+x}$

$u = \frac{1-x}{1+x} =$

$y = \arcsin u$

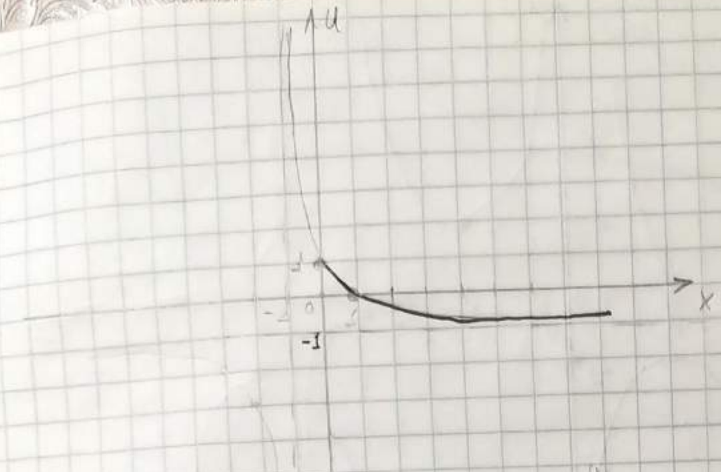
$$\begin{cases} \frac{1-x}{1+x} \le 1 \\ \frac{1-x}{1+x} \ge -1 \end{cases} \quad \begin{cases} \frac{1-x-(1+x)}{1+x} \le 0 \\ \frac{1-x+1+x}{1+x} \ge 0 \end{cases} \quad \begin{cases} -\frac{2x}{1+x} \le 0 \\ \frac{2}{1+x} \ge 0 \end{cases} \quad \begin{cases} \frac{2x}{1+x} \ge 0 \\ \frac{2}{1+x} \ge 0 \end{cases}$$

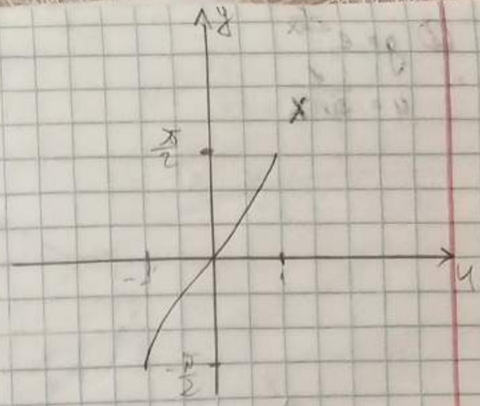


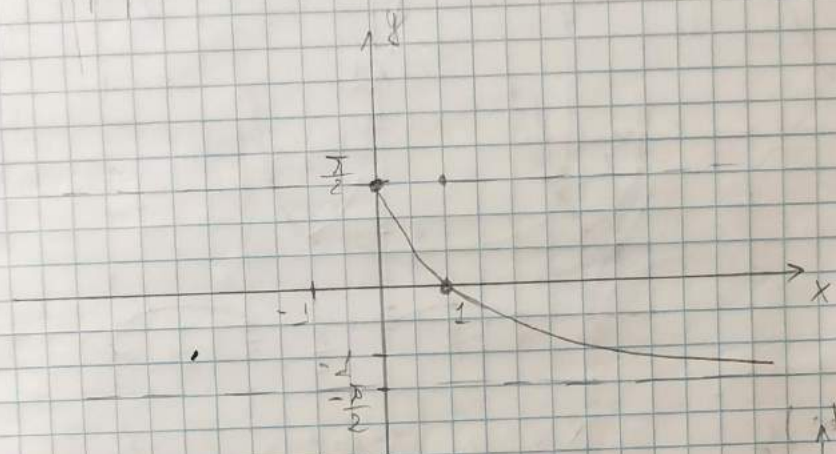
$\Rightarrow x \ge 0$

$-1 = \frac{1-x}{1+x}$

$-1 - x \ne 1 - x$





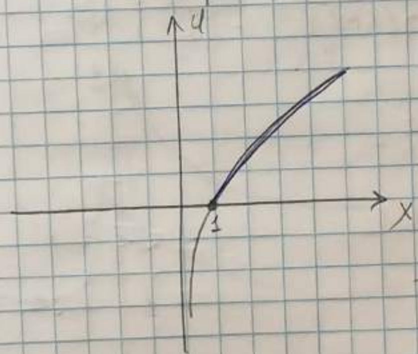


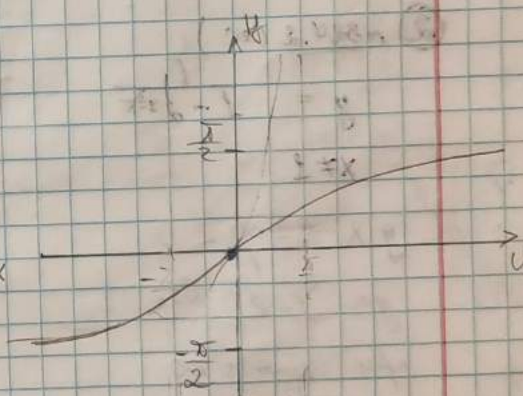
зад(8)

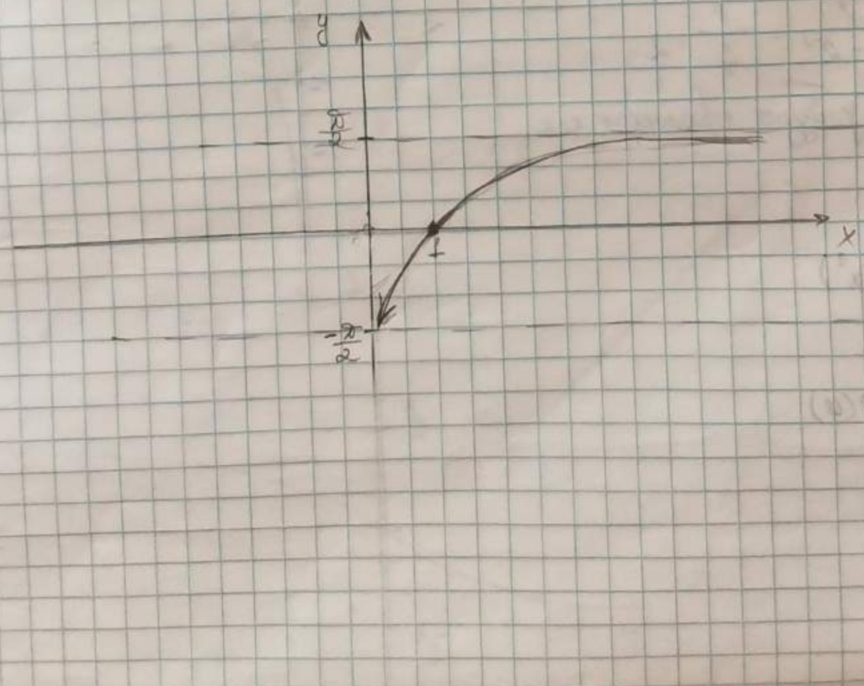
$y = \operatorname{arctg}(\ln x)$

$y = \operatorname{arctg}(u)$

$u = \ln x$

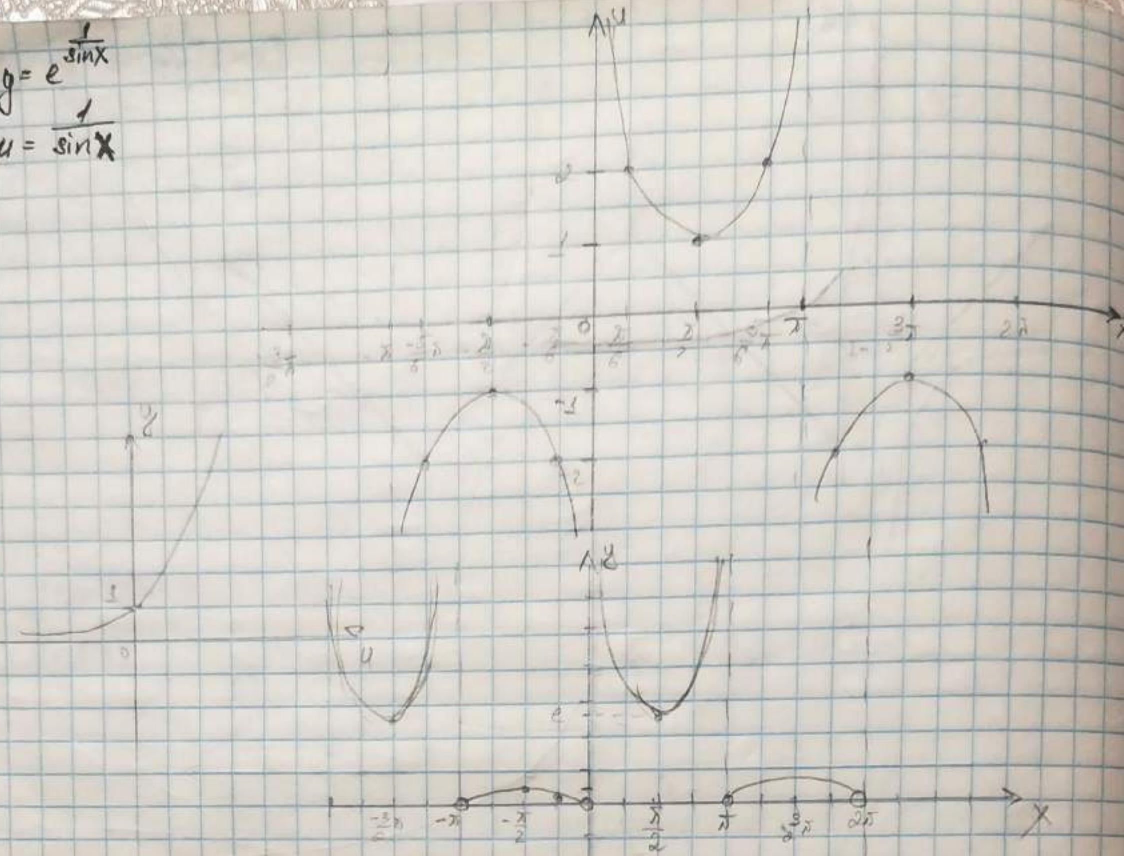






① $y = e^{\frac{1}{\sin x}}$

$u = \frac{1}{\sin x}$



② №824.2 (ж)

$y = \frac{1}{1 - 2^{\frac{x}{1-x}}}$

$x \neq 1$

$\frac{x}{-(x-1)} = \frac{(x-1)+1}{-(x-1)} = -1 - \frac{1}{x-1} = -1 + \frac{1}{1-x}$

? разобрать

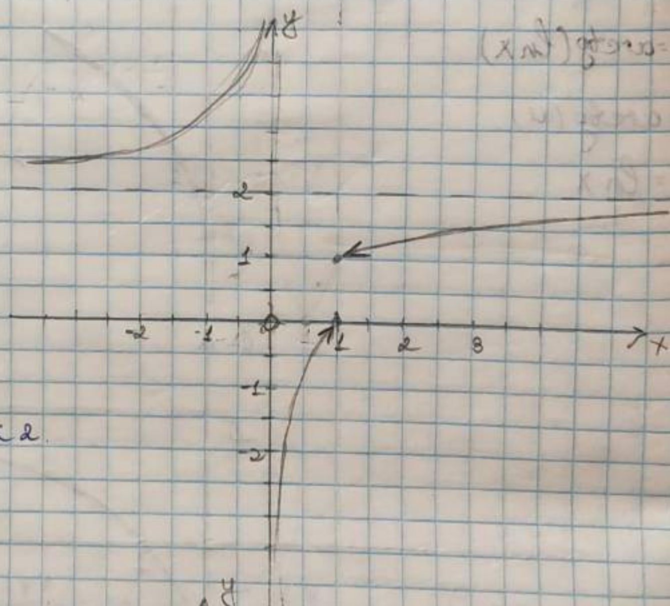
$x \to \pm\infty$

$y = \frac{1}{1 - 2^{-1 - \frac{1}{x-1}}}$

при $x \to \pm\infty$

$y = \frac{1}{1 - 2^{-1}} = \frac{1}{\frac{1}{2}} = 2$

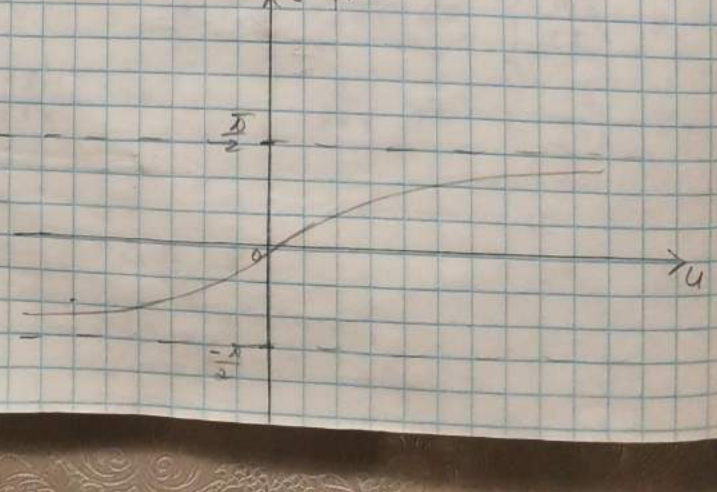
=> y будет стремиться к 2

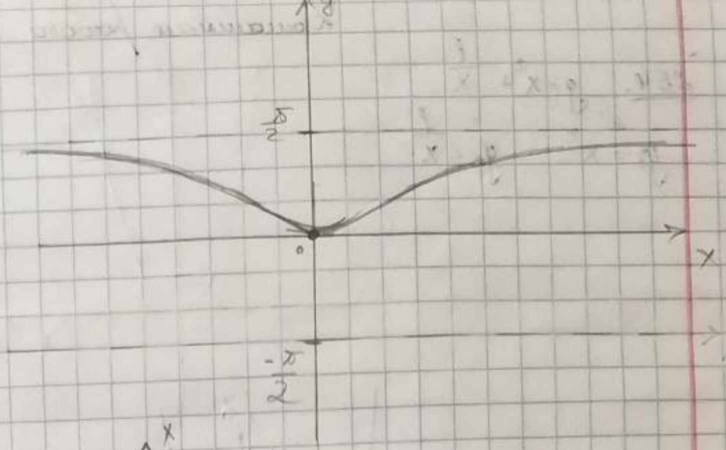


④ $y = arctg(x^2)$

$u = x^2$

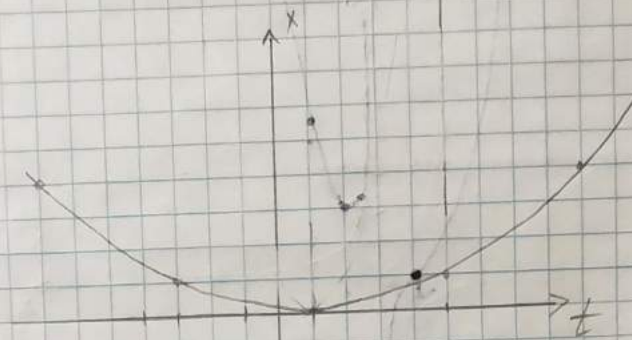
$y = arctg(u)$

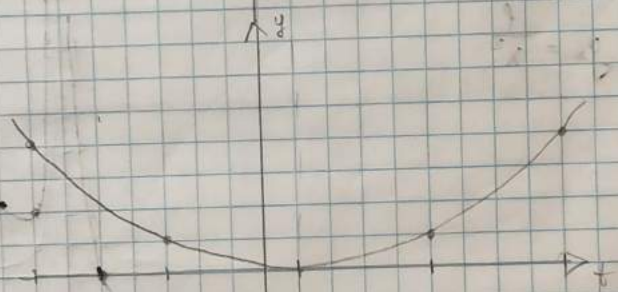


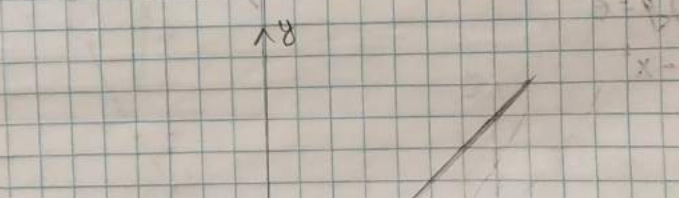


③ $\begin{cases} x = \dfrac{(t-1)^2}{4} \\ y = \dfrac{(t-1)^2}{4} \end{cases}$

$y = x$



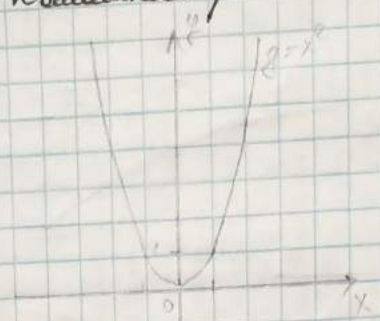


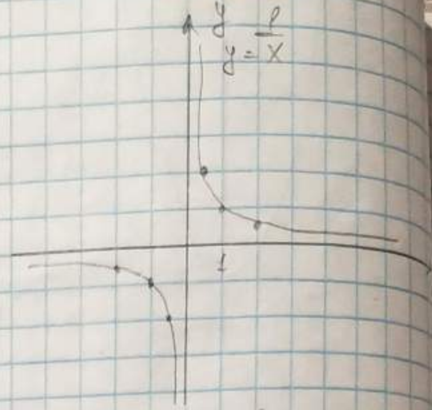


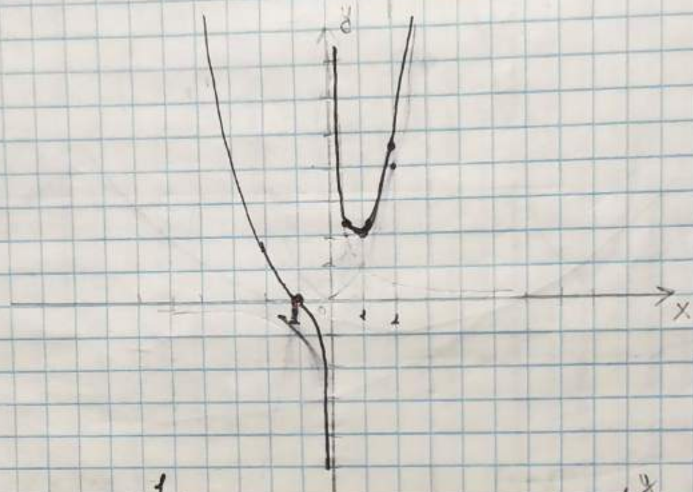
Домашняя работа.

254. $y = x^2 + \frac{1}{x}$

$y_1 = x^2 \qquad y_2 = \frac{1}{x}$

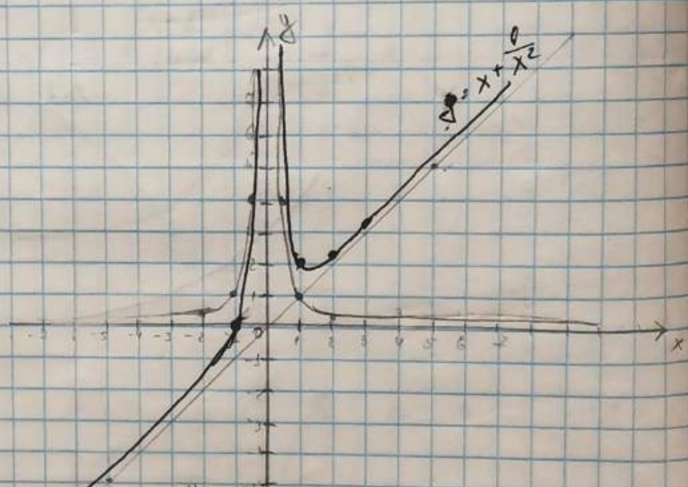






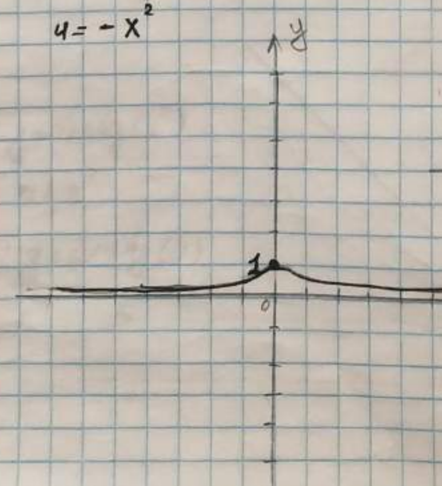
256. $y = x + \frac{1}{x^2}$

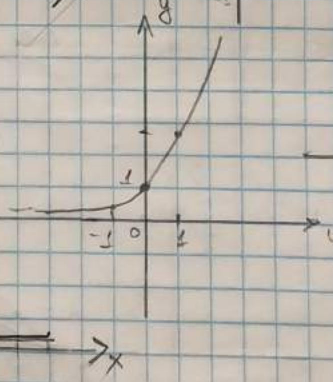
$y_1 = x \qquad y_2 = \frac{1}{x^2}$

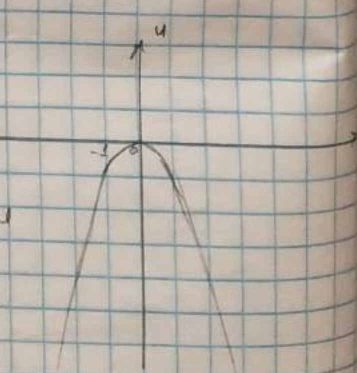


279. б) $y = e^u$

$u = -x^2$

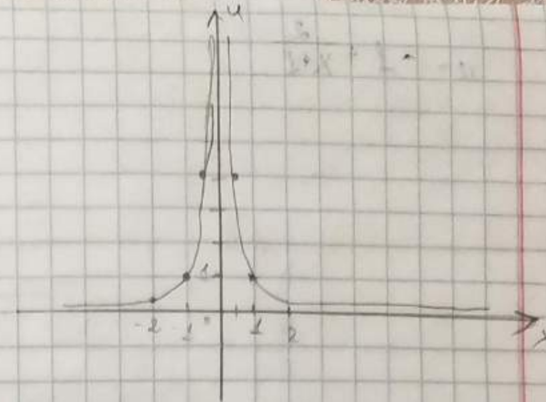


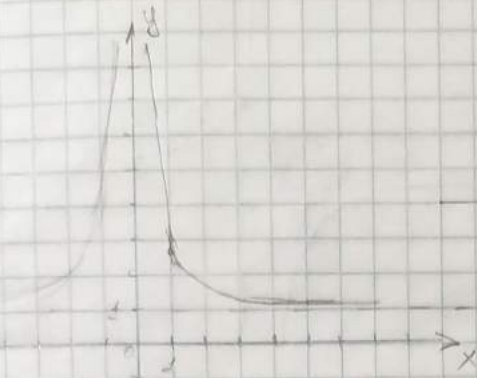




г) $y = e^u$

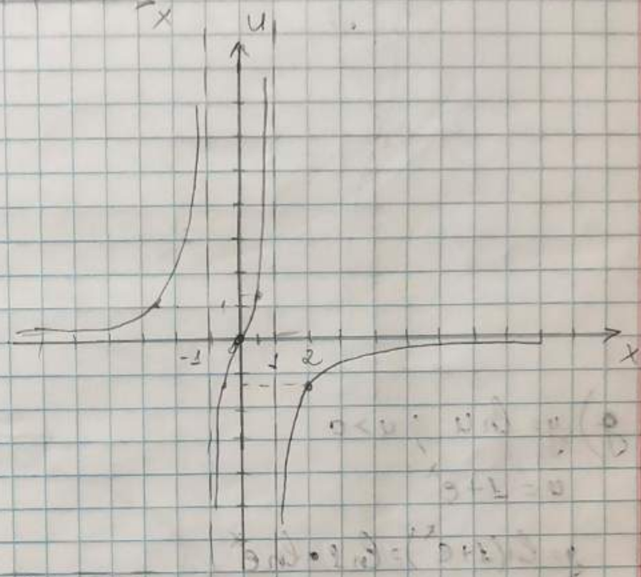
$u = \frac{1}{x^2}$

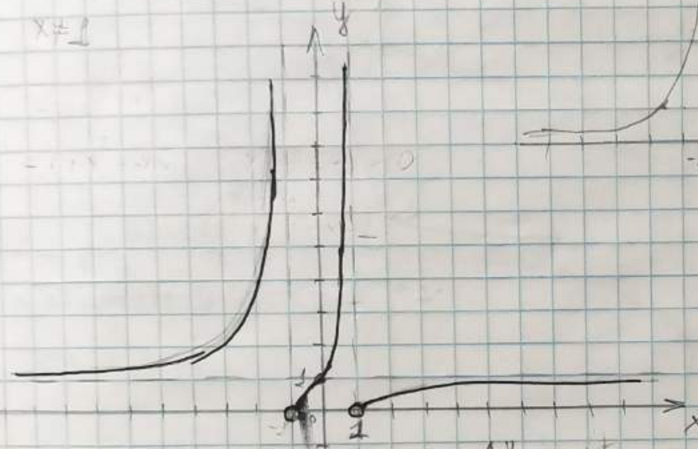




е) $y = e^u$

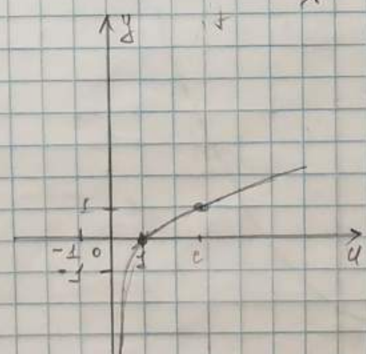
$u = \frac{2x}{1-x^2}$

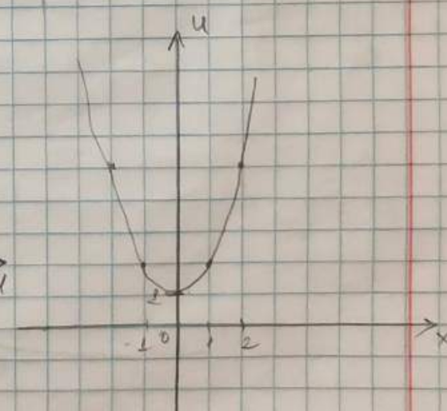


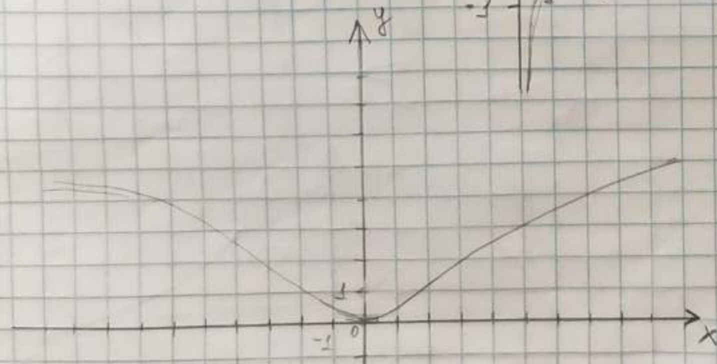


282. а) $y = \ln u \quad ; u > 0$

$u = 1 + x^2$



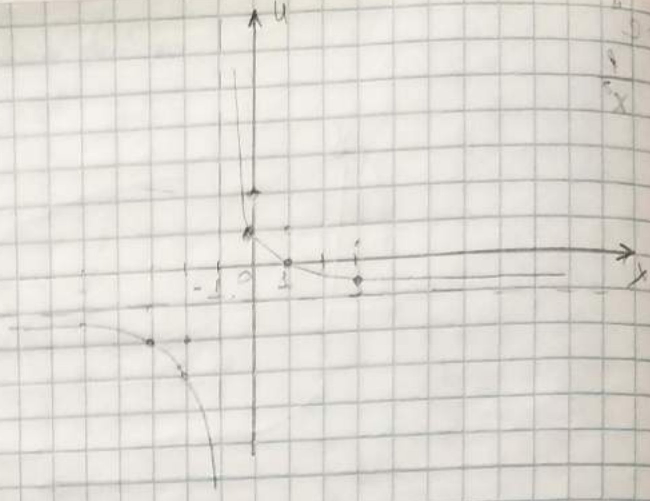


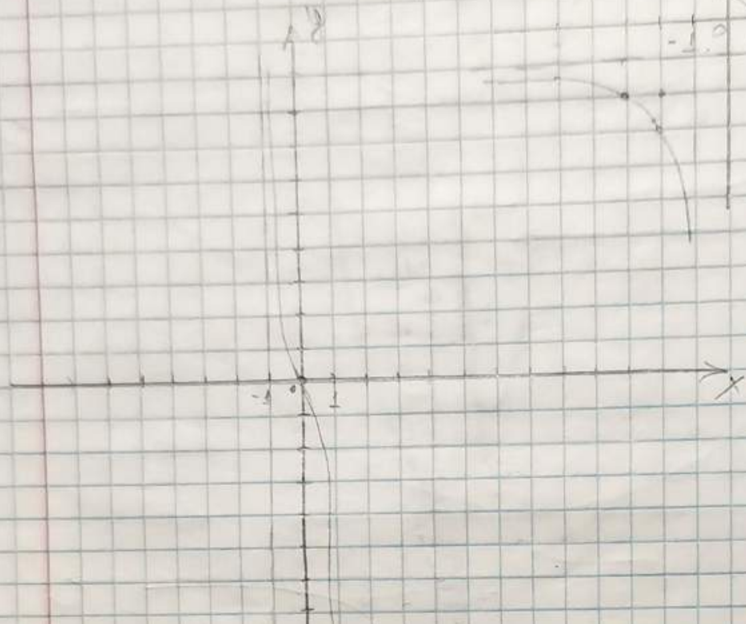


в) $y = \ln u \; ; \; u > 0$

$u = \frac{1-x}{1+x} = \frac{1-x+1-1}{1+x} = \frac{2-(x+1)}{x+1} = -1 + \frac{2}{x+1}$

$u = -1 + \frac{2}{x+1}$

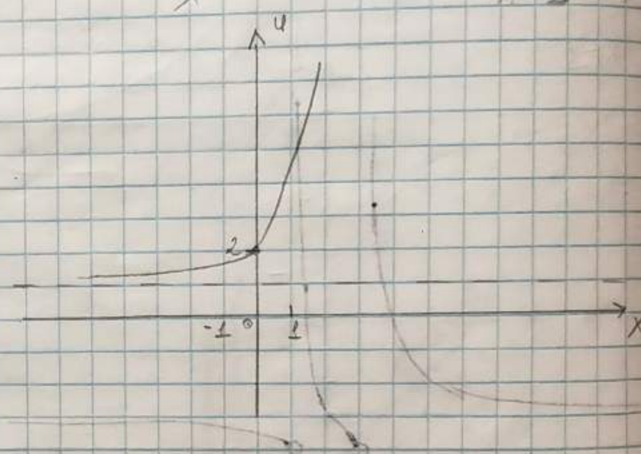


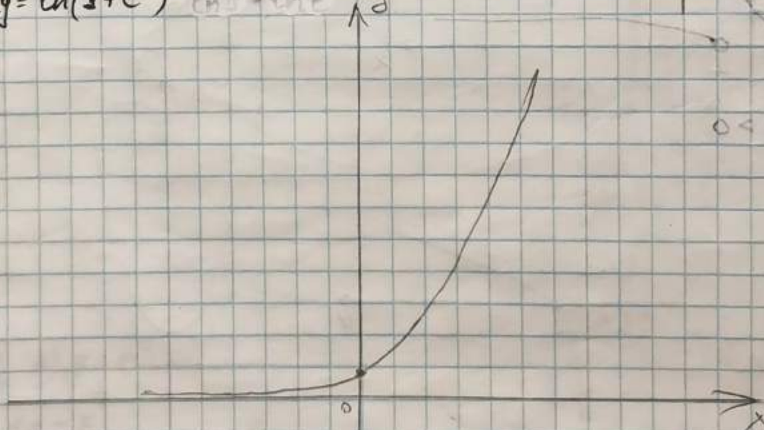


$g)\ y = \ln u\ ;\ u > 0$

$u = 1 + e^x$

$y = \ln(1 + e^x)$



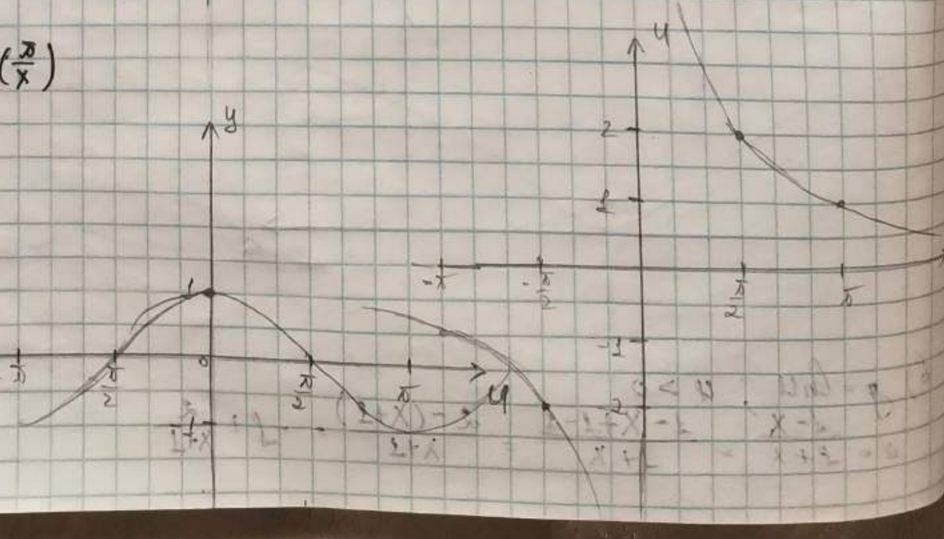


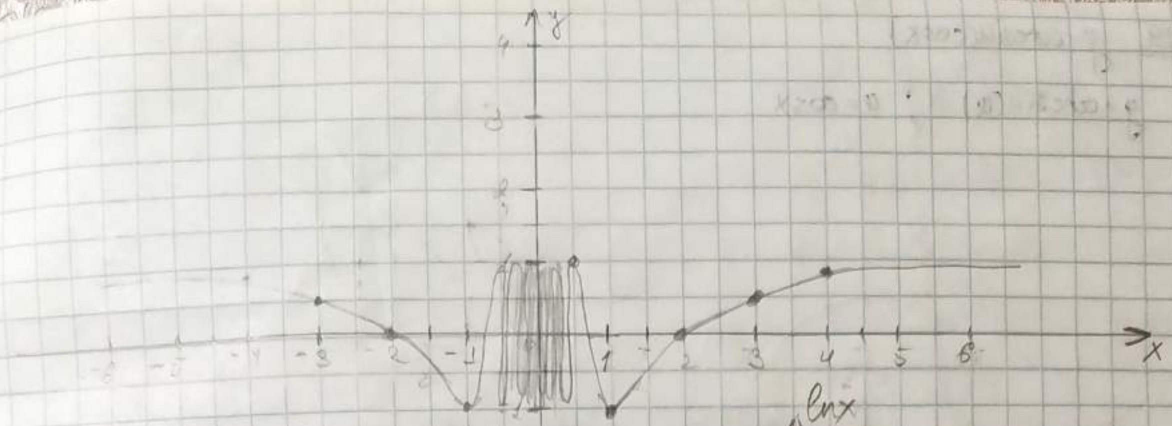
$\ln e^0 =$

300. $y = \cos\left(\frac{\pi}{x}\right)$

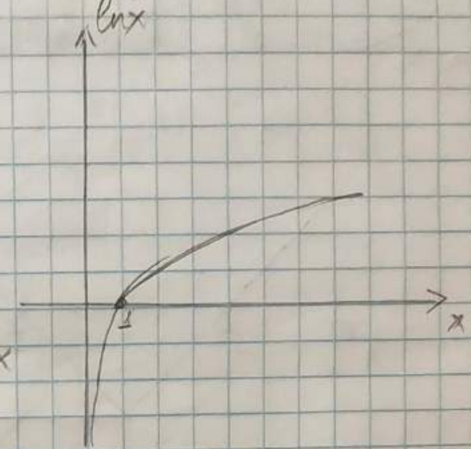
$u = \frac{\pi}{x}$

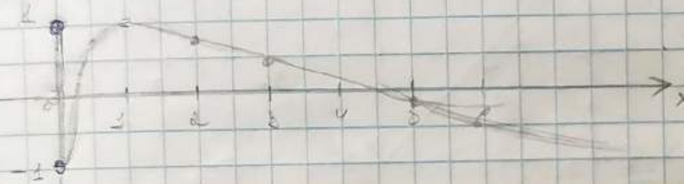
$y = \cos u$





309. $y = \cos(\ln x)$; $x > 0$

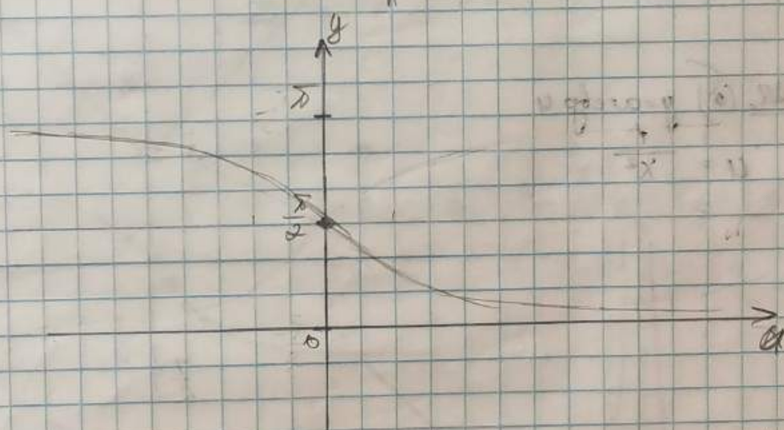


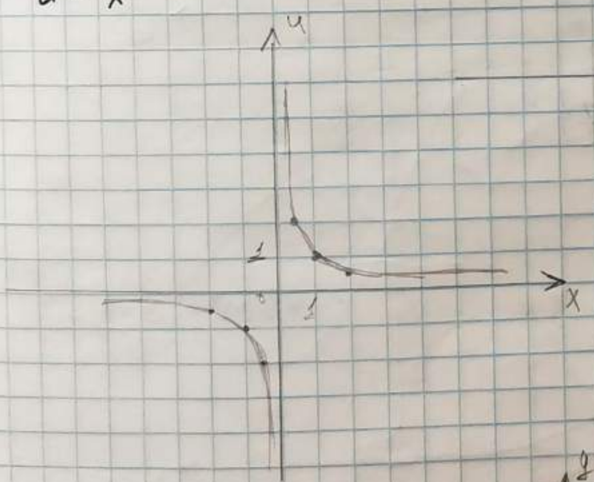


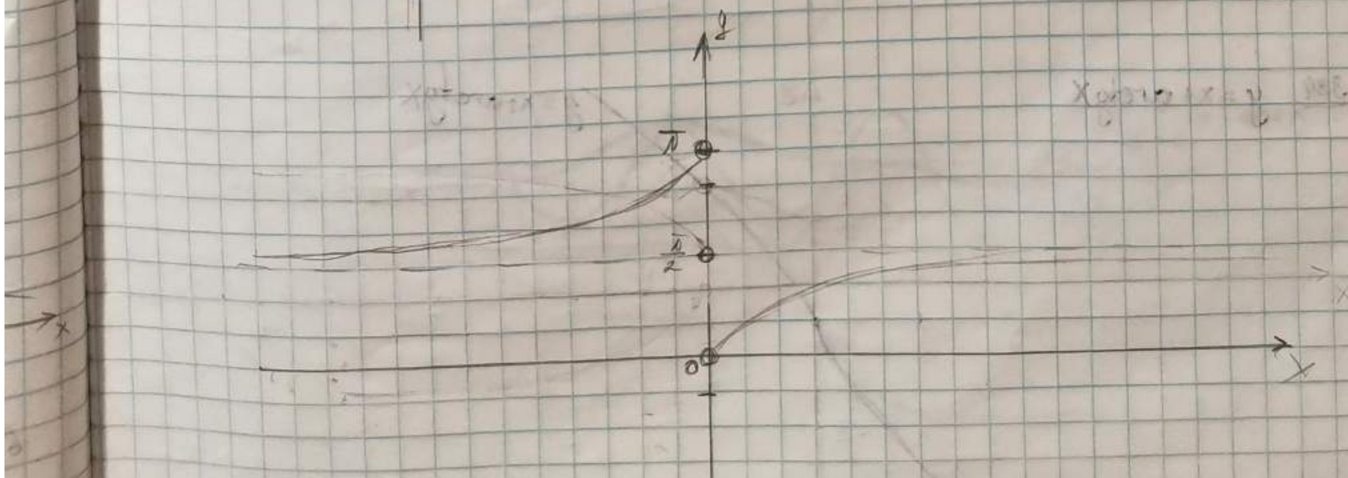
317. $y = \operatorname{arcctg}\left(\frac{1}{x}\right)$

$y = \operatorname{arcctg} u$

$u = \frac{1}{x}$

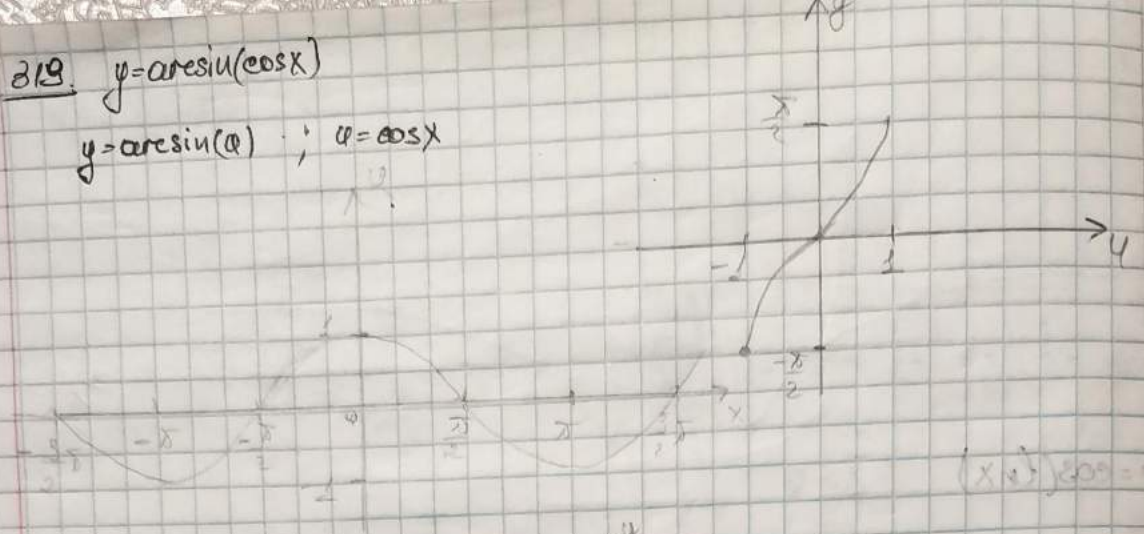


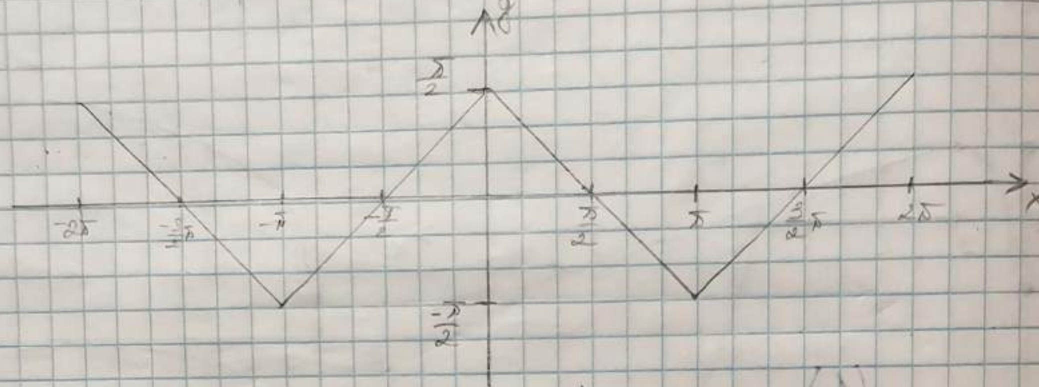




319. $y = \arcsin(\cos x)$

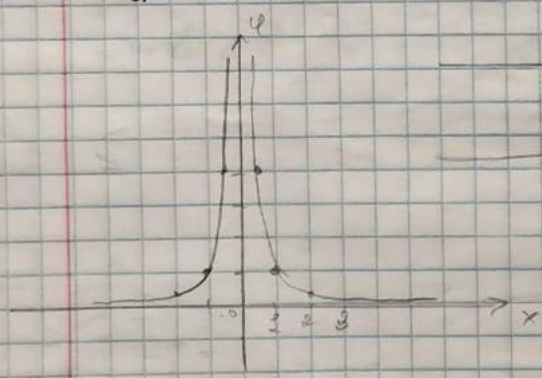
$y = \arcsin(u)$; $u = \cos x$

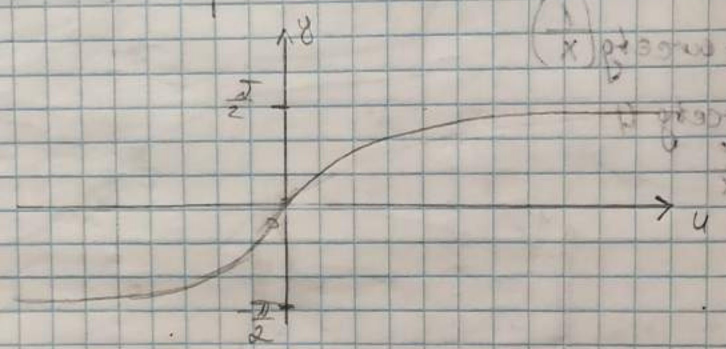


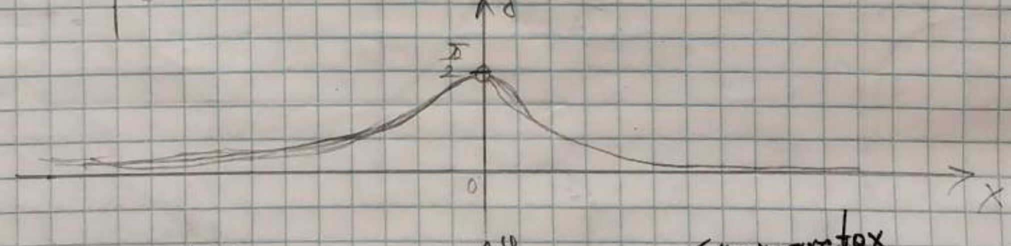


324. (б) $y = \operatorname{arctg} u$

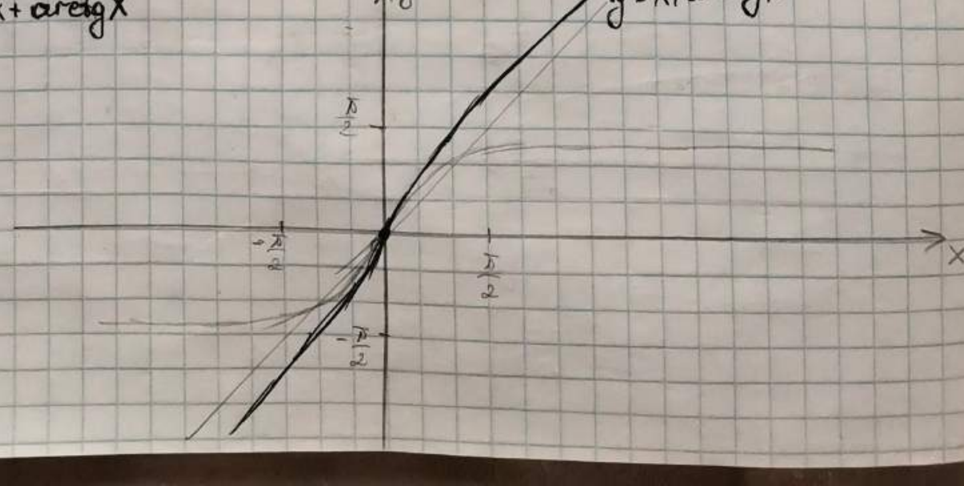
$u = \dfrac{1}{x^2}$





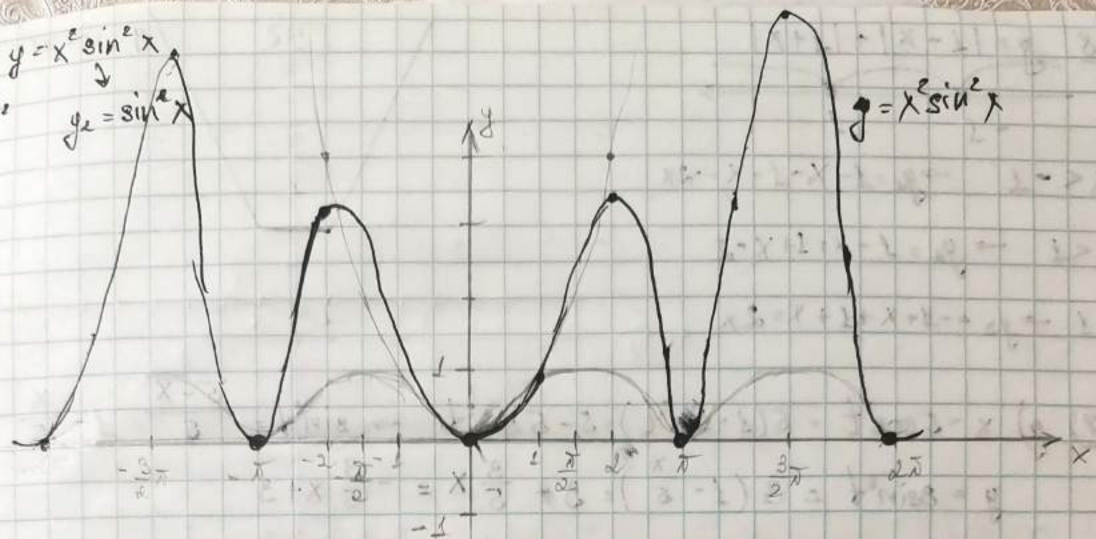


334. $y = x + \operatorname{arctg} x$

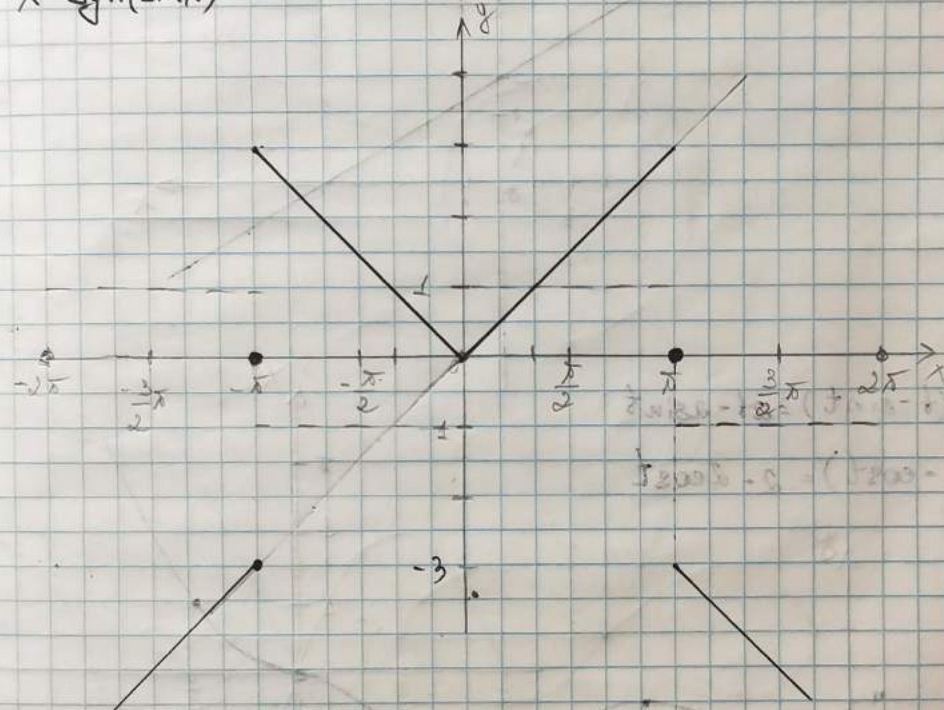


348. $y = x^2 \sin^2 x$

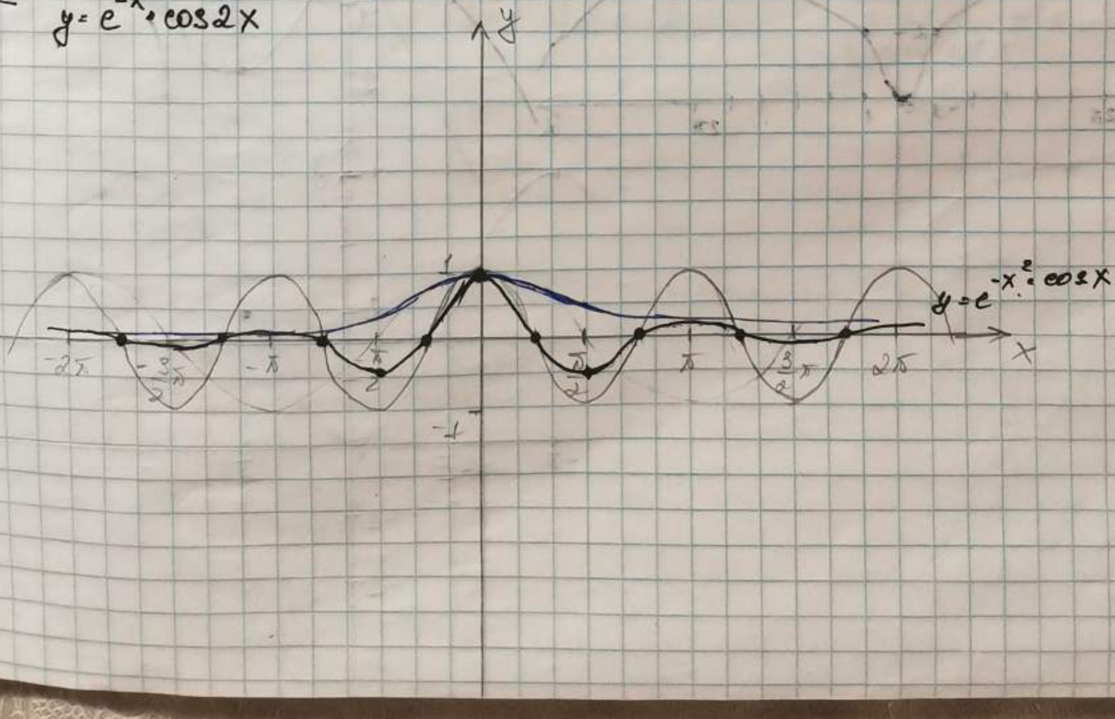
$y_1 = x^2$ $\quad$ $y_2 = \sin^2 x$



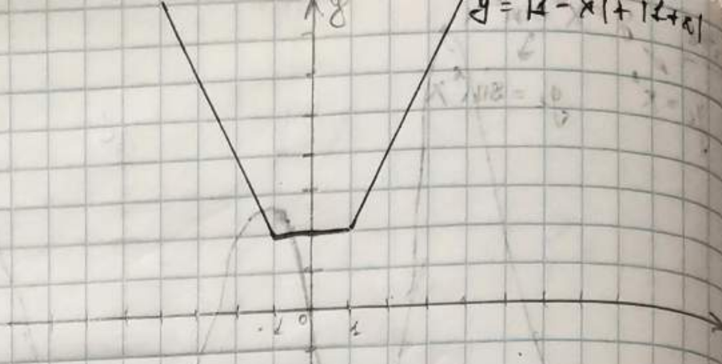
346. $y = x \cdot \mathrm{sgn}(\sin x)$



846. $y = e^{-x^2} \cdot \cos 2x$



338. $y=|1-x|+|1+x|$



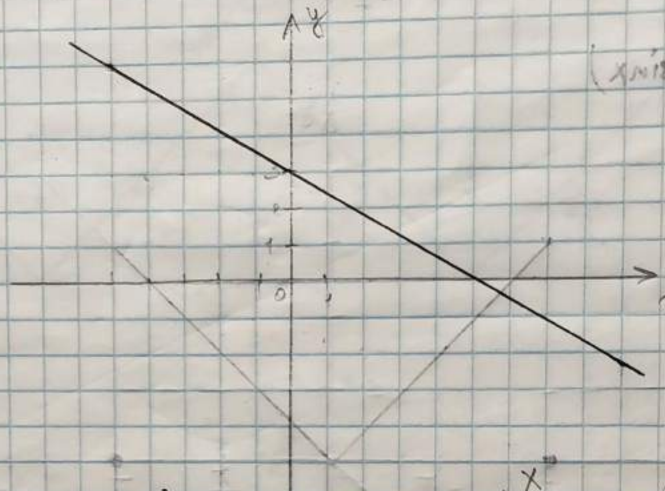
$x<-1 \rightarrow y_1 = 1-x-1-x=-2x$

$-1<x<1 \rightarrow y_2 = 1-x+1+x=2$

$x>1 \rightarrow y_3 = -1+x+1+x=2x$

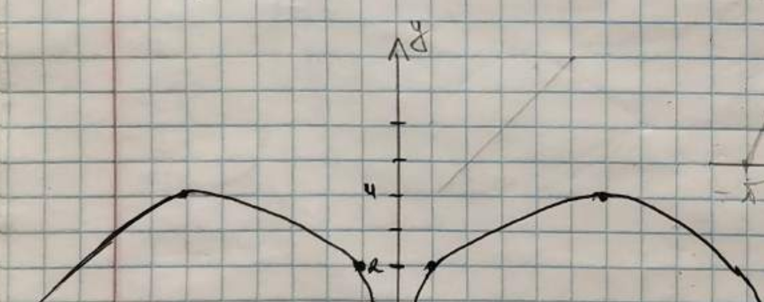
369. g) $x=5\cos^2 t = 5(1-\sin^2 t)=5-5\sin^2 t \rightarrow \sin^2 t = \frac{5-x}{5} = 1-\frac{x}{5}$

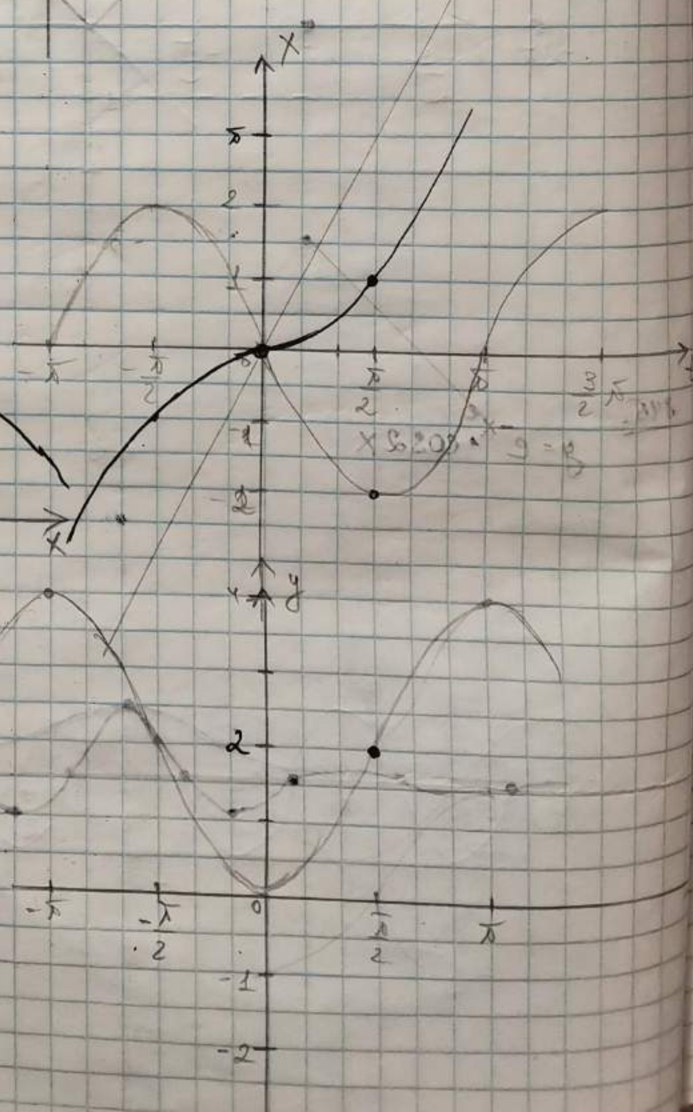
$y = 3\sin^2 t = 3\left(1-\frac{x}{5}\right) = 3-\frac{3}{5}x = -\frac{3}{5}x+3$



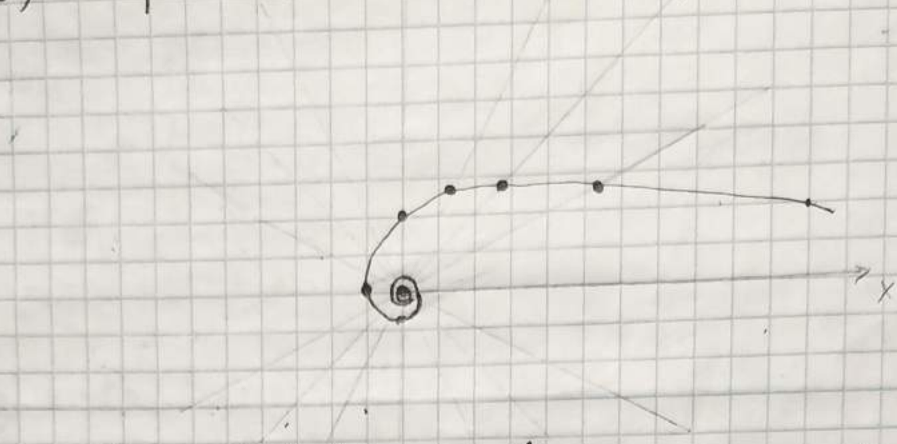
e) $x=2(t-\sin t)=2t-2\sin t$

$y=2(1-\cos t)=2-2\cos t$

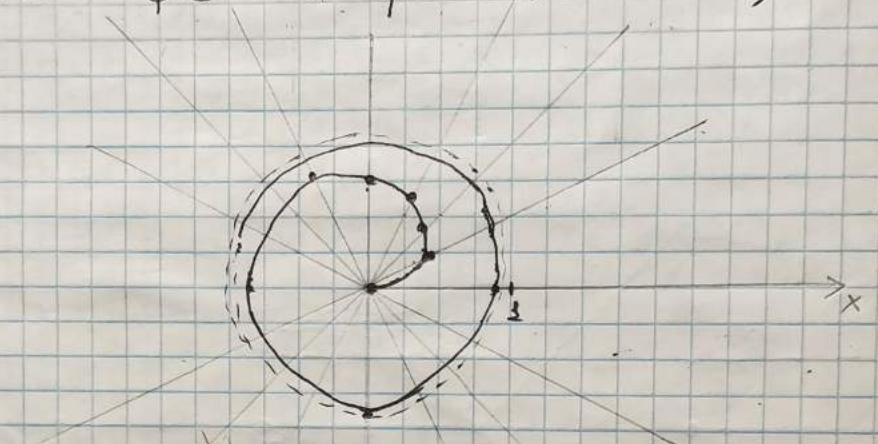




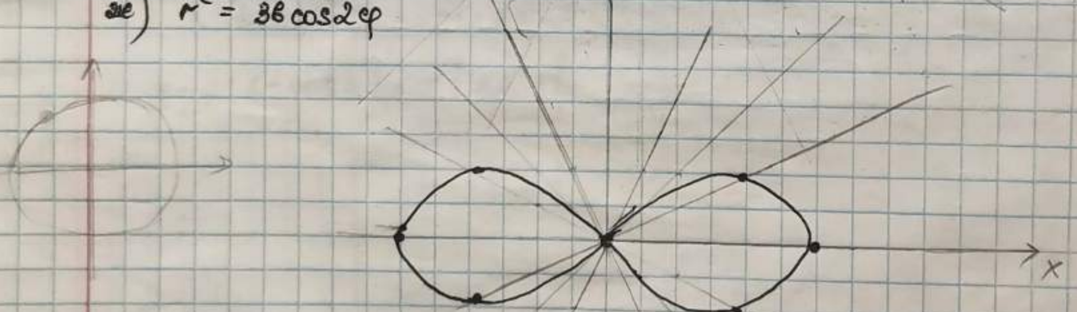
871. б) $r = \frac{\pi}{\varphi}$



в) $r = \frac{\varphi}{\varphi + 1} = \frac{\varphi + 1 - 1}{\varphi + 1} = 1 - \frac{1}{\varphi + 1} \quad (0 \leq \varphi < +\infty)$

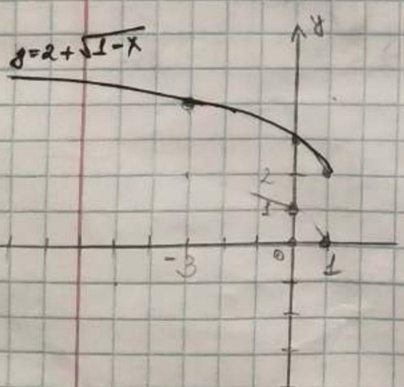


ж) $r^2 = 36\cos 2\varphi$

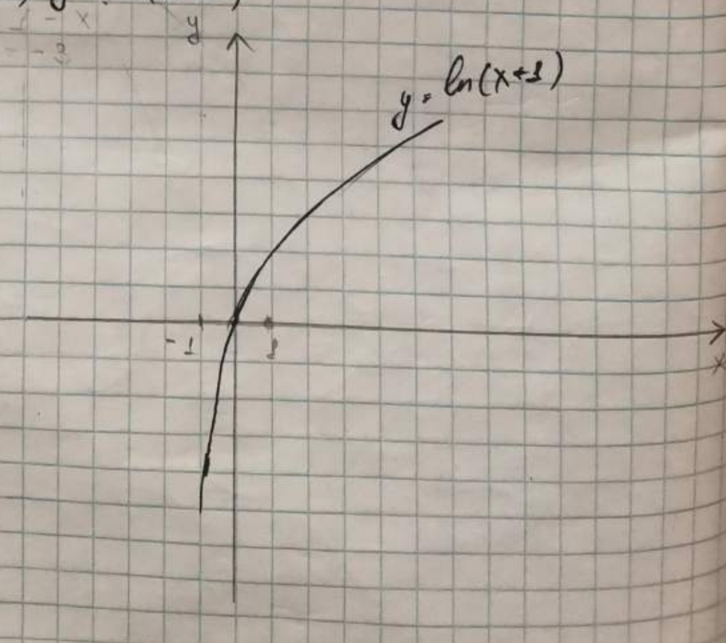


327. а) $y = 2 + \sqrt{1-x}$

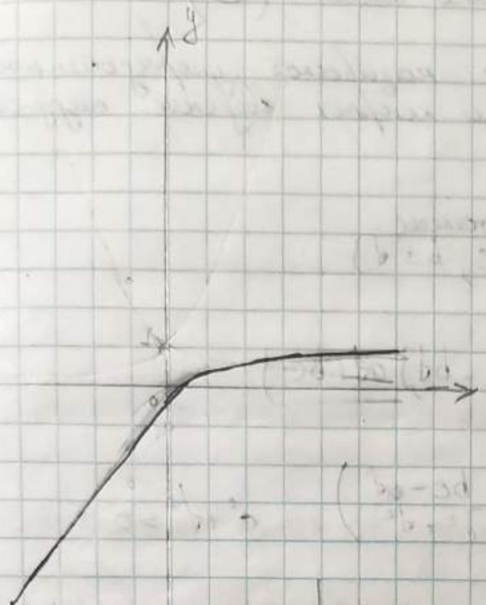
$1 - x \geq 0 \rightarrow x \leq 1$



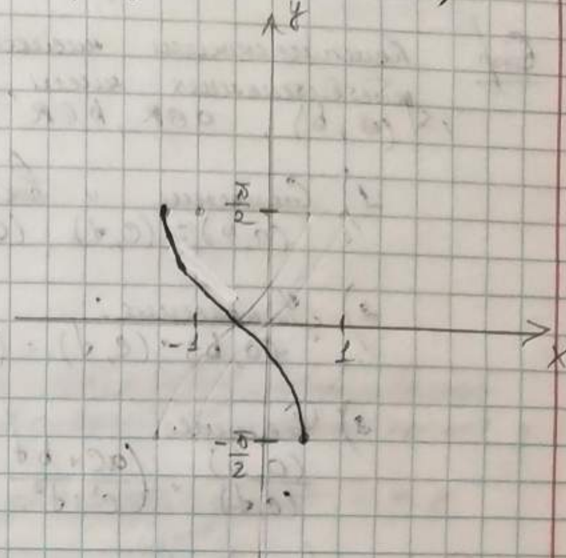
в) $y = \ln(x+1)$



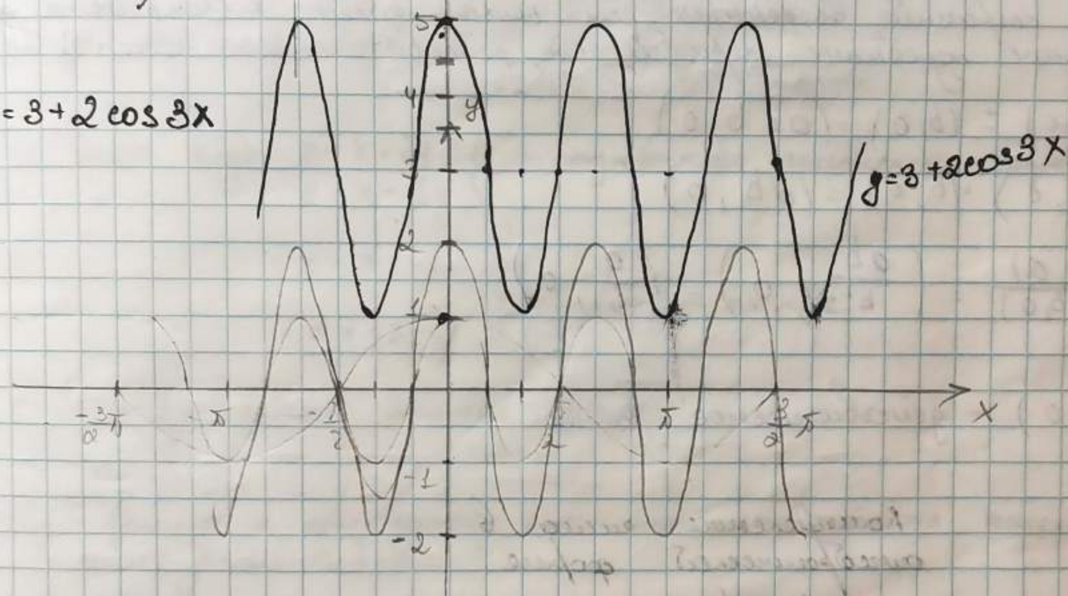
б) $y = 1 - e^{-x}$



г) $y = -\arcsin(x+1)$



д) $y = 3 + 2\cos 3x$



$\to \ ch^2 x = \frac{ch\,2x+1}{2}$, $sh^2 x = \frac{ch\,2x-1}{2}$, $th^2 x = \frac{ch\,2x-1}{ch\,2x+1}$

2.9 Формулы половинных углов следуют из формул понижения степени (2.8), если x заменить на $\frac{x}{2}$

$ch\frac{x}{2} = \sqrt{\frac{ch\,x+1}{2}}$; $sh\frac{x}{2} = \sqrt{\frac{ch\,x-1}{2}}$, $th\frac{x}{2} = \pm\sqrt{\frac{ch\,x-1}{ch\,x+1}}$

① $(2+3i)(4-5i)+(2-3i)(4+5i) = 8+\cancel{12i}-\cancel{10i}-15i^2+8+\cancel{10i}-\cancel{12i}-15i^2 =$
$= 16+15+15 = 46$

② $(x-1-i)(x-1+i)(x+1+i)(x+1-i) = ((x-1)^2-i^2)\cdot((x+1)^2-i^2) =$
$= (x^2-2x+1+1)(x^2+2x+1+1) = (x^2-2x+2)(x^2+2x+2) =$
$= x^4+\cancel{2x^3}+2x^2-\cancel{2x^3}-4x^2-\cancel{4x}+\cancel{2x^2}+\cancel{4x}+4 = x^4+4$

③ $\frac{1+2i}{3+5i} = \frac{(1+2i)\cdot(3-5i)}{(3+5i)\cdot(3-5i)} = \frac{(1+2i)(3-5i)}{34} = \frac{3-5i+6i-10i^2}{34} = \frac{13+i}{34} =$
$= \frac{13}{34}+\frac{i}{34}$

④ $\frac{2+3i}{(2-3i)^2} = \frac{2+3i}{4-12i+9i^2} = \frac{2+3i}{-5-12i} = -\frac{(2+3i)}{(5+12i)} = -\frac{(2+3i)(5-12i)}{25-144i^2} =$
$= -\frac{10-24i+15i-36i^2}{169} = -\frac{46-9i}{169} = -\frac{46}{169}+\frac{9i}{169}$

⑤ $\frac{5i}{(2+i)^3} = \frac{5i}{(4+4i+i^2)(2+i)} = \frac{5i}{(3+4i)(2+i)} = \frac{5i}{6+3i+8i+4i^2} =$
$= \frac{5i}{2+11i} = \frac{5i(2-11i)}{4-121i^2} = \frac{10i-55i^2}{125} = \frac{55+10i}{125} = \frac{11+2i}{25} =$
$= \frac{11}{25}+\frac{2i}{25}$

⑦ $x^2+x+1=0$

$D = 1-4 = -3$; $\sqrt{D} = \sqrt{-3} = \sqrt{-1\cdot 3} = \sqrt{3i^2}$

$x_1 = \frac{-1+\sqrt{3}i}{2} = -\frac{1}{2}+\frac{\sqrt{3}i}{2}$; $x_2 = -\frac{1}{2}-\frac{\sqrt{3}}{2}i$

$x^2+2x\cdot\frac{1}{2}+\frac{1}{4}-\frac{1}{4}+1=0$

$(x+\frac{1}{2})^2+\frac{3}{4}=0$

⑧ x

x

⑨

⑩

⑪

$(x+\frac{1}{2})^2-\frac{3i^2}{4}=0$

$\cancel{x+\frac{1}{2}=\frac{\sqrt{3}i}{2}}$ $\quad (x+\frac{1}{2}-\frac{\sqrt{3}i}{2})(x+\frac{1}{2}+\frac{i\sqrt{3}}{2})=0$

$x_{1,2}=-\frac{1}{2}\pm\frac{\sqrt{3}i}{2}$

1025
625
400

⑧ $x^2-50x+1025=0$

$x^2-2\cdot 25x+625-625+1025=0$

$(x-25)^2+400=0$

$(x-25)^2+20^2=0$

$(x-25)^2-20^2i^2=0$

$(x-25-20i)(x-25+20i)=0$

$\Rightarrow x_{1,2}=25\pm 20i$

⑨ $3x^2-2x+3=0 \quad |:3$

$x^2-2\cdot x\cdot\frac{1}{3}+\frac{1}{9}-\frac{1}{9}+1=0$

$(x-\frac{1}{3})^2+\frac{8}{9}=0$

$(x-\frac{1}{3})^2-(\frac{\sqrt{8}}{3})^2i^2=0$

$(x-\frac{1}{3}-\frac{\sqrt{8}}{3}i)(x-\frac{1}{3}+\frac{\sqrt{8}}{3}i)=0$

$x_{1,2}=\frac{1}{3}\pm\frac{\sqrt{8}}{3}i=\frac{1}{3}\pm\frac{2\sqrt{2}}{3}i$

⑩ $x_{1,2}=\frac{3}{5}\pm\frac{i}{2}$

$\cancel{(x-\frac{3}{5}+\frac{i}{2})(x-\frac{3}{5}-\frac{i}{2})}=\cancel{(x-\frac{3}{5})^2-\frac{i^2}{4}}$

$\cancel{x^2-\frac{2\cdot 3}{5}x+\frac{1}{4}=0}$

$\cancel{x^2-\frac{6}{5}x+\frac{1}{4}=0}$

$$\begin{cases} x_1+x_2=\frac{3}{5}+\frac{i}{2}+\frac{3}{5}-\frac{i}{2}=\frac{6}{5} \\ x_1\cdot x_2=(\frac{3}{5}+\frac{i}{2})(\frac{3}{5}-\frac{i}{2})=\frac{9}{25}-\frac{i^2}{4}=\frac{9}{25}+\frac{1}{4}=0{,}61 \end{cases}$$

$x^2-\frac{6}{5}x+\frac{9}{25}+\frac{1}{4}=0 \quad |\cdot 100$

$100x^2-120x+61=0$

⑪ $z=(3+5i)^2(1-i)=(9+30i+25i^2)(1-i)=(-16+30i)(1-i)=$

$=-16+16i+30i-30i^2=14+46i$

$Re(z)=14 \qquad Im(z)=46$

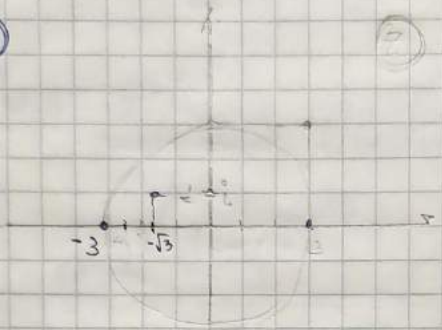
(12) $z = \frac{1+i}{(4+8i)^2} = \frac{1+i}{16+64i+64i^2} = \frac{1+i}{-48+64i} = \frac{(1+i)\cdot(-48-64i)}{(-48+64i)(-48-64i)} =$

$= \frac{-48-64i-48i-64i^2}{2304+4096} = \frac{16-112i}{6400} = \frac{1}{400} - \frac{i\cdot 112}{6400} = \frac{1}{400} - \frac{7i}{400}$

$\cancel{Re(z) = \frac{1}{400}}$; $\cancel{Im(z) = -\frac{1}{64}}$

$Re(z) = \frac{1}{400}$ $\qquad$ $Im(z) = -\frac{7}{400}$

(1)



$3 = 3(\cos 0 + i\sin 0) = 3\cdot e^{i0} = 3e^{i(0+2\pi k)}$

$3e^{2\pi k}$.

$|-3| = |-3|(\cos\pi + i\sin\pi) = 3e^{i\pi} = 3e^{i(\pi+2\pi k)}$, $k\in Z$

(2) $i = 1\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right) = e^{i\cdot\frac{\pi}{2}} = e^{i\left(\frac{\pi}{2}+2\pi k\right)}$

(3) $-\sqrt{3}+i = \sqrt{3+1}\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right) = 2\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right) = 2\cdot e^{i\frac{5\pi}{6}}$

$= 2\cdot e^{i\left(\frac{5}{6}\pi+2\pi k\right)}$, $k\in Z$

(4) $4\left(\cos\frac{\pi}{3} - i\sin\frac{\pi}{3}\right) = 4\left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right) = 2-2\sqrt{3}i$

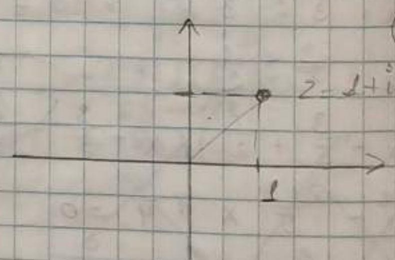
$4\left(\cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)\right) = 4\cdot e^{i\left(-\frac{\pi}{3}+2\pi k\right)}$, $k\in Z$

(5) $\left(\frac{1+i}{2}\right)^{10} = \frac{(1+i)^{10}}{2^{10}} = \frac{2^5 i}{2^{10}} = \frac{i}{2^5} = \frac{i}{32}$

$z = 1+i$

$|z| = \sqrt{1+1} = \sqrt{2}$

$z = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$



$z^{10} = (\sqrt{2})^{10}\left(\cos\frac{10\pi}{4} + i\sin\frac{10\pi}{4}\right) =$

$= (\sqrt{2})^{10}\left(\underbrace{\cos\frac{5}{2}\pi}_{0} + \underbrace{i\sin\frac{5}{2}\pi}_{i}\right) = 2^5 i = 32i$

Д/З: (1) $(3+4i)+(-5+3i) = -2+7i$

(2) $(\sqrt{3}-i)(\sqrt{2}+i\sqrt{3}) = \sqrt{6}+3i-i\sqrt{2}-\sqrt{3}i^2 = \sqrt{6}+\sqrt{3}+3i-i\sqrt{2} =$

$= \sqrt{3}(\sqrt{2}+1) + i(3-\sqrt{2})$

(3) $(4-5i)(4+5i) = 16-25i^2 = 16+25 = 41$.

(4) $\frac{-2+i}{1+3i} = \frac{(-2+i)(1-3i)}{(1-3i)(1+3i)} = \frac{-2+6i+i-3i^2}{1-9i^2} = \frac{1+7i}{10} = \frac{1}{10} + \frac{7}{10}i$

⑦ $x^2+25=0$

$x^2=-25$

$x=\pm\sqrt{-25}$

$x=\pm\sqrt{25i^2} \quad \rightarrow \quad x_{1,2}=\pm 5i$

⑧ $x^4+5x^2+4=0$

$x^2=t$

$t^2+5t+4=0$

$t^2+2\cdot\frac{5}{2}t+\frac{25}{4}-\frac{25}{4}+\frac{16}{4}=0$

$(t+\frac{5}{2})^2-\frac{9}{4}=0$

$(t+\frac{5}{2}-\frac{3}{2})(t+\frac{5}{2}+\frac{3}{2})=0$

$(t+1)(t+4)=0$

$\rightarrow t=-1 \qquad t=-4$

$x_{1,2}=\pm\sqrt{i^2} \qquad x_{3,4}=\pm\sqrt{4i^2}$

$x_{1,2}=\pm i \qquad x_{3,4}=\pm 2i$

Ответ: $x_{1,2}=\pm i$; $x_{3,4}=\pm 2i$

⑨ $(1+i)x+(1+2i)y=1+5i$

$x+xi+y+2yi=1+5i$

$x+y+xi+2yi=1+5i$

$(x+y)+(x+2y)i=1+5i$

$$-\begin{cases} x+y=1 \\ x+2y=5 \end{cases} \qquad \begin{matrix} -y=-4 \rightarrow y=4 \\ x=-3 \end{matrix}$$

Ответ: $x=-3$; $y=4$

⑩ $z=(1+2i)^2(2-3i)^3=(1+4i+4i^2)(4-12i+9i^2)(2-3i)=$

$=(4i-3)(2-3i)(-5-12i)=(8i-12i^2-6+9i)(-5-12i)=$

$=(6+17i)(-5-12i)=-30-72i-85i-204i^2=174-157i$

$Re\ z=174$; $Im=-157$

⑪ $z=\frac{a+bi}{c+di}=\frac{(a+bi)(c-di)}{c^2+d^2}=\frac{ac-adi+bci+bd}{c^2+d^2}=\frac{ac+bd}{c^2+d^2}+\frac{bc-ad}{c^2+d^2}i$

$Re\ z=\frac{ac+bd}{c^2+d^2}$; $Im=\frac{bc-ad}{c^2+d^2}i$

Д/З (1)

$-i = 1\left(\cos\frac{3}{2}\pi + i\sin\frac{3}{2}\pi\right) = e^{i\left(\frac{3}{2}\pi\right)} =$

$= e^{i\left(\frac{3}{2}\pi + 2\pi k\right)}, \ k \in Z$

(2) $1+i = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) = e^{i\frac{\pi}{4}}\cdot\sqrt{2} =$

$= \sqrt{2}\cdot e^{i\left(\frac{\pi}{4}+2\pi k\right)}, \ k \in Z$

(3) $1-i = \sqrt{2}\left(\cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4}\right) = e^{i\frac{7\pi}{4}}\cdot\sqrt{2} =$

$= \sqrt{2}\cdot e^{i\left(\frac{7}{4}\pi+2\pi k\right)}, \ k \in Z$

(4) $-1-i = \sqrt{2}\left(\cos\frac{5}{4}\pi + i\sin\frac{5}{4}\pi\right) = \sqrt{2}\cdot e^{i\frac{5}{4}\pi} = \sqrt{2}\cdot e^{i\left(\frac{5}{4}\pi+2\pi k\right)}, \ k\in Z$

(5) $\left((2+2i)(\sqrt{3}+i)\right)^{10} =$

$\cancel{(2+2i)(\sqrt{3}+i) = 2\sqrt{3}+2i}$

$(2+2i)^{10}(\sqrt{3}+i)^{10} = (2\sqrt{2})^{10}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)^{10}\cdot(2)^{10}\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)^{10} =$

$= (4\sqrt{2})^{10}\left(\cos\frac{10\pi}{4}+i\sin\frac{10\pi}{4}\right)\left(\cos\frac{10\pi}{6}+i\sin\frac{10\pi}{6}\right) =$

$= (4\sqrt{2})^{10}\left(\cos\frac{5\pi}{2}+i\sin\frac{5\pi}{2}\right)\left(\cos\frac{5\pi}{3}+i\sin\frac{5\pi}{3}\right) =$

$= (4\sqrt{2})^{10}(0+i)\left(\frac{1}{2}+\left(-\frac{\sqrt{3}}{2}\right)i\right) = 4^{10}\cdot 2^{5}\, i\left(\frac{1}{2}-\frac{\sqrt{3}}{2}i\right) =$

$= 2^{24}(\sqrt{3}+i)$

(6) $\left(\frac{2+2i}{\sqrt{3}+i}\right)^{10} = \frac{(2+2i)^{10}}{(\sqrt{3}+i)^{10}} = \frac{(2\sqrt{2})^{10}\left(\cos\frac{10\pi}{4}+i\sin\frac{10\pi}{4}\right)}{2^{10}\left(\cos\frac{5\pi}{3}+i\sin\frac{5\pi}{3}\right)} =$

$= \frac{\cancel{2^{10}}\cdot 2^{5}\cdot i}{\cancel{2^{10}}\left(\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)} = \frac{2^{5} i}{\frac{1}{2}-\frac{\sqrt{3}}{2}i} = \frac{32i\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)}{\frac{1}{4}-\frac{3}{4}i^{2}} =$

$= \frac{16i+16\sqrt{3}i^{2}}{1} = 16i - 16\sqrt{3} = 2^{4}(-\sqrt{3}+i)$

(7) $\sqrt[3]{-2+2i}$

$\sqrt[n]{z} = \sqrt[n]{|z|}\cdot e^{\frac{i(\varphi+2\pi k)}{n}}$

$-2+2i = 2\sqrt{2}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right) = 2\sqrt{2}\cdot e^{i\left(\frac{\pi}{4}+2\pi k\right)}$

$k = 0, 1, 2$

$z_1 = 2\sqrt{2}\cdot e^{i\frac{\pi}{4\cdot 3}} =$

$-1 = \sqrt[3]{-1} = (-1)^{\frac{1}{3}} = (-1)^{\frac{2}{6}} = \sqrt[6]{(-1)^{2}} = 1$

⑤ $\sqrt[6]{64}$

$z_1 = \sqrt[6]{2^6} \cdot e^0 = 2$

$z_2 = 2 \cdot e^{i\left(\frac{2\pi}{6}\right)} = 2 \cdot e^{i\frac{\pi}{3}} = 2 \cdot \left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right) = 1 + \sqrt{3}\, i$

$z_3 = 2 \cdot e^{\frac{4\pi}{6} i} = 2 \cdot e^{\frac{2}{3}\pi i} = 2 \cdot \left(\cos\frac{2}{3}\pi + i\sin\frac{2\pi}{3}\right) = -1 + \sqrt{3}\, i$

$z_4 = 2 \cdot e^{i\pi} = 2 \cdot (\cos\pi + i\sin\pi) = -2$

$z_5 = 2 \cdot e^{\frac{8\pi}{6} i} = 2\left(\cos\frac{4}{3}\pi + i\sin\frac{4}{3}\pi\right) = -1 - \sqrt{3}\, i$

$z_6 = 2 \cdot e^{\frac{10\pi}{6} i} = 2\left(\cos\frac{5}{3}\pi + i\sin\frac{5}{3}\pi\right) = 1 - \sqrt{3}\, i$

⑥ $\left(\frac{1+i\sqrt{3}}{1-i}\right)^{20} = \frac{\left((1+i\sqrt{3})(1+i)\right)^{20}}{2^{20}} = \frac{2^{20}\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)^{20}(\sqrt{2})^{20}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)^{20}}{2^{20}} =$

$\cancel{= 2^{10}\left(\frac{\sqrt{3}}{2} + \frac{1}{2} i\right)\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} i\right) = \left(2^9\sqrt{3} + 2^9 i\right)\left(\frac{\sqrt{2}}{2} + \sqrt{2}\right.}$

?

$= \left(\cos\frac{20\pi}{6} + i\sin\frac{20\pi}{6}\right) 2^{10}\left(\cos\frac{20\pi}{4} + i\sin\frac{20\pi}{4}\right) =$

$= 2^{10}(-1+0)\left(\cos\frac{10\pi}{3} + i\sin\frac{10\pi}{3}\right) = -2^{10}\left(\cos\left(3\pi + \frac{\pi}{3}\right) + i\sin\left(3\pi + \frac{\pi}{3}\right)\right) =$

$= -2^{10}\left(-\frac{1}{2} - \frac{\sqrt{3}}{2} i\right) = 2^9 + 2^9\sqrt{3}\, i$

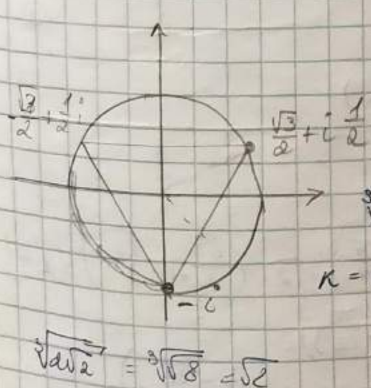
⑦ $i = 1\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right) = e^{i\left(\frac{\pi}{2} + 2\pi k\right)} = e^{i\frac{\pi}{2}}$

$\sqrt[3]{i} = 1 \cdot e^{\frac{i\left(\frac{\pi}{2} + 2\pi k\right)}{3}}$

$k = 0, 1, 2.$ $\to k = 0 \to z_1 = e^{\frac{\pi}{6} i} = \left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2} + \frac{1}{2} i = \frac{\sqrt{3}+i}{2}$

$k = 1 \to z_2 = e^{\frac{5}{6}\pi i} = \cos\frac{5}{6}\pi + i\sin\frac{5}{6}\pi = -\frac{\sqrt{3}}{2} + \frac{1}{2} i$

$k = 2 \to z_3 = e^{\frac{9}{6}\pi i} = e^{\frac{3}{2}\pi i} = \cos\frac{3}{2}\pi + i\sin\frac{3}{2}\pi = -i$



⑧ $2 - 2i = 2\sqrt{2}\left(\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right) = 2\sqrt{2} \cdot e^{i\left(-\frac{\pi}{4} + 2\pi k\right)}$

$\sqrt[3]{2-2i} = \sqrt{2} \cdot e^{\frac{i\left(-\frac{\pi}{4} + 2\pi k\right)}{3}}$

$k = 0 \to z_1 = \sqrt{2} \cdot e^{i\left(-\frac{\pi}{12}\right)} = \sqrt{2} \cdot \left(\cos\left(-\frac{\pi}{12}\right) + i\sin\left(-\frac{\pi}{12}\right)\right) =$

$= \sqrt{2} \cdot \left(\frac{\sqrt{6}+\sqrt{2}}{4} + \frac{\sqrt{2}-\sqrt{6}}{4} i\right) = \frac{2\sqrt{3}+2}{4} + \frac{2-2\sqrt{3}}{4} i =$

$= \frac{\sqrt{3}+1}{2} + \frac{1-\sqrt{3}}{2} i$

$\sqrt[3]{2\sqrt{2}} = \sqrt[3]{\sqrt{8}} = \sqrt{2}$

$k=1 \to z_2 = \sqrt{2}\cdot e^{\frac{i(-\frac{\pi}{4}+2\pi)}{3}} = \sqrt{2}\cdot e^{i\frac{7}{12}\pi} = \sqrt{2}\left(\cos\left(\frac{\pi}{2}+\frac{\pi}{12}\right) + i\sin\left(\frac{\pi}{2}+\frac{\pi}{12}\right)\right)$

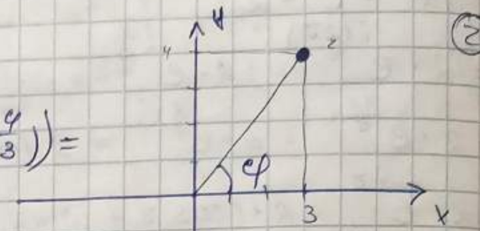
$= \sqrt{2}\left(-\sin\frac{\pi}{12} + i\cos\frac{\pi}{12}\right) = \sqrt{2}\left(\frac{\sqrt{2}-\sqrt{6}}{4} + i\frac{\sqrt{6}+\sqrt{2}}{4}\right) =$

$= \frac{2-2\sqrt{3}}{4} + \frac{2\sqrt{3}+2}{4}i = \frac{1-\sqrt{3}}{2} + \frac{\sqrt{3}+1}{2}i$

$k=2 \to z_3 = \sqrt{2}\cdot e^{\frac{i(-\frac{\pi}{4}+4\pi)}{3}} = \sqrt{2}\cdot e^{\frac{5}{4}\pi i} = \sqrt{2}\left(\cos\frac{5}{4}\pi + i\sin\frac{5}{4}\pi\right) =$

$= \sqrt{2}\left(-\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}\right) = -1-i$

(10) $\sqrt{3+4i}$



$|3+4i| = 5\left(\cos\left(\arctg\frac{4}{3}\right) + i\sin\left(\arctg\frac{4}{3}\right)\right) = 5\cdot e^{i(\arctg\frac{4}{3}+2\pi k)}$

$\varphi = \arctg\frac{4}{3}$

$\sqrt{z} = \sqrt{5}\cdot e^{\frac{i(\arctg\frac{4}{3}+2\pi k)}{2}}, \quad k=0,1.$

$k=0 \to z_1 = \sqrt{5}\cdot e^{\frac{i(\arctg\frac{4}{3})}{2}} = \sqrt{5}\cdot\left(\cos\left(\frac{\arctg\frac{4}{3}}{2}\right) + i\sin\left(\frac{\arctg\frac{4}{3}}{2}\right)\right)$

Пусть $\arctg\frac{4}{3} = \alpha \to \tg\alpha = \frac{4}{3}$

$1+\tg^2\alpha = \frac{1}{\cos^2\alpha}$ $\quad 1+\frac{16}{9} = \frac{1}{\cos^2\alpha}$ $\quad \cos\alpha = \frac{3}{5}$

$\cos^2\frac{\alpha}{2} = \frac{1+\cos\alpha}{2}$

$\to \cos\frac{\alpha}{2} = \frac{2}{\sqrt{5}} \to \sin\frac{\alpha}{2} = \frac{1}{\sqrt{5}}$

$z_1 = \sqrt{5}\left(\frac{2}{\sqrt{5}} + i\frac{1}{\sqrt{5}}\right) = 2+i$

$\sqrt{3+4i} = x+iy$

$3+4i = x^2+2xyi-y^2$

$\begin{cases} x^2-y^2=3 \\ 2xy=4 \end{cases} \iff \begin{cases} x^2-y^2=3 \\ xy=2 \end{cases}$ $\quad \begin{cases} x=2 \\ y=1 \end{cases}$ $\quad \begin{cases} x=-2 \\ y=-1 \end{cases}$

$z_1 = 2+i$; $z_2 = -2-i$

(7) $\sqrt[3]{-2+2i}$

$-2+2i = 2\sqrt{2}\left(\cos\frac{3}{4}\pi + i\sin\frac{3}{4}\pi\right) = 2\sqrt{2}\cdot e^{i(\frac{3}{4}\pi+2\pi k)}$

$\sqrt[3]{-2+2i} = \sqrt{2}\cdot e^{\frac{i(\frac{3}{4}\pi+2\pi k)}{3}}, \quad k=0,1,2.$

$k=0 \to z_1 = \sqrt{2}\cdot e^{i\frac{3\pi}{12}} = \sqrt{2}\cdot e^{i\frac{\pi}{4}} = \sqrt{2}\cdot\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) = \sqrt{2}\cdot\left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right) = 1+i$

$k=1 \to z_2 = \sqrt{2}\cdot e^{i\frac{11}{12}\pi} = \sqrt{2}\cdot\left(\cos\left(\pi-\frac{\pi}{12}\right) + i\sin\left(\pi-\frac{\pi}{12}\right)\right) =$

$= \sqrt{2}\left(-\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}\right) = \sqrt{2}\cdot\left(\frac{-\sqrt{6}-\sqrt{2}}{4} + i\frac{\sqrt{6}-\sqrt{2}}{4}\right) =$

$= \frac{-2\sqrt{3}-2}{4} + \frac{i(2\sqrt{3}-2)}{4} = -\frac{\sqrt{3}+1}{2} + i\frac{\sqrt{3}-1}{2}$

$k=2 \Rightarrow z_3 = \sqrt{2}\cdot e^{i\frac{19}{12}\pi} = \sqrt{2}\left(\cos\left(\frac{3}{2}\pi+\frac{\pi}{12}\right) + i\sin\left(\frac{3}{2}\pi+\frac{\pi}{12}\right)\right) =$

$= \sqrt{2}\left(\sin\frac{\pi}{12} - i\cos\frac{\pi}{12}\right) = \sqrt{2}\left(\frac{\sqrt{6}-\sqrt{2}}{4} - i\,\frac{\sqrt{6}+\sqrt{2}}{4}\right) =$

$= \frac{2\sqrt{3}-2}{4} - i\,\frac{2\sqrt{3}+2}{4} = \frac{\sqrt{3}-1}{2} - i\,\frac{\sqrt{3}+1}{2}$

(9) $\sqrt[4]{-4}$

$-4 = 4(\cos\pi + i\sin\pi) = 4\cdot e^{i(\pi+2\pi k)}, \quad k\in Z$

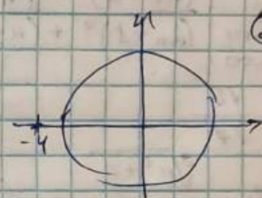
$\sqrt[4]{-4} = \sqrt{2}(\cos\pi + i\sin\pi) = \sqrt{2}\cdot e^{i\frac{(\pi+2\pi k)}{4}}, \quad k = 0, 1, 2, 3$

$k=0 \rightarrow z_1 = \sqrt{2}\cdot e^{i\frac{\pi}{4}} = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) = 1+i$

$k=1 \rightarrow z_2 = \sqrt{2}\cdot e^{\frac{3}{4}\pi} = \sqrt{2}\left(\cos\frac{3}{4}\pi + i\sin\frac{3}{4}\pi\right) = -1+i$ — неверно

$k=2 \rightarrow z_3 = \sqrt{2}\cdot e^{\frac{5}{4}\pi} = \sqrt{2}\left(\cos\left(\pi+\frac{\pi}{4}\right) + i\sin\left(\pi+\frac{\pi}{4}\right)\right) = -1-i$

$k=3 \rightarrow z_4 = \sqrt{2}\cdot e^{\frac{4}{4}\pi} = 1-i$



(2)

$\sqrt[4]{-4}$

$z = 4(\cos\pi + i\sin\pi) = 4\cdot e^{i(\pi+2\pi k)}, \quad k\in Z$

$z_k = \sqrt[4]{z} = \sqrt[4]{|z|}\left(\cos\left(\frac{\pi+2\pi k}{4}\right) + i\sin\left(\frac{\pi+2\pi k}{4}\right)\right), \quad k\in Z$

$k=0 \rightarrow z_0 = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) = 1+i$

$k=1 \rightarrow z_1 = \sqrt{2}\left(\cos\frac{3}{4}\pi + i\sin\frac{3}{4}\pi\right) = -1+i$ — верно

$k=2 \rightarrow z_2 = \sqrt{2}\left(\cos\left(\pi+\frac{\pi}{4}\right) + i\sin\left(\pi+\frac{\pi}{4}\right)\right) = -1-i$

$k=3 \rightarrow z_3 = \sqrt{2}\left(\cos\left(2\pi-\frac{\pi}{4}\right) + i\sin\left(2\pi-\frac{\pi}{4}\right)\right) = 1-i$

29.09.2020

Предел последовательности.

1) $\lim\limits_{n\to+\infty}\frac{n+1}{n}=\left\{\frac{\infty}{\infty}\right\}=\lim\limits_{n\to+\infty}\left(1+\frac{1}{n}\right)=1.$

2) $\lim\limits_{n\to+\infty}\frac{(n+1)^2}{2n^2}=\lim\limits_{n\to+\infty}\frac{n^2\left(1+\frac{1}{n}\right)^2}{2n^2}=\frac{1}{2}$

3) $\lim\limits_{n\to+\infty}\frac{n^3-100n^2+1}{100n^2+15n}=\left\{\frac{\infty}{\infty}\right\}=\lim\limits_{n\to+\infty}\frac{n^3\left(1-\frac{100}{n}+\frac{1}{n^3}\right)}{n^2\left(100+\frac{15}{n}\right)}=+\infty$

4) $\lim\limits_{n\to+\infty}\frac{1000n^3+3n^2}{0{,}001n^4-100n^2+1}=\lim\limits_{n\to+\infty}\frac{n^3\left(1000+\frac{3}{n}\right)}{n^4\left(0{,}001-\frac{100}{n^2}+\frac{1}{n^4}\right)}=\left\{\frac{\infty}{\infty}\right\}=0$

Если степень числителя > степени знамен. $\to +\infty$

Если степень числителя < степени знамен. $\to 0$

Если одинак. $\to$ отношение

5) $\lim\limits_{n\to+\infty}\frac{(2n+1)^4-(n-1)^4}{(2n+1)^4+(n+1)^4}=\left\{\frac{\infty-\infty}{\infty}\right\}=\lim\limits_{n\to+\infty}\frac{n^4\left(\left(2+\frac{1}{n}\right)^4-\left(1-\frac{1}{n}\right)^4\right)}{n^4\left(\left(2+\frac{1}{n}\right)^4+\left(1+\frac{1}{n}\right)^4\right)}$

$=\frac{15}{17}$

6) $\lim\limits_{n\to+\infty}\frac{\sqrt[3]{n^2+n}}{n+1}=\lim\limits_{n\to+\infty}\frac{\sqrt[3]{n^2\left(1+\frac{1}{n}\right)}}{n\left(1+\frac{1}{n}\right)}=\left\{\frac{\infty}{\infty}\right\}=\lim\limits_{n\to+\infty}\frac{\left(1+\frac{1}{n}\right)^{\frac{1}{3}}}{n^{\frac{1}{3}}\left(1+\frac{1}{n}\right)}$

$=\lim\limits_{n\to+\infty}\frac{\sqrt[3]{1+\frac{1}{n}}}{\sqrt[3]{n}\left(1+\frac{1}{n}\right)}=0$

7) $\lim\limits_{n\to+\infty}\frac{\sqrt{n^3-2n^2+1}+\sqrt[3]{n^4+1}}{\sqrt[4]{n^8+6n^5+2}-\sqrt[5]{n^7+3n^3+1}}=\lim\limits_{n\to+\infty}\frac{\frac{\sqrt{n^3-2n^2+1}}{\sqrt{n^3}}+\frac{\sqrt[3]{n^4+1}}{\sqrt{n^3}}}{\frac{\sqrt[4]{n^8+6n^5+2}}{\sqrt{n^3}}-\frac{\sqrt[5]{n^7+3n^3+1}}{\sqrt{n^3}}}=$

$=\lim\limits_{n\to+\infty}\frac{\sqrt{1-\frac{2}{n}+\frac{1}{n^3}}+\sqrt[6]{\frac{(n^4+1)^2}{n^9}}}{\sqrt[4]{\frac{n^8\left(1+\frac{6}{n^3}+\frac{2}{n^8}\right)}{n^6}}-\sqrt[10]{\frac{n^{14}\left(1+\frac{3}{n^4}+\frac{1}{n^7}\right)^2}{n^{15}}}}=1$

8) $\lim\limits_{n\to+\infty}\frac{\sqrt[4]{n^5+2}-\sqrt[3]{n^2+1}}{\sqrt[5]{n^4+2}-\sqrt{n^3+1}}=\lim\limits_{n\to+\infty}\frac{\frac{\sqrt[4]{n^5+2}}{\sqrt[4]{n^5}}-\frac{\sqrt[3]{n^2+1}}{\sqrt[4]{n^5}}}{\frac{\sqrt[5]{n^4+2}}{\sqrt{n^3}}-\frac{\sqrt{n^3+1}}{\sqrt{n^3}}}=\frac{\sqrt[4]{1+\frac{2}{n^5}}-\sqrt[12]{\frac{(n^2+1)^4}{n^{15}}}}{\sqrt[10]{\frac{(n^4+2)^2}{n^{15}}}-\sqrt{1+\frac{1}{n^3}}}=$

$=\frac{\sqrt[4]{1+\frac{2}{n^5}}-\sqrt[12]{\frac{n^8\left(1+\frac{1}{n^2}\right)^4}{n^{15}}}}{\sqrt[10]{\frac{n^8\left(1+\frac{2}{n^4}\right)^2}{n^{15}}}-\sqrt{1+\frac{1}{n^3}}}$

9) $\lim\limits_{n\to+\infty}\frac{(n+2)!+(n+1)!}{(n+2)!-(n+1)!}=\lim\limits_{n\to+\infty}\frac{(n+1)!\,(n+2+1)}{(n+1)!\,(n+2-1)}=\lim\limits_{n\to+\infty}\frac{n+3}{n+1}=1$

10) $\lim\limits_{n\to+\infty}\frac{1+\frac{1}{2}+\frac{1}{4}+\ldots+\frac{1}{2^n}}{1+\frac{1}{3}+\frac{1}{9}+\ldots+\frac{1}{3^n}}=\lim\limits_{n\to+\infty}\frac{1\left(1-\frac{1}{2}\right)^n\left(1-\frac{1}{3}\right)}{\left(1-\frac{1}{2}\right)1\cdot\left(1-\frac{1}{3}\right)^n}=\lim\limits_{n\to+\infty}\frac{2\cdot2\left(1-\frac{1}{2}\right)^n}{3\left(1-\frac{1}{3}\right)^n}=\lim\limits_{n\to+\infty}\frac{2\cdot2\cdot\left(\frac{1}{2}\right)^n}{3\left(\frac{2}{3}\right)^n}$

$=\frac{4}{3}$

11) $\lim\limits_{n\to+\infty}\left(\frac{1+2+3+\ldots+n}{n+2}-\frac{n}{2}\right)=\lim\limits_{n\to+\infty}\frac{\frac{1+n}{2}\cdot n}{n+2}-\frac{n}{2}=\lim\limits_{n\to+\infty}\left(\frac{n(n+1)}{2(n+2)}-\frac{n}{2}\right)=$

$= \lim_{n\to+\infty} \frac{n}{2}\left(\frac{n+1}{n+2} - 1\right) = \lim_{n\to+\infty} \frac{n}{2}\left(\frac{1+n-n-2}{n+2}\right) = \lim_{n\to+\infty} \frac{n}{2}\left(-\frac{1}{n+2}\right) =$

$= \lim_{n\to+\infty} -\frac{n}{2(n+2)} = \lim_{n\to+\infty} \frac{-n}{2n(1+\frac{2}{n})} = -\frac{1}{2}$

$\lim_{n\to+\infty} \frac{1}{2}\left(\frac{3-1}{1\cdot 3} + \frac{5-3}{3\cdot 5} + \ldots + \frac{(2n+1)-(2n-1)}{(2n+1)(2n-1)}\right) = \frac{1}{2}\lim_{n\to+\infty}\left(1 - \frac{1}{2n+1}\right) = \frac{1}{2}$

в) Ограниченное $\sin$ → величина ограниченная

10) $\lim_{n\to+\infty}\left(\sqrt{n^2+1} - \sqrt{n^2-1}\right) = \{\infty - \infty\} = \lim_{n\to+\infty} \frac{(\sqrt{n^2+1} - \sqrt{n^2-1})(\sqrt{n^2+1} + \sqrt{n^2-1})}{(\sqrt{n^2+1} + \sqrt{n^2-1})} =$

$= \lim_{n\to+\infty} = \frac{n^2+1-n^2+1}{(\sqrt{n^2+1} + \sqrt{n^2-1})} = \frac{2}{\{\infty - \infty\}} = 0$

Домашняя работа

1) $\lim\limits_{n\to+\infty} \dfrac{(n+1)^3-(n-1)^3}{(n+1)^2+(n-1)^2} = \lim\limits_{n\to+\infty} \dfrac{n^3+3n^2+3n+1-n^3+3n^2-3n+1}{2n^2+2} = \lim\limits_{n\to+\infty} \dfrac{6n^2+2}{2(n^2+1)} =$

$= \lim\limits_{n\to+\infty} \dfrac{3n^2+1}{n^2+1} = \lim\limits_{n\to+\infty} \dfrac{n^2\left(3+\frac{1}{n^2}\right)}{n^2\left(1+\frac{1}{n^2}\right)} = 3$

2) $\lim\limits_{n\to+\infty} \dfrac{\sqrt[3]{n^3+2n-1}}{n+2} = \lim\limits_{n\to+\infty} \dfrac{\sqrt[3]{1+\frac{2}{n^2}-\frac{1}{n^3}}}{1+\frac{2}{n}} = 1.$

3) $\lim\limits_{n\to+\infty} \dfrac{(\sqrt{n^2+1}+n)^2}{\sqrt[3]{n^6+1}} = \lim\limits_{n\to+\infty} \dfrac{2n^2+1+2n\sqrt{n^2+1}+n^2}{\sqrt[3]{n^6+1}} = \lim\limits_{n\to+\infty} \dfrac{2n^2+1+\sqrt{4n^4+4n^2}}{\sqrt[3]{n^6+1}} =$

$= \lim\limits_{n\to+\infty} \dfrac{\dfrac{2n^2+1+\sqrt{4n^4+4n^2}}{\sqrt{n^4}}}{\dfrac{\sqrt[3]{n^6+1}}{\sqrt[3]{n^6}}} = \lim\limits_{n\to+\infty} \dfrac{2+\frac{1}{n^2}+\sqrt{4+\frac{4}{n^2}}}{\sqrt[3]{1+\frac{1}{n^2}}} = \dfrac{2+2}{1} = 4$

4) $\lim\limits_{n\to+\infty} \dfrac{(n+2)!+(n+1)!}{(n+3)!} = \lim\limits_{n\to+\infty} \dfrac{(n+1)!\,(n+2+1)}{(n+1)!\,(n+2)(n+3)} = \lim\limits_{n\to+\infty} \dfrac{1}{n+2} = 0$

5) $\lim\limits_{n\to+\infty} (\sqrt{n+1}-\sqrt{n}) = \lim\limits_{n\to+\infty} \dfrac{(\sqrt{n+1}-\sqrt{n})(\sqrt{n+1}+\sqrt{n})}{\sqrt{n+1}+\sqrt{n}} = \lim\limits_{n\to+\infty} \dfrac{1}{\sqrt{n+1}+\sqrt{n}} = 0$

6) $\lim\limits_{n\to+\infty} \dfrac{\sqrt[3]{n^2}\sin(n!)}{n+1} = \dfrac{\sqrt[3]{\frac{1}{n}}\sin(n!)}{1+\frac{1}{n}} = 0$

7) $\lim\limits_{n\to+\infty} \dfrac{1+a+a^2+\ldots+a^n}{1+b+b^2+\ldots+b^n} = \lim\limits_{n\to+\infty} \dfrac{(1-a)^n(1-b)}{(1-b)^n(1-a)} = \dfrac{1-b}{1-a}$

$|a|<1$; $|b|<1$.

8) $\lim\limits_{n\to+\infty} \left(\dfrac{1}{n^2}+\dfrac{2}{n^2}+\dfrac{3}{n^2}+\ldots+\dfrac{n-1}{n^2}\right) = \lim\limits_{n\to+\infty} \dfrac{\frac{1+n-1}{2}\cdot n}{n^2} = \dfrac{1}{2}$

10) $\lim\limits_{n\to+\infty} \left[\dfrac{1}{1\cdot 2}+\dfrac{1}{2\cdot 3}+\ldots+\dfrac{1}{n(n+1)}\right] = \lim\limits_{n\to+\infty} \dfrac{n}{n+1} = \lim\limits_{n\to+\infty} \dfrac{1}{1+\frac{1}{n}} = 1$

? 9) $\lim\limits_{n\to+\infty} \left[\dfrac{1^2}{n^3}+\dfrac{2^2}{n^3}+\ldots+\dfrac{(n-1)^2}{n^3}\right] = \lim\limits_{n\to+\infty} \dfrac{(n-1)n\cdot 2n}{6n^3} = \lim\limits_{n\to+\infty} \dfrac{n^3-n^2}{3n^3} = \lim\limits_{n\to+\infty} \dfrac{n^3\left(1-\frac{1}{n}\right)}{3n^3} = \dfrac{1}{3}$

Из пособия:

15) $\lim\limits_{n\to+\infty} (\sqrt[3]{n+1}-\sqrt[3]{n}) = \lim\limits_{n\to+\infty} \dfrac{(\sqrt[3]{n+1}-\sqrt[3]{n})(\sqrt[3]{n+1}+\sqrt[3]{n})}{\sqrt[3]{n+1}+\sqrt[3]{n}} = \lim\limits_{n\to+\infty} \dfrac{\sqrt[3]{(n+1)^2}-\sqrt[3]{n^2}}{\sqrt[3]{n+1}+\sqrt[3]{n}} =$

$= \lim\limits_{n\to+\infty} \dfrac{\sqrt[3]{n^2+2n+1}-\sqrt[3]{n^2}}{\sqrt[3]{n+1}+\sqrt[3]{n}} = \lim\limits_{n\to+\infty} \dfrac{\sqrt[3]{1+\frac{2}{n}+\frac{1}{n^2}}-\sqrt[3]{1}}{\sqrt[3]{\frac{1}{n}+\frac{1}{n^2}}+\sqrt[3]{\frac{1}{n}}} = 0.$

16) $\lim\limits_{n\to+\infty} (\sqrt[4]{n+1}-\sqrt[4]{n-2}) = \lim\limits_{n\to+\infty} \dfrac{\sqrt[4]{(n+1)^2}-\sqrt[4]{(n-2)^2}}{\sqrt[4]{n+1}+\sqrt[4]{n-2}} = \lim\limits_{n\to+\infty} \dfrac{\sqrt[4]{n^2+2n+1}-\sqrt[4]{n^2-2n+4}}{\sqrt[4]{n+1}+\sqrt[4]{n-2}} =$

$= \lim\limits_{n\to+\infty} \dfrac{\sqrt[4]{1+\frac{2}{n}+\frac{1}{n^2}}-\sqrt[4]{1-\frac{2}{n}+\frac{1}{n^2}}}{\sqrt[4]{\frac{1}{n}+\frac{1}{n^2}}+\sqrt[4]{\frac{1}{n}-\frac{2}{n^2}}} = 0$

Предел функции.

№ 411, 413, 416, 419, 425, 435, 443, 449, 461, 462

411

а) $\lim\limits_{x\to 0}\dfrac{x^2-1}{2x^2-x-1} = 1.$

б) $\lim\limits_{x\to 1}\dfrac{x^2-1}{2x^2-x-1} = \left\{\dfrac{0}{0}\right\} = \lim\limits_{x\to 1}\dfrac{(x-1)(x+1)}{2(x-1)(x+\frac{1}{2})} = \lim\limits_{x\to 1}\dfrac{x+1}{2x+1} = \dfrac{2}{3}$

$x \to 1$

$x-1 \to 0$

в) $\lim\limits_{x\to \infty}\dfrac{x^2-1}{2x^2-x-1} = \lim\limits_{x\to\infty}\dfrac{x^2(1-\frac{1}{x^2})}{x^2(2-\frac{1}{x}-\frac{1}{x^2})} = \dfrac{1}{2}$

413 $\lim\limits_{x\to 0}\dfrac{(1+x)^5-(1+5x)}{x^2+x^5} = \left\{\dfrac{0}{0}\right\} =$

$= \lim\limits_{x\to 0}\dfrac{\cancel{1}+\cancel{5x}+10x^2+10x^3+5x^4+x^5-\cancel{1}-\cancel{5x}}{x^2(1+x^3)} = \lim\limits_{x\to 0}\dfrac{x^2(10+10x+5x^2+x^3)}{x^2(1+x^3)} = 10$

416 $\lim\limits_{x\to\infty}\dfrac{(2x-3)^{20}(3x+2)^{30}}{(2x+1)^{50}} = \left\{\dfrac{\infty}{\infty}\right\} = \lim\limits_{x\to\infty}\dfrac{x^{20}(2-\frac{3}{x})^{20}\,x^{30}(3+\frac{2}{x})^{30}}{x^{50}(2+\frac{1}{x})^{50}} =$

$= \dfrac{2^{20}\cdot 3^{30}}{2^{50}} = \left(\dfrac{3}{2}\right)^{30}$

419 $\lim\limits_{x\to 1}\dfrac{x^3-3x+2}{x^4-4x+3} = \left\{\dfrac{0}{0}\right\} =$

	1	0	−3	2
1	1	1	−2	0

	1	0	0	−4	3
1	1	1	1	−3	0

$= \lim\limits_{x\to 1}\dfrac{(x-1)(x^2+x-2)}{(x-1)(x^3+x^2+x-3)} = \lim\limits_{x\to 1}\dfrac{(x-1)(x+2)}{(x-1)(x^2+2x+3)} = \lim\limits_{x\to 1}\dfrac{x+2}{x^2+2x+3} = \dfrac{3}{6} = \dfrac{1}{2}$

435 $\lim\limits_{x\to+\infty}\dfrac{\sqrt{x+\sqrt{x+\sqrt{x}}}}{\sqrt{x+1}} = \lim\limits_{x\to+\infty}\dfrac{\frac{\sqrt{x+\sqrt{x+\sqrt{x}}}}{\sqrt{x}}}{\frac{\sqrt{x+1}}{\sqrt{x}}} = \lim\limits_{x\to+\infty}\dfrac{\sqrt{1+\frac{\sqrt{x+\sqrt{x}}}{x}}}{\sqrt{1+\frac{1}{x}}} =$

$= \lim\limits_{x\to+\infty}\dfrac{\sqrt{1+\sqrt{\frac{1}{x}+\frac{\sqrt{x}}{x^2}}}}{\sqrt{1+\frac{1}{x}}} = \lim\limits_{x\to+\infty}\dfrac{\sqrt{1+\sqrt{\frac{1}{x}+\sqrt{\frac{1}{x^3}}}}}{\sqrt{1+\frac{1}{x}}} = 1$

443 $\lim\limits_{x\to 8}\dfrac{\sqrt{9+2x}-5}{\sqrt[3]{x}-2} = \left\{\dfrac{0}{0}\right\} = \lim\limits_{x\to 8}\dfrac{(\sqrt{9+2x}-5)(\sqrt{9+2x}+5)(\sqrt[3]{x^2}+2\sqrt[3]{x}+4)}{(\sqrt{9+2x}+5)(\sqrt[3]{x}-2)(\sqrt[3]{x^2}+2\sqrt[3]{x}+4)} =$

$= \lim\limits_{x\to 8}\dfrac{2\cancel{(x-8)}(\sqrt[3]{x^2}+2\sqrt[3]{x}+4)}{(\sqrt{9+2x}+5)\cancel{(x-8)}} = \dfrac{2(4+4+4)}{10} = \dfrac{12}{5} = 2{,}4$

449 $\lim\limits_{x\to 7}\dfrac{\sqrt{x+2}-\sqrt[3]{x+20}}{\sqrt[4]{x+9}-2} = \left\{\dfrac{0}{0}\right\} = \lim\limits_{x\to 7}\dfrac{\sqrt{x+2}-3+3-\sqrt[3]{x+20}}{\sqrt[4]{x+9}-2} =$

$= \lim\limits_{x\to 7}\left(\dfrac{\sqrt{x+2}-3}{\sqrt[4]{x+9}-2} - \dfrac{\sqrt[3]{x+20}-3}{\sqrt[4]{x+9}-2}\right) = \dfrac{16}{3} - \dfrac{32}{27} = \dfrac{112}{27}$

1) $\frac{\sqrt{x+2}-3}{\sqrt[4]{x+9}-2} = \frac{x+2-9}{(\sqrt[4]{x+9}-2)(\sqrt{x+2}+3)} = \frac{(x+2-9)(\sqrt[4]{x+9}+2)}{(\sqrt{x+2}+3)(\sqrt{x+9}-4)} =$

$= \frac{(x-7)(\sqrt[4]{x+9}+2)(\sqrt{x+9}+4)}{(x+9-16)(\sqrt{x+2}+3)} \to \frac{16}{3}$

2) $\frac{\sqrt[3]{x+20}-3}{\sqrt[4]{x+9}-2} = \frac{(\sqrt[3]{x+20}-3)(\sqrt[4]{x+9}+2)}{(\sqrt{x+9}-4)} = \frac{(\sqrt[3]{x+20}-3)(\sqrt[4]{x+9}+2)(\sqrt{x+9}+4)}{(x+9-16)} =$

$= \frac{(\sqrt[3]{x+20}-3)(\sqrt[3]{(x+20)^2}+3\sqrt[3]{x+20}+9)(\sqrt[4]{x+9}+2)(\sqrt{x+9}+4)}{(x-7)(\sqrt[3]{(x+20)^2}+3\sqrt[3]{x+20}+9)} =$

$= \frac{(x+20-27)(\sqrt[4]{x+9}+2)(\sqrt{x+9}+4)}{(x-7)(\sqrt[3]{(x+20)^2}+3\sqrt[3]{x+20}+9)} \to \frac{4\cdot 8}{9+9+9} \to \frac{32}{27}$

461) $\lim\limits_{x\to\infty}(\sqrt[3]{x^3+x^2+1}-\sqrt[3]{x^3-x^2+1}) = \lim\limits_{x\to\infty} (\sqrt[3]{x^3+x^2+1}-\sqrt[3]{x^3-x^2+1})(\sqrt[3]{(x^3+x^2+1)^2}+\sqrt[3]{(x^3+x^2+1)}) =$

$= \lim\limits_{x\to\infty}\frac{(\sqrt[3]{x^3+x^2+1}-\sqrt[3]{x^3-x^2+1})(\sqrt[3]{(x^3+x^2+1)^2}+\sqrt[3]{(x^3+x^2+1)(x^3-x^2+1)}+\sqrt[3]{(x^3-x^2+1)^2})}{\sqrt[3]{(x^3+x^2+1)^2}+\sqrt[3]{(x^3+x^2+1)(x^3-x^2+1)}+\sqrt[3]{(x^3-x^2+1)^2}} =$

$= \lim\limits_{x\to\infty}\frac{2x^2}{\sqrt[3]{(x^3+x^2+1)^2}+\sqrt[3]{(x^3+x^2+1)(x^3-x^2+1)}+\sqrt[3]{(x^3-x^2+1)^2}} =$

$= \lim\limits_{x\to\infty}\frac{2}{\underset{\downarrow 1}{\sqrt[3]{(x^3+x^2+1)^2}}+\underset{\downarrow 1}{\sqrt[3]{(x^3+x^2+1)(x^3-x^2+1)}}+\underset{\downarrow 1}{\sqrt[3]{(x^3-x^2+1)^2}}} = \frac{2}{3}$

462) $\lim\limits_{x\to+\infty}(\sqrt[3]{x^3+3x^2}-\sqrt{x^2-2x}) = \{\infty-\infty\} =$

$= \lim\limits_{x\to+\infty}(\sqrt[3]{x^3+3x^2}-x+x-\sqrt{x^2-2x}) =$

$= \lim\limits_{x\to+\infty}\frac{x^3+3x^2-x^3}{\sqrt[3]{(x^3+3x^2)^2}+x\sqrt[3]{x^3+3x^2}+x^2} + \frac{x^2-x^2+2x}{x+\sqrt{x^2-2x}} = 1+1=2$

Д/З ~~415~~, ~~418~~, ~~420~~, ~~438~~, ~~440~~, ~~441~~, 444, ~~450~~.

Домашняя работа

415.

$$\lim_{x\to\infty}\frac{(x-1)(x-2)(x-3)(x-4)(x-5)}{(5x-1)^5}=\lim_{x\to\infty}\frac{x^5-15x^4+85x^3-225x^2+274x-120}{(5x-1)^5}=$$

$$=\lim_{x\to\infty}\frac{x^5-15x^4+85x^3-225x^2+274x-120}{5^5x^5-5^5x^4+1250x^3-250x^2+25x-1}=$$

$$=\lim_{x\to\infty}\frac{x^5\left(1-\frac{15}{x}+\frac{85}{x^2}-\frac{225}{x^3}+\frac{274}{x^4}-\frac{120}{x^5}\right)}{x^5\left(5^5-\frac{5^5}{x}+\frac{1250}{x^2}-\frac{250}{x^3}+\frac{25}{x^4}-\frac{1}{x^5}\right)}=\frac{1}{5^5}=\frac{1}{3125}$$

418.

$$\lim_{x\to3}\frac{x^2-5x+6}{x^2-8x+15}=\lim_{x\to3}\frac{(x-3)(x-2)}{(x-5)(x-3)}=\lim_{x\to3}\frac{x-2}{x-5}=\frac{1}{-2}=-\frac{1}{2}$$

$x^2-5x+6=0 \quad \rightarrow x^2-5x+6=(x-3)(x-2)$

$\left.\begin{matrix}x_1+x_2=5\\ x_1\cdot x_2=6\end{matrix}\right\}\ \begin{matrix}x_1=3\\ x_2=2\end{matrix}$

$\left.\begin{matrix}x_1+x_2=8\\ x_1\cdot x_2=15\end{matrix}\right\}\ \begin{matrix}x_1=5\\ x_2=3\end{matrix} \quad \rightarrow x^2-8x+15=(x-5)(x-3)$

420.

$$\lim_{x\to1}\frac{x^4-3x+2}{x^5-4x+3}=\lim_{x\to1}\frac{(x-1)(x^3+x^2+x-2)}{(x-1)(x^4+x^3+x^2+x-3)}=\frac{1}{4-3}=1$$

x^4-3x+2

	1	0	0	-3	2
1	1	1	1	-2	0

$x^4-3x+2=(x-1)(x^3+x^2+x-2)$

x^5-4x+3

	1	0	0	0	-4	3
1	1	1	1	1	-3	0

$x^5-4x+3=(x-1)(x^4+x^3+x^2+x-3)$

438.

$$\lim_{x\to-8}\frac{\sqrt{1-x}-3}{2+\sqrt[3]{x}}=\frac{1-x-9}{(2+\sqrt[3]{x})(\sqrt{1-x}+3)}=$$

$$=\lim_{x\to-8}\frac{-1(x+8)(4-2\sqrt[3]{x}+\sqrt[3]{x^2})}{(x+8)(\sqrt{1-x}+3)}=\lim_{x\to-8}\frac{2\sqrt[3]{x}-\sqrt[3]{x^2}-4}{\sqrt{1-x}+3}=\frac{-4-4-4}{6}=-\frac{12}{6}=-2$$

440.

$$\lim_{x\to3}\frac{\sqrt{x+13}-2\sqrt{x+1}}{x^2-9}=\lim_{x\to3}\frac{x+13-4(x+1)}{(x-3)(x+3)}=\lim_{x\to3}\frac{x-4x-4+13}{(x-3)(x+3)}=$$

$$=\lim_{x\to3}\frac{-3x+9}{(x-3)(x+3)}=\lim_{x\to3}\frac{-3\cancel{(x-3)}}{(x+3)\cancel{(x-3)}}=\frac{-3}{6}=-\frac{1}{2}$$

442.

$$\lim_{x\to16}\frac{\sqrt[4]{x}-2}{\sqrt{x}-4}=\lim_{x\to16}\frac{(\sqrt{x}-4)}{(\sqrt{x}-4)(\sqrt[4]{x}+2)}=\lim_{x\to16}\frac{1}{\sqrt[4]{x}+2}=\frac{1}{2+2}=\frac{1}{4}$$

441. $$\lim_{x\to -2}\frac{\sqrt[3]{x-6}+2}{x^3+8}=\lim_{x\to -2}\frac{\sqrt[3]{x-6}+2}{(x+2)(x^2-2x+4)}=\lim_{x\to -2}\frac{x-6+8}{(x+2)(x^2-2x+4)\left(\sqrt[3]{(x-6)^2}-2\sqrt[3]{x-6}+4\right)}$$

$$=\lim_{x\to -2}\frac{1}{(x^2-2x+4)\left(\sqrt[3]{(x-6)^2}-2\sqrt[3]{x-6}+4\right)}=\frac{1}{(4+4+4)(4+4+4)}=\frac{1}{144}$$

458. $$\lim_{x\to +\infty}\left(\sqrt{x+\sqrt{x+\sqrt{x}}}-\sqrt{x}\right)=\lim_{x\to +\infty}\frac{\cancel{x}+\sqrt{x+\sqrt{x}}-\cancel{x}}{\sqrt{x+\sqrt{x+\sqrt{x}}}+\sqrt{x}}=$$

$$=\lim_{x\to +\infty}\frac{\frac{\sqrt{x+\sqrt{x}}}{\sqrt{x}}}{\frac{\sqrt{x+\sqrt{x+\sqrt{x}}}}{\sqrt{x}}+\frac{\sqrt{x}}{\sqrt{x}}}=\lim_{x\to +\infty}\frac{\sqrt{1+\sqrt{\frac{1}{x}}}}{\sqrt{1+\sqrt{\frac{1}{x}+\sqrt{\frac{1}{x^3}}}}+1}=\frac{1}{2}$$

444. $$\lim_{x\to 0}\frac{\sqrt[n]{1+x}-1}{x}=\lim_{x\to 0}\frac{\frac{\sqrt[n]{1+x}}{\sqrt[n]{x}}-\frac{1}{\sqrt[n]{x}}}{\frac{x}{\sqrt[n]{x}}}=\lim_{x\to 0}\frac{\overbrace{\sqrt[n]{\frac{1}{x}+1}-\frac{1}{\sqrt[n]{x}}}^{\to 1}}{\sqrt[n]{\frac{x^n}{x}}}=$$

425. m и n - натуральные числа.

$$\lim_{x\to 1}\frac{x^m-1^m}{x^n-1^n}=\lim_{x\to 1}\frac{(x-1)\overbrace{(x^{m-1}+x^{m-2}+\dots+x+1)}^{\to m}}{(x-1)\underbrace{(x^{n-1}+x^{n-2}+\dots+x+1)}_{\to n}}=\frac{m}{n}$$

444. $$\lim_{x\to 0}\frac{\sqrt[n]{1+x}-1}{x}=\lim_{x\to 0}\frac{\left(\sqrt[n]{1+x}-\sqrt[n]{1}\right)\left(\sqrt[n]{(1+x)^{n-1}}+\sqrt[n]{(1+x)^{n-2}}+\dots+1\right)}{x\left(\sqrt[n]{(1+x)^{n-1}}+\sqrt[n]{(1+x)^{n-2}}+\dots+1\right)}=$$

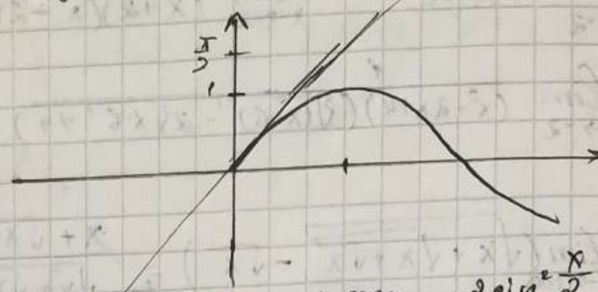
$$=\lim_{x\to 0}\frac{1+x-1}{x\underbrace{\left(\sqrt[n]{(1+x)^{n-1}}+\sqrt[n]{(1+x)^{n-2}}+\dots+1\right)}_{\to n \text{ членов по } 1}}=\frac{1}{n}$$

Первый и второй замечательный предел.

$$\lim_{x\to 0}\frac{\sin x}{x}=1 \qquad \text{— I}$$

$$\lim_{x\to 0}(1+x)^{\frac{1}{x}}=e \qquad \text{— II}$$

$$\lim_{x\to \infty}\left(1+\frac{1}{x}\right)^{x}=e$$



$$1-\cos x = 2\sin^2\frac{x}{2}$$

471, 473, 477, 480, 500, 505, 506, 512, 514, 522.

(471) $$\lim_{x\to 0}\frac{\sin 5x}{x}=\lim_{x\to 0}\frac{5\overbrace{\sin 5x}^{\to 1}}{5x}=5$$

(473.) $$\lim_{x\to \pi}\frac{\sin(mx)}{\sin(nx)}=\left\{\begin{array}{l}x-\pi=t\\ x=t+\pi\end{array}\right\}=\lim_{t\to 0}\frac{\sin m(\pi+t)}{\sin n(\pi+t)}=\lim_{t\to 0}\frac{\sin(\pi m+mt)}{\sin(\pi n+nt)}=$$

$$=\lim_{t\to 0}\frac{(-1)^m\sin mt}{(-1)^n\sin nt}=\lim_{t\to 0}(-1)^{m-n}\frac{(\sin mt)\cdot nt\cdot n}{mt(\sin nt)\cdot n}=(-1)^{m-n}\cdot\frac{m}{n}$$

(477.) $$\lim_{x\to 0}\frac{\cos x-\cos 3x}{x^2}=\left\{\frac{0}{0}\right\}=\lim_{x\to 0}\frac{-2\sin\frac{x+3x}{2}\sin\frac{x-3x}{2}}{x^2}=$$

$$=\lim_{x\to 0}\frac{2\sin 2x\sin x}{x^2}=\frac{2\sin 2x\cdot\sin x\cdot 2}{2x\cdot x}=4$$

(480.) $$\lim_{x\to 1}(1-x)\,tg\frac{\pi x}{2}=\left\{\begin{array}{l}x-1=t\\ x=t+1\end{array}\right\}=\lim_{t\to 0}-t\cdot tg\frac{\pi}{2}(t+1)=$$

$$=-\lim_{t\to 0}t\,tg\left(\frac{\pi}{2}+\frac{\pi}{2}t\right)=\lim_{t\to 0}ctg\frac{\pi}{2}t\cdot t=\frac{2}{\pi}\lim_{t\to 0}\left(\frac{\pi}{2}t\,\frac{\cos\frac{\pi}{2}t}{\sin\frac{\pi}{2}t}\right)=\frac{2}{\pi}$$

(500) $$\lim_{x\to 0}\frac{x^2}{\sqrt{1+x\sin x}-\sqrt{\cos x}}\cdot\lim_{x\to 0}\frac{x^2(\sqrt{1+x\sin x}+\sqrt{\cos x})}{1-\cos x+x\sin x}=$$

$$=\lim_{x\to 0}\frac{x^2(\sqrt{1+x\sin x}+\sqrt{\cos x})}{2\sin^2\frac{x}{2}+x\sin x}=\lim_{x\to 0}\frac{(\sqrt{1+x\sin x}+\sqrt{\cos x})}{\frac{2\sin^2\frac{x}{2}}{\frac{1}{4}x^2\cdot 4}+\frac{x\sin x}{x^2}}=\frac{4}{3}$$

(505.) $$\lim_{x\to +\infty}(\sin\sqrt{x+1}-\sin\sqrt{x})=\lim_{x\to +\infty}2\sin\frac{\sqrt{x+1}-\sqrt{x}}{2}\cos\frac{\sqrt{x+1}+\sqrt{x}}{2}=$$

$$=2\lim_{x\to +\infty}\underbrace{\sin\frac{x+1-x}{2\sqrt{x+1}+\sqrt{x}}}_{\to 0}\;\underbrace{\cos\frac{\sqrt{x+1}+\sqrt{x}}{2}}_{\text{огр.}}=0$$

508. а) $\lim\limits_{x\to 0}\left(\frac{1+x}{2+x}\right)^{\frac{1-\sqrt{x}}{1-x}} = \frac{1}{2}$

б) $\lim\limits_{x\to 1}\left(\frac{1+x}{2+x}\right)^{\frac{1-\sqrt{x}}{(1-\sqrt{x})(1+\sqrt{x})}} = \lim\limits_{x\to 1}\left(\frac{1+x}{2+x}\right)^{\frac{1}{1+\sqrt{x}}} = \sqrt{\frac{2}{3}}$

в) $\lim\limits_{x\to +\infty}\left(\frac{1+x}{2+x}\right)^{\frac{1-\sqrt{x}}{1-x}} = 1^0 = 1$

512. $\lim\limits_{x\to\infty}\left(\frac{x^2+1}{x^2-2}\right)^{x^2} = \{1^\infty\} = \lim\limits_{x\to\infty}\left(\frac{x^2-2+3}{x^2-2}\right)^{x^2} = \lim\limits_{x\to\infty}\left(1+\frac{3}{x^2-2}\right)^{x^2} =$

$= \lim\limits_{x\to\infty}\left(1+\frac{x^2+1}{x^2-2}-1\right)^{x^2} = \lim\limits_{x\to\infty}\left(\left(\underbrace{\left(1+\frac{3}{x^2-2}\right)^{\frac{x^2-2}{3}}}_{\to e}\right)^{\frac{3}{x^2-2}}\right)^{x^2} =$

$= e^{\lim\limits_{x\to+\infty}\frac{3x^2}{x^2-2}} = e^3$

514. $\lim\limits_{x\to\infty}\sqrt[x]{1-2x} = \lim\limits_{x\to\infty}(1-2x)^{\frac{1}{x}} = \lim\limits_{x\to\infty}\left(\left(\underbrace{(1-2x)^{-\frac{1}{2x}}}_{\to e}\right)^{-2x}\right)^{\frac{1}{x}} =$

$= e^{\lim\limits_{x\to\infty}\frac{-2x}{x}} = e^{-2} = \frac{1}{e^2}$

521. $\lim\limits_{x\to 0}\left(\frac{\cos x}{\cos 2x}\right)^{\frac{1}{x^2}} = \{1^\infty\} = \lim\limits_{x\to 0}\left(1+\frac{\cos x}{\cos 2x}-1\right)^{\frac{1}{x^2}} =$

$= \lim\limits_{x\to 0}\left(\left(\underbrace{\left(1+\frac{\cos x-\cos 2x}{\cos 2x}\right)^{\frac{\cos 2x}{\cos x-\cos 2x}}}_{\to e}\right)^{\frac{\cos x-\cos 2x}{\cos 2x}}\right)^{\frac{1}{x^2}} =$

$= e^{\lim\limits_{x\to 0}\frac{\cos x-\cos 2x}{x^2\cos 2x}} = e^{\frac{3}{2}}$

$\lim\limits_{x\to 0}\frac{\cos x-\cos 2x}{x^2\cos 2x} = \frac{+2\sin\frac{3x}{2}\sin\frac{x}{2}}{x^2\cdot\cos 2x} = \frac{2\sin\frac{3}{2}x\,\sin\frac{x}{2}}{\frac{2}{3}\cdot x\cdot\frac{3}{2}\cdot 2\cdot\frac{x}{2}\cos 2x} = \frac{3}{2}$

Домашняя работа.

476.
$$\lim_{x\to0}\frac{\sin5x-\sin3x}{\sin x} = \lim_{x\to0}\frac{2\sin x\cos4x}{\sin x} = 2$$

479.
$$\lim_{x\to\frac{\pi}{4}} tg2x\, tg\left(\frac{\pi}{4}-x\right) = \left\{\begin{matrix}x-\frac{\pi}{4}=t\\ x=t+\frac{\pi}{4}\end{matrix}\right\} = \lim_{t\to0} tg\left(2t+\frac{\pi}{2}\right) tg\left(\frac{\pi}{4}-t-\frac{\pi}{4}\right) =$$
$$= +\lim_{t\to0} ctg2t\, tgt = \lim_{t\to0}\frac{\cos2t\,\sin t}{\sin2t\,\cos t} = \lim_{t\to0}\frac{\cos2t\cdot2\sin t\cdot t\cdot t}{\cos t\cdot2\sin2t\cdot t\cdot t} =$$
$$= \frac{1}{2}$$

499.
$$\lim_{x\to0}\frac{\sqrt{1+tgx}-\sqrt{1+\sin x}}{x^3} = \frac{1+tgx-1+\sin x}{x^3(\sqrt{1+tgx}+\sqrt{1+\sin x})} = \lim_{x\to0}\frac{\frac{\sin x}{\cos x}-\sin x}{x\cdot x^2(\sqrt{1+tgx}-\sqrt{1+\sin x})} =$$
$$= \lim_{x\to0}\frac{\sin x\left(\frac{1}{\cos x}-1\right)}{x\cdot x^2(\sqrt{1+tgx}-\sqrt{1+\sin x})} = 1\lim_{x\to0}\frac{\frac{1}{\cos x}-1}{(\sqrt{1+tgx}-\sqrt{1+\sin x})x^2} =$$
$$= \lim_{x\to0}\frac{1-\cos x}{x^2(\sqrt{1+tgx}-\sqrt{1+\sin x})} = \lim_{x\to0}\frac{2\sin\frac{x}{2}\sin\frac{x}{2}}{\frac{x}{2}\cdot\frac{x}{2}\cdot2\cdot2(\sqrt{1+tgx}+\sqrt{1+\sin x})} =$$
$$= \frac{1}{4}$$

513.
$$\lim_{x\to\infty}\left(\frac{x^2+2x-1}{2x^2-3x-2}\right)^{\frac{1}{x}} = \lim_{x\to\infty}\left(\frac{x^2\left(1+\frac{2}{x}-\frac{1}{x^2}\right)}{x^2\left(2-\frac{3}{x}-\frac{2}{x^2}\right)}\right)^{\frac{1}{x}} = \left(\frac{1}{2}\right)^0 = 1.$$

522.
$$\lim_{x\to\frac{\pi}{4}}(tgx)^{tg2x} = \left\{\begin{matrix}x-\frac{\pi}{4}=t\\ x=t+\frac{\pi}{4}\end{matrix}\right\} = \lim_{t\to0}\left(tg\left(t+\frac{\pi}{4}\right)\right)^{tg\left(2t+\frac{\pi}{2}\right)} =$$
$$= \lim_{t\to0}\left(tg\left(t+\frac{\pi}{4}\right)\right)^{-ctg2t} = \lim_{t\to0}\frac{1}{tg\left(t+\frac{\pi}{4}\right)^{ctg2t}}$$
$$\lim_{x\to\frac{\pi}{4}}(1+tgx-1)^{tg2x} = \lim_{x\to\frac{\pi}{4}}(1+(tgx-1))^{\frac{2tgx}{1-tg^2x}} = \lim_{x\to\frac{\pi}{4}}\left(\underbrace{\left(1+(tgx-1)\right)^{\frac{1}{tgx-1}}}_{e}\right)^{\frac{tgx-1}{1}\cdot\frac{2tgx}{1-tg^2x}} =$$
$$= e^{\lim\limits_{x\to\frac{\pi}{4}}\frac{(tgx-1)2tgx}{1-tg^2x}} = \lim_{x\to\frac{\pi}{4}}\left(1+(tgx-1)\right)^{\frac{1}{tgx-1}\cdot\frac{-2tgx}{1+tgx}} =$$
$$= e^{\lim\limits_{x\to\frac{\pi}{4}}\frac{-2tgx}{1+tgx}} = e^{-1}.$$

523.
$$\lim_{x\to\frac{\pi}{2}}(\sin x)^{tgx} = \{1^{+\infty}\} = \lim_{x\to\frac{\pi}{2}}(1+(\sin x-1))^{\frac{\sin x}{\cos x}} =$$
$$= \left\{\begin{matrix}x-\frac{\pi}{2}=t\\ x=t+\frac{\pi}{2}\end{matrix}\right\} = \lim_{t\to0}\left(1+\sin\left(t+\frac{\pi}{2}\right)-1\right)^{\frac{\sin\left(t+\frac{\pi}{2}\right)}{\cos\left(t+\frac{\pi}{2}\right)}} =$$
$$= \lim_{t\to0}(1+\cos t-1)^{-\frac{\cos t}{\sin t}} = \lim_{t\to0}\left(\left(\underbrace{(1+(\cos t-1))^{\frac{1}{\cos t-1}}}_{e}\right)^{\cos t-1}\right)^{-ctg\,t} =$$

$$= e^{\lim\limits_{t\to0}(\cos t-1)\cdot ctg\, t} = e^{0} = 1.$$

$$-\lim_{t\to0}(\cos t-1)\,ctg\,t = -\lim_{t\to0}\frac{\cos^2 t-\cos t}{\sin t} = -\lim_{t\to0}\frac{(1-\sin^2 t-\cos t)\cdot t}{\sin t\cdot t} = \quad \frac{t}{\sin t} \to 1.$$

$$= -\lim_{t\to0}\frac{1-\sin^2 t-\cos t}{t} = -\lim_{t\to0}\frac{\cancel{t}\left(\frac{1}{t}-\frac{\cos t}{t}-\frac{\sin^2 t}{t}\right)}{\cancel{t}} =$$

$$= \lim_{t\to0}\underbrace{\frac{\sin t}{t}}_{\to 1}\cdot\underbrace{\frac{\sin t}{t}}_{\to 1}\cdot t = 0$$

529) $$\lim_{x\to0}\frac{\ln(1+x)}{x} = \lim_{x\to0}\frac{\ln\left((1+x)^{\frac{1}{x}}\right)^{x}}{x} = \lim_{x\to0}\ln(1+x)^{\frac{1}{x}} = \ln e = 1.$$

581) $$\lim_{x\to a}\frac{\ln x-\ln a}{x-a} = \lim_{x\to a}\frac{\ln\left(\frac{x}{a}\right)}{x-a} = \left\{\begin{matrix}x-a=t\\ x=t+a\end{matrix}\right\} = \lim_{t\to0}\frac{\ln\left(\frac{t+a}{a}\right)}{t} =$$

$(a>0)$ $$= \lim_{t\to0}\frac{\ln\left(1+\frac{t}{a}\right)}{t} = \lim_{t\to0}\frac{\ln\left(\left(1+\frac{t}{a}\right)^{\frac{a}{t}}\right)^{\frac{t}{a}}}{t} = \frac{\ln e}{a} = \frac{1}{a}$$

Самостоятельная работа.

1) $\lim\limits_{x\to -1} \frac{x^3-2x-1}{x^5-2x-1} = \lim\limits_{x\to -1} \frac{(x+1)(x^2-x-1)}{(x+1)(x^4-x^3+x^2-x-1)} = \lim\limits_{x\to -1} \frac{x^2-x-1}{x^4-x^3+x^2-x-1} =$

$= \frac{1+1-1}{1+1+1+1-1} = \frac{1}{3}$

2) $\lim\limits_{x\to +\infty} \frac{\sqrt{x}+\sqrt[3]{x}+\sqrt[4]{x}}{\sqrt{2x+1}} = \lim\limits_{x\to +\infty} \frac{\frac{\sqrt{x}+\sqrt[3]{x}+\sqrt[4]{x}}{\sqrt{x}}}{\frac{\sqrt{2x+1}}{\sqrt{x}}} = \lim\limits_{x\to +\infty} \frac{1+\sqrt[6]{\frac{1}{x}}+\sqrt[8]{\frac{1}{x^2}}}{\sqrt{2+\frac{1}{x}}} =$

$= \frac{1}{\sqrt{2}}$

3) $\lim\limits_{x\to 16} \frac{\sqrt[4]{x}-2}{\sqrt{x}-4} = \lim\limits_{x\to 16} \frac{(\sqrt{x}-4)}{(\sqrt{x}-4)(\sqrt[4]{x}+2)} = \lim\limits_{x\to 16} \frac{1}{\sqrt[4]{x}+2} = \frac{1}{4}$

4) $\lim\limits_{x\to 0} \frac{\sqrt[3]{8+3x-x^2}-2}{x+x^2} = \lim\limits_{x\to 0} \frac{(\sqrt[3]{8+3x-x^2}-2)(\sqrt[3]{(8+3x-x^2)^2}+2\sqrt[3]{8+3x-x^2}+4)}{(x+x^2)(\sqrt[3]{(8+3x-x^2)^2}+2\sqrt[3]{8+3x-x^2}+4)} =$

$= \lim\limits_{x\to 0} \frac{8+3x-x^2-8}{(x+x^2)(\sqrt[3]{(8+3x-x^2)^2}+2\sqrt[3]{8+3x-x^2}+4)} = \lim\limits_{x\to 0} \frac{x(3-x)}{x(1+x)(\sqrt[3]{(8+3x-x^2)^2}\ldots}$

$= \lim\limits_{x\to 0} \frac{3-x}{(1+x)(\sqrt[3]{(8+3x-x^2)^2}+2\sqrt[3]{8+3x-x^2}+4)} = \frac{3}{(4+4+4)} = \frac{3}{3\cdot 4} = \frac{1}{4}$

5) $\lim\limits_{x\to 0} \frac{tg x - \sin x}{\sin^3 x} = \lim\limits_{x\to 0} \left(\frac{\sin x}{\cos x \cdot \sin^3 x} - \frac{1}{\sin x \cdot \sin x}\right) =$

$= \lim\limits_{x\to 0} \left(\frac{1}{\sin x \cdot \sin x \cdot \cos x} - \frac{1}{\sin x \cdot \sin x}\right) = \lim\limits_{x\to 0} \frac{1-\cos x}{\sin x \cdot \sin x \cdot \cos x} =$

$= \lim\limits_{x\to 0} \frac{2\sin\frac{x}{2}\cdot\sin\frac{x}{2}}{\sin x\cdot\sin x\cdot\cos x} = \lim\limits_{x\to 0} \frac{\sin\frac{x}{2}\cdot\sin\frac{x}{2}\cdot\left(\frac{x}{2}\right)\left(\cdot\frac{x}{2}\right)\cdot 2}{\sin x\left(\frac{x}{2}\right)\sin x\left(\frac{x}{2}\right)\cos x} = \frac{1}{2}$

1

6) $\lim\limits_{x\to a} \frac{\cos x - \cos a}{x-a} = \lim\limits_{x\to a} \frac{-2\sin\frac{x+a}{2}\sin\frac{x-a}{2}}{x-a} = \left\{\begin{matrix} x-a=t \\ x=t+a \end{matrix}\right\} =$

$= \lim\limits_{t\to 0} \frac{-2\sin\frac{(t+2a)}{2}\sin\frac{t}{2}}{t} = \lim\limits_{t\to 0} \frac{-2(\sin\frac{t+2a}{2}\sin\frac{t}{2}\cdot\cancel{\frac{(t+2a)}{2}}}{\frac{t}{2}\cdot 2\cdot\cancel{\frac{t+2a}{2}}} = ?$

1

$= \quad -\sin a$

7) $\lim\limits_{x\to 0} (1+x^2)^{ctg^2 x} = \lim\limits_{x\to 0} \left(\left(\underbrace{(1+x^2)^{\frac{1}{x^2}}}_{e}\right)^{x^2}\right)^{ctg^2 x} = e^{\lim\limits_{x\to 0} x^2 ctg^2 x} = e$

$\lim\limits_{x\to 0} x^2 ctg^2 x = \frac{x\cdot x\cdot\cos x\cdot\cos x}{\sin x\cdot\sin x} = 1$

8)

9)

10)

8) $\lim\limits_{x\to 1}(1+\sin\pi x)^{ctg\pi x} = \left\{\begin{matrix} x-1=t \\ x=t+1 \end{matrix}\right\} = \lim\limits_{t\to 0}\left(1+\sin\pi(t+1)\right)^{ctg(\pi t+\pi)}$

$= \lim\limits_{t\to 0}\left(1+\sin(\pi t+\pi)\right)^{ctg(\pi t+\pi)} = \lim\limits_{t\to 0}\Bigg(\underbrace{\left(1+\sin(\pi t+\pi)\right)^{\frac{1}{\sin(\pi t+\pi)}}}_{e}\Bigg)^{\sin(\pi t+\pi)\cdot ctg(\pi t+\pi)} =$

$= e^{\lim\limits_{t\to 0}\frac{\sin(\pi t+\pi)\cdot\cos(\pi t+\pi)}{\sin(\pi t+\pi)}} = e^{\lim\limits_{t\to 0}(\cos(\pi t+\pi))} = \frac{1}{e}$

9) $\lim\limits_{x\to 0}\sqrt[x]{\cos\sqrt{x}} = \lim\limits_{x\to 0}(\cos\sqrt{x})^{\frac{1}{x}} = \lim\limits_{x\to 0}\left(1+(\cos\sqrt{x}-1)\right)^{\frac{1}{x}} =$

$= \lim\limits_{x\to 0}\Bigg(\underbrace{\left(1+(\cos\sqrt{x}-1)\right)^{\frac{1}{\cos\sqrt{x}-1}}}_{e}\Bigg)^{(\cos\sqrt{x}-1)\cdot\frac{1}{x}} = e^{\lim\limits_{x\to 0}\frac{\cos\sqrt{x}-1}{x}} = e^{-\frac{1}{2}} = \frac{1}{\sqrt{e}}$

$\lim\limits_{x\to 0}\frac{\cos\sqrt{x}-1}{x} = \lim\limits_{x\to 0}\frac{\cos^2\sqrt{x}-1}{x(\cos\sqrt{x}+1)} = \lim\limits_{x\to 0}\frac{\cancel{1}-\sin^2\sqrt{x}\cancel{-1}}{x(\cos\sqrt{x}+1)} =$

$= -\lim\limits_{x\to 0}\frac{\sin\sqrt{x}\cdot\sin\sqrt{x}\cdot\cancel{\sqrt{x}\cdot\sqrt{x}}}{\cancel{x}\cdot\sqrt{x}\cdot\sqrt{x}(\cos\sqrt{x}+1)} = -\frac{1}{2}$

$1 \quad 1$

10) $\lim\limits_{n\to\infty}\left(\frac{n+x}{n-1}\right)^n = \lim\limits_{n\to\infty}\left(\frac{n-1+1+x}{n-1}\right)^n = \lim\limits_{n\to\infty}\left(1+\frac{1+x}{n-1}\right)^n =$

$= \lim\limits_{n\to\infty}\Bigg(\underbrace{\left(1+\frac{1+x}{n-1}\right)^{\frac{n-1}{1+x}}}_{e}\Bigg)^{\frac{1+x}{n-1}\cdot n} = e^{\lim\limits_{n\to\infty}\frac{n(1+x)}{n-1}} = e^{(1+x)}$

$\lim\limits_{n\to\infty}\frac{n(1+x)}{n-1} = \lim\limits_{n\to\infty}\frac{\cancel{n}(1+x)}{\cancel{n}(1-\frac{1}{n})} = \lim\limits_{n\to\infty}\frac{1+x}{1-\frac{1}{n}} = 1+x$

Задача 1.

$$8)\ \lim_{n\to\infty}\frac{4n-3}{2n+1}=\lim_{n\to\infty}\frac{n\left(4-\frac{3}{n}\right)}{n\left(2+\frac{1}{n}\right)}=\frac{1}{2}\qquad =a,\quad \text{ч.т.д.}$$

$$16)\ \lim_{n\to\infty}\frac{3n^3}{n^3-1}=\lim_{n\to\infty}\frac{3}{1-\frac{1}{n^3}}=3\qquad =a,\quad \text{ч.т.д.}$$

~~Задача 2.~~

$$\cancel{4)\ \lim_{n\to\infty}\frac{(1-n)^4-(1+n)^4}{(1+n)^3-(1-n)^3}=\lim_{n\to\infty}\frac{n^4\left(\left(\frac{1}{n}-1\right)^4-\left(\frac{1}{n}+1\right)^4\right)}{n^3\left(\left(\frac{1}{n}+1\right)^3-\left(\frac{1}{n}-1\right)^3\right)}=}$$

$$\cancel{=\lim_{n\to\infty}\frac{n^4\left(\frac{1}{n}-1\right)^4-n^4\left(\frac{1}{n}+1\right)^4}{n^3\left(\frac{1}{n}+1\right)^3-n^3\left(\frac{1}{n}-1\right)^3}=\lim_{n\to\infty}\frac{n^4\left[\left(\frac{1}{n}-1\right)^4-\left(\frac{1}{n}+1\right)^4\right]}{n^3\left[\left(\frac{1}{n}+1\right)^3-\left(\frac{1}{n}-1\right)^3\right]}=\infty}$$

$=$

$$\cancel{18)\ \lim_{n\to\infty}\frac{(n+10)^2+(3n+1)^2}{(n+6)^3-(n+1)^3}=\lim_{n\to\infty}\frac{n^2\left(1+\frac{10}{n}\right)^2+n^2\left(3+\frac{1}{n}\right)^2}{n^3\left(1+\frac{6}{n}\right)^3-n^3\left(1+\frac{1}{n}\right)^3}=}$$

$$\cancel{=\lim_{n\to\infty}\frac{\left(1+\frac{10}{n}\right)^2+\left(3+\frac{1}{n}\right)^2}{n\left(\left(1+\frac{6}{n}\right)^3-\left(1+\frac{1}{n}\right)^3\right)}}$$

~~Задача 3-11(,13)~~

$$\cancel{11)\ \lim_{n\to\infty}\frac{n\sqrt[4]{3n+1}+\sqrt{81n^4-n^2+1}}{(n+\sqrt[3]{n})\cdot\sqrt{5-n+n^2}}}$$

Задача 2.

$$4)\ \lim_{n\to\infty}\frac{(1-n)^4-(1+n)^4}{(1+n)^3-(1-n)^3}=\lim_{n\to\infty}\frac{(1-2n+n^2-1-2n-n^2)\cdot(1-2n+n^2+1+2n+n^2)}{6n+2n^3}=$$

$$=\lim_{n\to\infty}\frac{-4n(2+2n^2)}{2n(3+n^2)}=\lim_{n\to\infty}-2\cdot\frac{n^2\left(\frac{2}{n^2}+2\right)}{n^2\left(\frac{3}{n^2}+1\right)}=-4$$

$$18)\ \lim_{n\to\infty}\frac{(n+10)^2+(3n+1)^2}{(n+6)^3-(n+1)^3}=\lim_{n\to\infty}\frac{10n^2+26n+101}{15n^2+105n+43\cdot5}=\lim_{n\to\infty}\frac{n^2\left(10+\frac{26}{n}+\frac{101}{n^2}\right)}{n^2\left(15+\frac{105}{n}+\frac{215}{n^2}\right)}=\frac{2}{3}$$

Задача 3.

$$11)\ \lim_{n\to\infty}\frac{n\sqrt[4]{3n+1}+\sqrt{81n^4-n^2+1}}{(n+\sqrt[3]{n})\sqrt{5-n+n^2}}=\lim_{n\to\infty}\frac{n^2\left(\sqrt[4]{\frac{3}{n^3}+\frac{1}{n^4}}+\sqrt{81-\frac{1}{n^2}+\frac{1}{n^4}}\right)}{n^2\left(1+\sqrt[3]{\frac{1}{n^2}}\right)\sqrt{\frac{5}{n^2}-\frac{1}{n}+1}}=9$$

$$13)\ \lim_{n\to\infty}\frac{\sqrt{n^3+3}-\sqrt{n-3}}{\sqrt[5]{n^5-4}-\sqrt{n-3}}=\lim_{n\to\infty}\frac{n^{\frac{3}{2}}\left(\sqrt{1+\frac{3}{n^3}}-\sqrt{\frac{1}{n^2}-\frac{3}{n^3}}\right)}{n\left(\sqrt[5]{1-\frac{4}{n^5}}-\sqrt{\frac{1}{n}-\frac{3}{n}}\right)}=\infty$$

Задача 4. (7, 12)

7) $\lim\limits_{n\to\infty}$ (

$=$

12) $\lim\limits_{n\to\infty}$

$=\lim\limits_{n\to}$

Задача

2) $\lim\limits_{n\to\infty}$

$=\lim\limits_{n\to}$

14) $\lim\limits_{n\to\infty}$

15) $\lim\limits_{n\to}$

$=$

27)

31)

$=$

Задача

22)

31)

$=$

7) $\lim\limits_{n\to\infty}(\sqrt[3]{n^3}+\sqrt[3]{4-n^3}) = \lim\limits_{n\to\infty}\frac{n^3+4-n^3}{n^2-n\sqrt[3]{(4-n^3)}+\sqrt[3]{(4-n^3)^2}} = \lim\limits_{n\to\infty}\frac{4}{n^2\left(1-\sqrt[3]{\frac{4}{n^3}-1}+\sqrt[3]{\left(\frac{4}{n^3}-1\right)^2}\right)} =$

$= 0$

12) $\lim\limits_{n\to\infty} n^2(\sqrt[3]{5+n^3}-\sqrt[3]{3+n^3}) = \lim\limits_{n\to\infty}\frac{n^2(5+n^3-3-n^3)}{\sqrt[3]{(5+n^3)^2}+\sqrt[3]{(5+n^3)(3+n^3)}+\sqrt[3]{(3+n^3)^2}} =$

$= \lim\limits_{n\to\infty}\frac{2n^2}{\sqrt[3]{n^6+10n^3+25}+\sqrt[3]{n^6+8n^6+15}+\sqrt[3]{n^6+6n^3+9}} = \frac{2}{3}$

Задача 5. (2, 12, 15, 27, 31)

2) $\lim\limits_{n\to\infty}\frac{(2n+1)!+(2n+2)!}{(2n+3)!} = \lim\limits_{n\to\infty}\frac{(2n+1)!\,(1+2n+2)}{(2n+1)!\,(2n+2)(2n+3)} =$

$= \lim\limits_{n\to\infty}\frac{2n+3}{n^2(2+\frac{2}{n})(2+\frac{3}{n})} = 0$

12) $\lim\limits_{n\to\infty}\frac{1+\frac{1}{3}+\frac{1}{3^2}+\ldots+\frac{1}{3^n}}{1+\frac{1}{5}+\frac{1}{5^2}+\ldots+\frac{1}{5^n}} = \lim\limits_{n\to\infty}\frac{1\cdot(1-\frac{1}{3^n})(1-\frac{1}{5})}{(1-\frac{1}{3})(1-\frac{1}{5^n})} = \frac{4\cdot3}{5\cdot2} = \frac{6}{5}$

15) $\lim\limits_{n\to\infty}\frac{\sqrt[3]{n^3+5}-\sqrt{3n^4+2}}{1+3+5+\ldots+(2n-1)} = \lim\limits_{n\to\infty}\frac{\sqrt[3]{n^3+5}-\sqrt{3n^4+2}}{n^2} = \lim\limits_{n\to\infty}\frac{n^2\left(\sqrt[3]{\frac{1}{n^3}+\frac{5}{n^6}}-\sqrt{3+\frac{2}{n^4}}\right)}{n^2} =$

$= -\sqrt{3}$

27) $\lim\limits_{n\to\infty}\frac{2^n+7^n}{2^n-7^{n-1}} = \lim\limits_{n\to\infty}\frac{7^n\left(\left(\frac{2}{7}\right)^n+1\right)}{7^n\left(\left(\frac{2}{7}\right)^n-\frac{1}{7}\right)} = -7$

31) $\lim\limits_{n\to\infty}\left(\frac{2+4+\ldots+2n}{n+3}-n\right) = \lim\limits_{n\to\infty}\left(\frac{\frac{2+2n}{2}\cdot n}{n+3}-n\right) = \lim\limits_{n\to\infty}\frac{n+\cancel{n^2}-\cancel{n^2}-3n}{n+3} =$

$= \lim\limits_{n\to\infty}\frac{-2\cancel{n}}{\cancel{n}(1+\frac{3}{n})} = -2$

Задача 6 (22, 31)

22) $\lim\limits_{n\to\infty}\left(\frac{n+3}{n+1}\right)^{-n^2} = \lim\limits_{n\to\infty}\left(\frac{n+1+2}{n+1}\right)^{-n^2} = \lim\limits_{n\to\infty}\Bigg(\underbrace{\left(1+\frac{2}{n+1}\right)^{\frac{n+1}{2}}}_{e}\Bigg)^{\frac{-2n^2}{n+1}} = e^{\lim\limits_{n\to\infty}\frac{-2n^2}{n(1+\frac{1}{n})}} =$

$= e^{-\infty} = 0$

31) $\lim\limits_{n\to\infty}\left(\frac{4n^2+4n-1}{4n^2+2n+3}\right)^{1-2n} = \lim\limits_{n\to\infty}\left(\left(1+\frac{2n-4}{4n^2+2n+3}\right)^{\frac{4n^2+2n+3}{2n-4}}\right)^{\frac{(2n-4)(1-2n)}{4n^2+2n+3}} = e^{\lim\limits_{n\to\infty}\frac{-4n^2+10n-4}{4n^2+2n+3}} =$

$= e^{-1} = \frac{1}{e}$

Задача 9 (21, 31)

21) $\lim\limits_{x\to -1} \frac{x^3-3x-2}{x^2+2x+1} = \lim\limits_{x\to-1} \frac{(x+1)^2(x-2)}{(x+1)^2} = \lim\limits_{x\to -1}(x-2) = -3$

31) $\lim\limits_{x\to 3} \frac{x^3-4x^2-3x+18}{x^3-5x^2+3x+9} = \frac{\cancel{(x-3)^2}(x+2)}{\cancel{(x-3)^2}(x+1)} = \frac{5}{4}$

Задача 10 (10, 30)

10) $\lim\limits_{x\to 0} \frac{\sqrt[3]{27+x}-\sqrt[3]{27-x}}{x+2\sqrt[3]{x^4}} = \lim\limits_{x\to 0} \frac{2x}{(x+2\sqrt[3]{x^4})(\sqrt[3]{(27+x)^2}+\sqrt[3]{(27+x)(27-x)}+\sqrt[3]{(27-x)^2})}$

$= \lim\limits_{x\to 0} \frac{2\cancel{x}}{\cancel{x}(1+2\sqrt[3]{x})(\sqrt[3]{(27+x)^2}+\sqrt[3]{(27+x)(27-x)}+\sqrt[3]{(27-x)^2})} = \frac{2}{27}$

30) $\lim\limits_{x\to -8} \frac{10-x-6\sqrt{1-x}}{2+\sqrt[3]{x}} = \lim\limits_{x\to -8} \frac{(10-x)^2-36(1-x)}{2+\sqrt[3]{x}\,(10-x+6\sqrt{1-x})} =$

$= \lim\limits_{x\to -8} \frac{(100-20x+x^2-36+36x)(4-2\sqrt[3]{x}+\sqrt[3]{x^2})}{(8+x)(10-x+6\sqrt{1-x})} =$

$= \lim\limits_{x\to -8} \frac{(x^2+16x+64)(4-2\sqrt[3]{x}+\sqrt[3]{x^2})}{(x+8)(10-x+6\sqrt{1-x})} = \lim\limits_{x\to -8} \frac{(x+8)^2(4-2\sqrt[3]{x}+\sqrt[3]{x^2})}{(x+8)(10-x+6\sqrt{1-x})} =$

$= \lim\limits_{x\to -8} \frac{(x+8)(4-2\sqrt[3]{x}+\sqrt[3]{x^2})}{(10-x+6\sqrt{1-x})} = \lim\limits_{x\to -8} \frac{(2+\sqrt[3]{x})(4-2\sqrt[3]{x}+\sqrt[3]{x^2})^2}{10-x+6\sqrt{1-x}} = 0$

Задача 16 (10, 22.)

10) $\lim\limits_{x\to 0} (1+\sin^2 3x)^{\frac{1}{\ln\cos x}} = \lim\limits_{x\to 0} \left((1+\sin^2 3x)^{\frac{1}{\sin^2 3x}}\right)^{\frac{\sin^2 3x}{\ln\cos x}} = e^{\lim\limits_{x\to 0}\frac{3x\cdot\sin 3x\cdot\sin 3x\cdot 3x}{\ln\cos x\cdot 3x\cdot 3x}} =$

$18x^2$

$= e^{\lim\limits_{x\to 0}\frac{9x^2}{\ln\cos x}} = e^{\lim\limits_{x\to 0}\frac{9x^2\cdot 2}{\ln(1-\sin^2 x)}} = e^{\lim\limits_{x\to 0}} = e^{-18}$

$\lim\limits_{x\to 0} \frac{\sin^2 3x}{\ln\cos x} = \lim\limits_{x\to 0} \frac{\sin 3x\cdot\sin 3x\cdot 3x\cdot 3x\cdot 2}{3x\cdot 3x\;\ln\left(1-\frac{\sin x\cdot\sin x\cdot x^2}{x\cdot x}\right)} = \lim\limits_{x\to 0}\frac{18x^2}{\ln(1-x^2)} =$

$= \lim\limits_{x\to 0} \frac{18x^2}{\ln\left((1+(-x^2))^{-\frac{1}{x^2}}\right)^{-x^2}} = -18$

22) $\lim\limits_{x\to 0} \left(3-\frac{2}{\cos x}\right)^{\operatorname{cosec}^2 x} = \lim\limits_{x\to 0}\left(3-\frac{2}{\cos x}\right)^{\frac{1}{\sin^2 x}} =$?

$= \lim\limits_{x\to 0} \left(\left(e^{\ln\left(3-\frac{2}{\cos x}\right)}\right)^{\frac{1}{\sin^2 x}}\right) = \lim\limits_{x\to 0} e^{\frac{\ln\left(3-\frac{2}{\cos x}\right)}{\sin^2 x}}$

Предел функции.

401. $\lim\limits_{x\to 2} x^2 = 4$

$\lim\limits_{x\to a} f(x) = A \Leftrightarrow \forall\varepsilon>0\ \exists\delta=\delta(\varepsilon)>0,\ \forall x\in E\ \text{ch}\ 0<|x-a|<\delta:\ |f(x)-A|<\varepsilon$

$|x^2-4|<\varepsilon$

$|x-2|\cdot|x+2|<\varepsilon$

$|x^2-4| = |x-2|\cdot|x+2| = |x-2|\cdot|x-2+4| \le |x-2|(|x-2|+4) < \varepsilon$

$|x-2| = t,\ t>0$

$t(t+4)<\varepsilon$

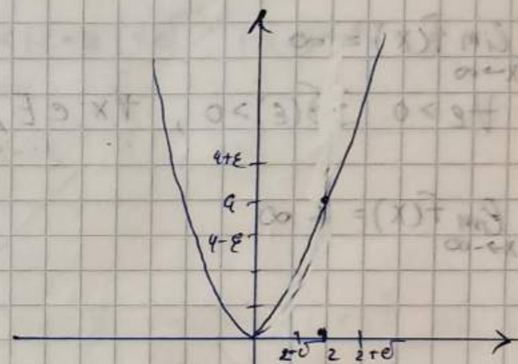
$t^2+4t-\varepsilon<0$

$t_{1,2} = -2 \pm \sqrt{4+\varepsilon}$

$-2-\sqrt{4+\varepsilon} < t < \sqrt{4+\varepsilon}-2$

$0<t<\sqrt{4+\varepsilon}-2$

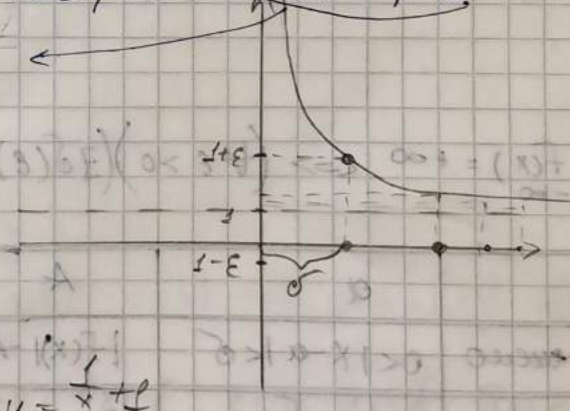
$0<|x-2|<\underbrace{\sqrt{4+\varepsilon}-2}_{\delta}$



$\varepsilon=0{,}1 \to \delta=\sqrt{4+0{,}1}-2 \approx 0{,}025$

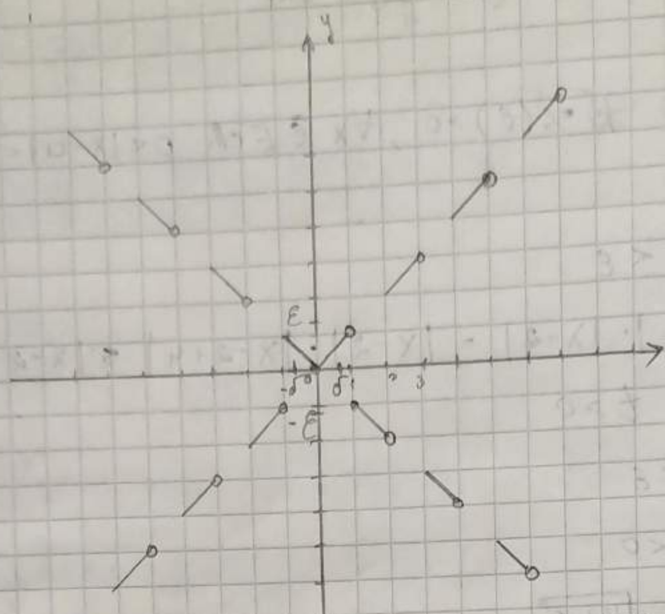
$\lim\limits_{x\to+\infty} = A\in\mathbb{R} \Leftrightarrow \forall\varepsilon>0\ \exists\delta(\varepsilon)>0,\ \forall x\in E,\ x>\delta:\ |f(x)-A|<\varepsilon$

$x\to+\infty \qquad A-\varepsilon<f(x)<A+\varepsilon$



$y=\frac{1}{x}+1$

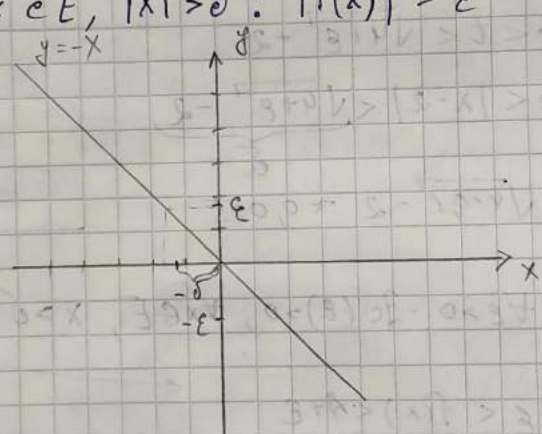
$y = x \cdot (-1)^{[x]}$



$\lim\limits_{x\to\infty} f(x) = \infty$

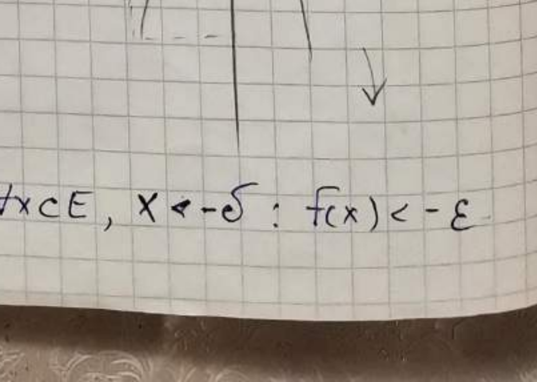
$\forall \varepsilon > 0 \ \ \exists \delta(\varepsilon) > 0, \ \ \forall x \in E, \ |x| > \delta : \ |f(x)| > \varepsilon$ (когда x и $f(x) \to \infty$)

$\lim\limits_{x\to-\infty} f(x) = +\infty$



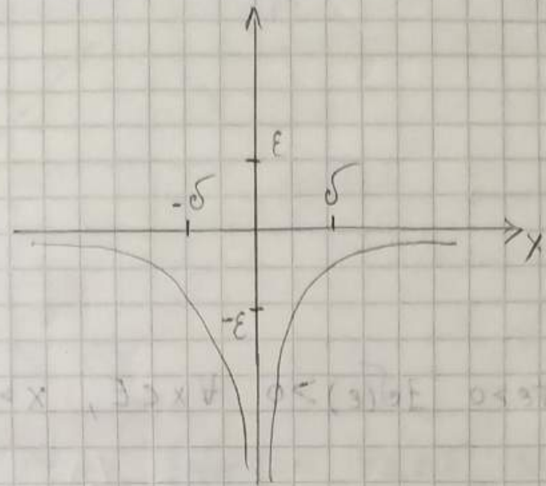
$\lim\limits_{x\to-\infty} f(x) = +\infty \iff (\forall \varepsilon > 0)(\exists \delta(\varepsilon) > 0)(\forall x \in E, \ x < -\delta) : f(x) > \varepsilon$

	a	A
число	$0 < \lvert x - a \rvert < \delta$	$\lvert f(x) - A \rvert < \varepsilon$
$+\infty$	$x > \delta$	$f(x) > \varepsilon$
$-\infty$	$x < -\delta$	$f(x) < -\varepsilon$
∞	$\lvert x \rvert > \delta$	$\lvert f(x) \rvert > \varepsilon$



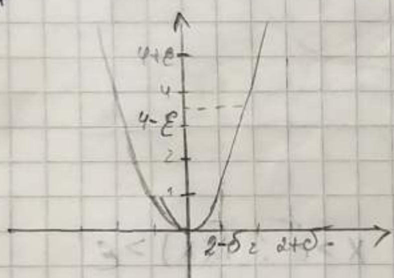
$\lim\limits_{x\to-\infty} f(x) = -\infty \iff \forall \varepsilon > 0 \ \ \exists \delta(\varepsilon) > 0 \ \forall x \in E, \ x < -\delta : \ f(x) < -\varepsilon$

$\lim\limits_{x\to a} f(x) = -\infty \iff \forall \varepsilon > 0 \quad \exists \delta(\varepsilon) > 0 \quad \forall x \subset E, \; 0<|x-a|<\delta : f(x) < -\varepsilon$



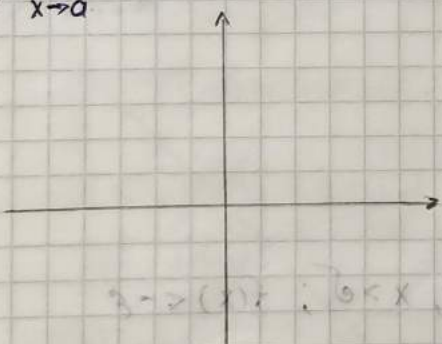
Домашняя работа.

1) $\lim\limits_{x\to a} f(x) = A \iff \forall \varepsilon > 0 \quad \exists \delta(\varepsilon) \geq 0 \quad \forall x \subset E, \quad 0<|x-a|<\delta : |f(x) - A| < \varepsilon$

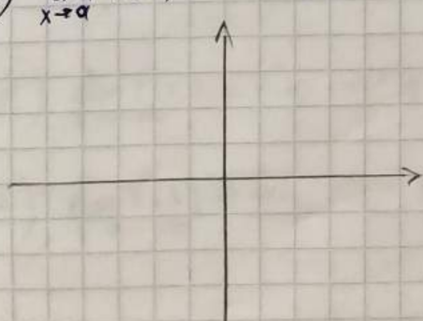


$\lim\limits_{x\to 2} x^2 = 4$

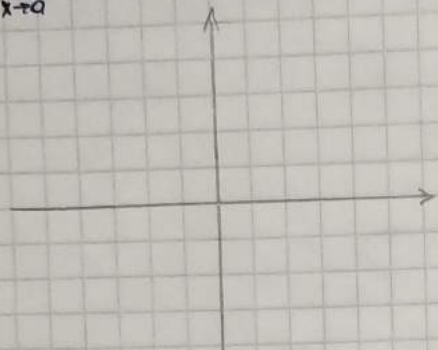
2) $\lim\limits_{x\to a} f(x) = +\infty \iff \forall \varepsilon > 0 \quad \exists \delta(\varepsilon) > 0 \quad \forall x \subset E, \; 0<|x-a|<\delta : f(x) > \varepsilon$



3) $\lim\limits_{x\to a} f(x) = -\infty \iff \forall \varepsilon > 0 \quad \exists \delta(\varepsilon) > 0 \quad \forall x \subset E, \; 0<|x-a|<\delta : f(x) < -\varepsilon$

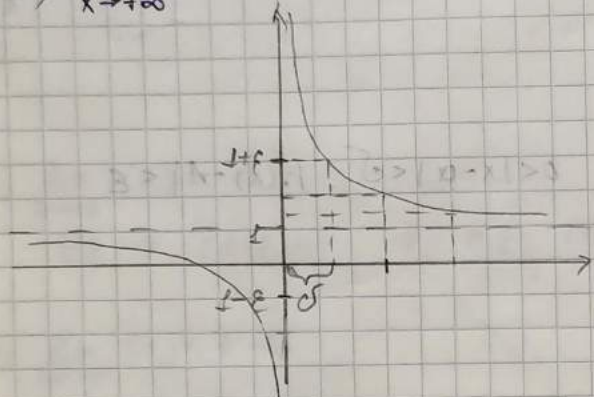


4) $\lim\limits_{x \to a} f(x) = \infty \iff \forall \varepsilon > 0 \ \exists \delta(\varepsilon) > 0 \ \ \forall x \subset E, \ \ 0 < |x - a| < \delta : |f(x)| > \varepsilon$



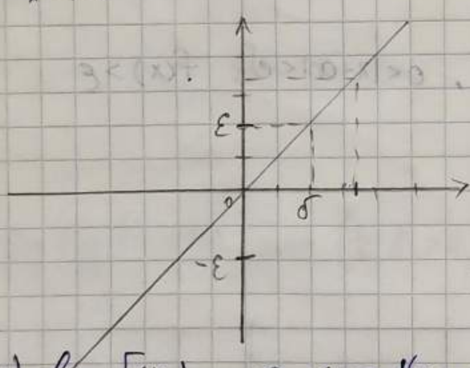
5) $\lim\limits_{x \to +\infty} f(x) = A \iff \forall \varepsilon > 0 \ \exists \delta(\varepsilon) > 0 \ \ \forall x \subset E, \ \ x > \delta : \ |f(x) - A| < \varepsilon$

$\lim\limits_{x \to +\infty} \left(1 + \frac{1}{x}\right) = 1$



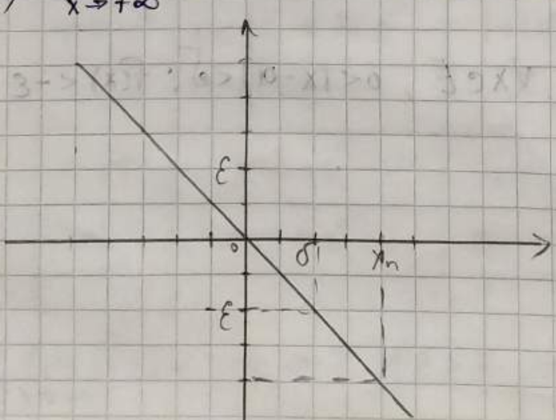
6) $\lim\limits_{x \to +\infty} f(x) = +\infty \iff \forall \varepsilon > 0 \ \exists \delta(\varepsilon) > 0 \ \ \forall x \subset E, \ \ x > \delta : \ f(x) > \varepsilon$

$\lim\limits_{x \to +\infty} (x) = +\infty$

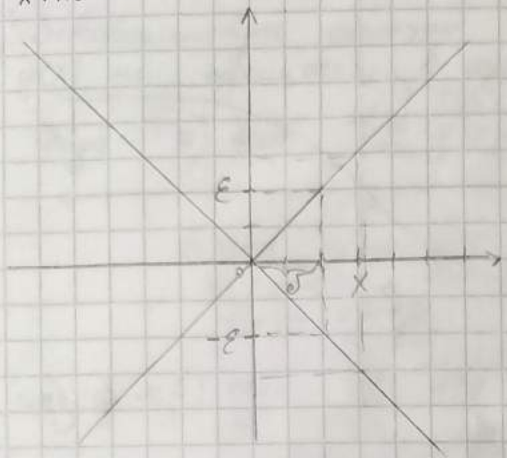


7) $\lim\limits_{x \to +\infty} f(x) = -\infty \iff \forall \varepsilon > 0 \ \exists \delta(\varepsilon) > 0 \ \ \forall x \subset E, \ x > \delta ; \ f(x) < -\varepsilon$

$\lim\limits_{x \to +\infty} (-x) = -\infty$



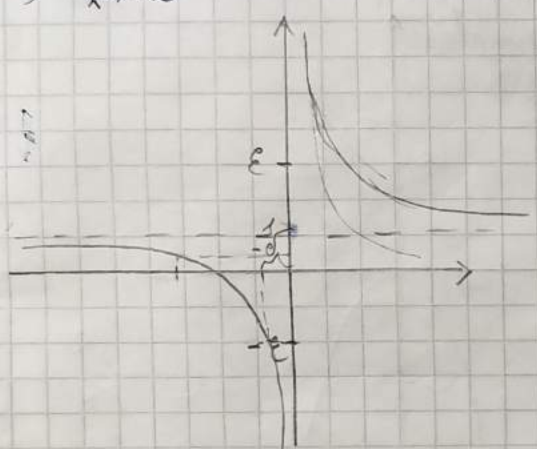
8) $\lim\limits_{x \to +\infty} f(x) = \infty \iff \forall \varepsilon > 0 \quad \exists \delta(\varepsilon) > 0 \quad \forall x \in E,\ x > \delta : |f(x)| > \varepsilon$



$|y| = |x|$

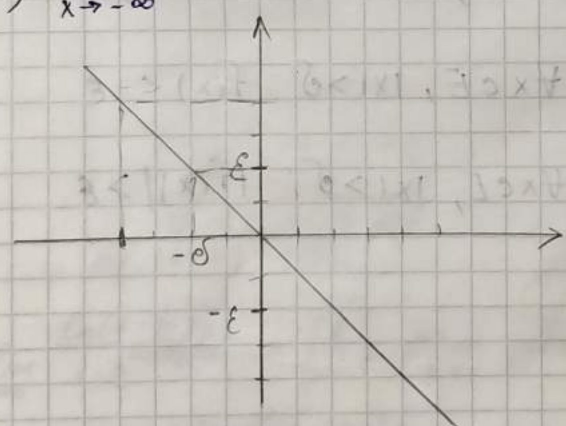
9) $\lim\limits_{x \to -\infty} f(x) = A \iff \forall \varepsilon > 0 \quad \exists \delta(\varepsilon) > 0 \quad \forall x \in E,\ x < -\delta : |fx - A| < \varepsilon$

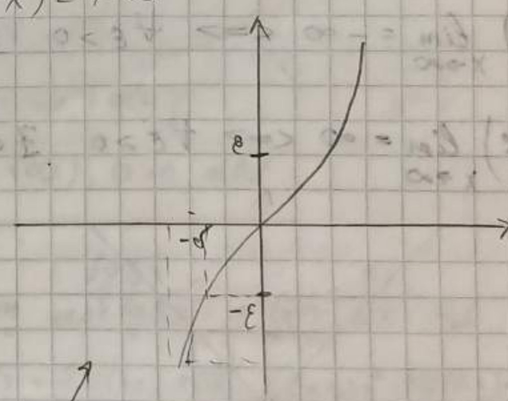
$\lim\limits_{x \to -\infty} \left(1 + \frac{1}{x}\right) = 1$



10) $\lim\limits_{x \to -\infty} f(x) = +\infty \iff \forall \varepsilon > 0 \quad \exists \delta(\varepsilon) > 0 \quad \forall x \in E,\ x < -\delta : f(x) > \varepsilon$

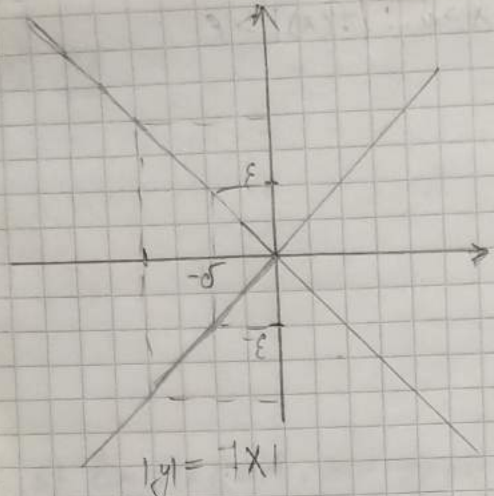
$\lim\limits_{x \to -\infty} (-x) = +\infty$

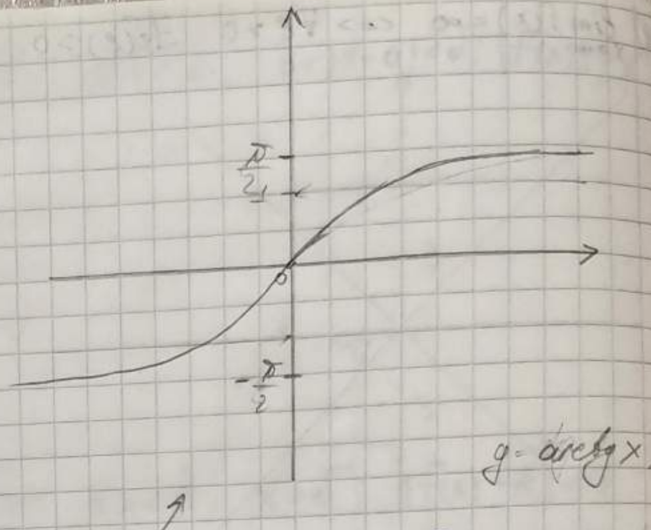




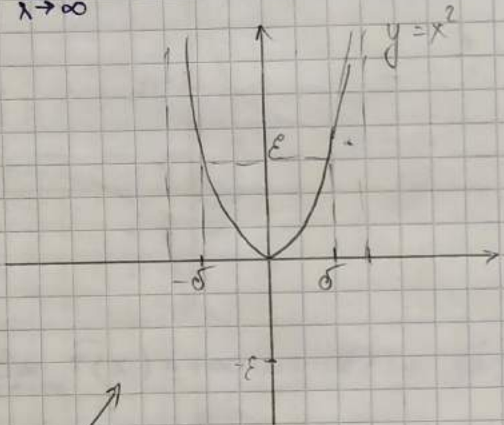
11) $\lim\limits_{x \to -\infty} f(x) = -\infty \iff \forall \varepsilon > 0 \quad \exists \delta(\varepsilon) > 0 \quad \forall x \in E,\ x < -\delta : f(x) < -\varepsilon$

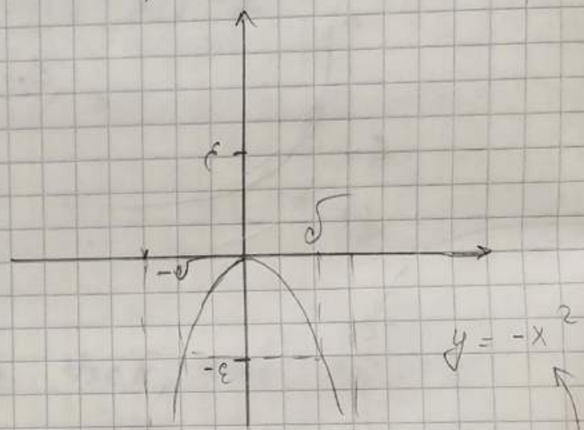
12) $\lim\limits_{x \to -\infty} f(x) = \infty \iff \forall \varepsilon > 0 \quad \exists \delta(\varepsilon) > 0 \quad \forall x \in E,\ x < -\delta : |f(x)| > \varepsilon$





13) $\lim\limits_{x\to\infty} = A \iff \forall \varepsilon>0 \ \ \exists\, \delta(\varepsilon)>0 \ \ \forall x \in E, \ |x|>\delta : \ |f(x)-A|<\varepsilon$

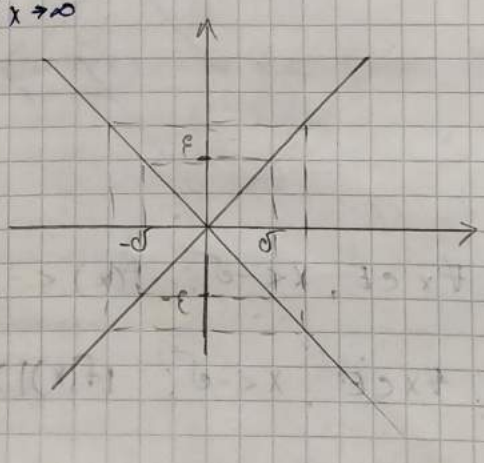




14) $\lim\limits_{x\to\infty} = +\infty \iff \forall \varepsilon>0 \ \ \exists\, \delta(\varepsilon)>0 \ \ \forall x \in E, \ |x|>\delta : \ f(x)>\varepsilon$

15) $\lim\limits_{x\to\infty} = -\infty \iff \forall \varepsilon>0 \ \ \exists\, \delta(\varepsilon)>0 \ \ \forall x \in E, \ |x|>\delta : \ f(x)<-\varepsilon$

16) $\lim\limits_{x\to\infty} = \infty \iff \forall \varepsilon>0 \ \ \exists\, \delta(\varepsilon)>0 \ \ \forall x \in E, \ |x|>\delta : \ |f(x)|>\varepsilon$



Сравнение функций.

1)° Эквивалентные функции. Символы $O(f)$ и $o(f)$. Пусть функция $g(x)$ не обращается в нуль в некоторой проколотой окрестности точки x_0.

Тогда: а) если $\lim\limits_{x \to x_0} \frac{f(x)}{g(x)} = 1 \Rightarrow f(x)$ эквивалентна $g(x)$ $(f(x) \sim g(x))$ при $x \to x_0$

б) если существует число $C > 0$, где справедливо $\left|\frac{f(x)}{g(x)}\right| \leq C$, то говорят, что $f(x)$ есть O большое от $g(x)$ при $x \to x_0$
$f(x) = O(g(x))$

в) если $\lim\limits_{x \to x_0} \frac{f(x)}{g(x)} = 0$ говорят, что $f(x)$ есть о малое от $g(x)$ при $x \to x_0$
, $f(x) = o(g(x))$ $x \to x_0$

2 Таблица эквивалентных функций.

Эквивалентность при $x \to 0$	Равенство при $x \to 0$
$\sin x \sim x$	$\sin x = x + o(x)$
$\operatorname{sh} x \sim x$	$\operatorname{sh} x = x + o(x)$
$\operatorname{tg} x \sim x$	$\operatorname{tg} x = x + o(x)$
$\arcsin x \sim x$	$\arcsin x = x + o(x)$
$\operatorname{arctg} \sim x$	$\operatorname{arctg} x = x + o(x)$
$1 - \cos x \sim \frac{x^2}{2}$	$1 - \cos x = \frac{x^2}{2} + o(x^2)$
$\operatorname{ch} x - 1 \sim \frac{x^2}{2}$	$\operatorname{ch} x - 1 = \frac{x^2}{2} + o(x^2)$
$e^x - 1 \sim x$	$e^x - 1 = x + o(x)$
$\ln(1+x) \sim x$	$\ln(1+x) = x + o(x)$
$(1+x)^\alpha - 1 \sim \alpha x$	$(1+x)^\alpha = 1 + \alpha x + o(x)$
$a^x - 1 \sim x \ln a$	$a^x = 1 + x\ln a + o(x)$, $a > 0$, $a \neq 1$
$\log_a(1+x) \sim \frac{x}{\ln a}$ *	

$$\lim_{x \to 0} \frac{\ln(1 - \operatorname{tg} 3x)}{1 - e^{\operatorname{arctg} 3x}} = \left\{\frac{0}{0}\right\} = -\lim_{x \to 0} \frac{-\operatorname{tg} 3x}{e^{\operatorname{arctg} 3x} - 1} = \lim_{x \to 0} \frac{\operatorname{tg} 3x}{\operatorname{arctg} 3x} = \lim_{x \to 0} \frac{3x}{3x} = 1.$$

$$\lim_{x \to 0} \frac{\sin 2x + 2\operatorname{arctg} 3x + 3x^2}{\ln(1 + 3x + \sin^2 x) + xe^x} = \lim_{x \to 0} \frac{2x + o(2x) + 6x + 2o(3x) + 3x^2}{3x + o(3x) + x + o(x)} = \lim_{x \to 0} \frac{8x + 3x^2 + o(2x) + 2o(3x)}{4x + o(3x) + o(x)} =$$

$\sin 2x = 2x + o(2x)$

$$= \lim_{x \to 0} \frac{x\left(8 + \frac{o(2x)}{x} + \frac{2\,o(3x)}{x} + 3x\right)}{x\left(4 + \frac{o(3x)}{x} + \frac{o(x)}{x}\right)} = 2$$

$\operatorname{arctg} 3x = 3x + o(3x)$

$\ln(1 + 3x + \sin^2 x) = 3x + \cancel{\sin^2 x} + o \Rightarrow 3x + o(3x)$

$xe^x = 1x + x^2 + xo(x) = x + o(x)$ Но! $o(x) - o(x) = o(x)$

2) $\lim\limits_{x\to 0} \dfrac{\operatorname{arctg}(3x-x^2)}{\ln(e^3-x)-3} = \left\{\dfrac{0}{0}\right\} = \lim\limits_{x\to0} \dfrac{\cancel{3x+o(3x)}}{-\frac{x}{e^3}}$

$\operatorname{arctg}(3x\cdot x^2) = 3x - x^2 + o(3x-x^2) = 3x + o(3x)$

$\ln(e^3-x) - \ln e^3 = \ln\left(1-\dfrac{x}{e^3}\right) = -\dfrac{x}{e^3}$

$$\lim_{x\to0}\frac{\operatorname{arctg}(3x-x^2)}{\ln(e^3-x)-3} = \left\{\frac{0}{0}\right\} = \lim_{x\to0}\frac{3x-x^2}{-\frac{x}{e^3}} = -3e^3$$

3) $\lim\limits_{x\to\frac{\pi}{5}} \dfrac{\sqrt{1+\cos 5x}}{x^2-\frac{\pi^2}{25}} = \left\{\begin{matrix} x-\frac{\pi}{5}=t \\ x=t+\frac{\pi}{5}\end{matrix}\right\} = \lim\limits_{t\to0}\dfrac{\sqrt{1+\cos\left(5\left(t+\frac{\pi}{5}\right)\right)}}{t^2+\frac{2\pi t}{5}+\frac{\pi^2}{25}-\frac{\pi^2}{25}} =$

$= \lim\limits_{t\to0}\dfrac{\sqrt{1+\cos(5t+\pi)}}{t^2+\frac{2\pi t}{5}} = \lim\limits_{t\to0}\dfrac{\sqrt{1-\cos 5t}}{t^2+\frac{2\pi t}{5}} = \left\{\dfrac{0}{0}\right\} =$

$\cancel{(1-\cos 5t)^{\frac{1}{2}} = 1-\frac{1}{2}\cos 5t}$

$1-\cos 5t \sim \dfrac{(5t)^2}{2} = \dfrac{25t^2}{2}$

$= \lim\limits_{t\to0}\dfrac{\sqrt{\frac{25t^2}{2}}}{t^2+\frac{2\pi t}{5}} = \lim\limits_{t\to0}\dfrac{t\sqrt{\frac{25}{2}}}{t\left(t+\frac{2\pi}{5}\right)} =$

$= \lim\limits_{t\to0}\dfrac{\sqrt{25}}{\sqrt{2}\left(t+\frac{2\pi}{5}\right)} = \dfrac{5\cdot5}{\sqrt{2}\cdot2\pi} = \dfrac{25}{2\sqrt{2}\pi}$

$\lim\limits_{t\to0}\dfrac{\sqrt{2\sin^2\frac{5t}{2}}}{t\left(t+\frac{2\pi}{5}\right)} = \lim\limits_{t\to0}\dfrac{\sqrt{2}\sin\frac{5t}{2}\cdot\frac{5}{2}}{\frac{5}{2}t\left(t+\frac{2\pi}{5}\right)} = \dfrac{25\sqrt{2}}{4\pi}$

1

Убедиться в том, что при $x\to0$ бесконечно малые величины $1-x$ и $1-\sqrt[3]{x}$ будут одного порядка малости? Будут ли они эквивалентны.

$\lim\limits_{x\to1}\dfrac{1-x}{1-\sqrt[3]{x}} = \lim\limits_{x\to1}\dfrac{(1-\sqrt[3]{x})(1+\sqrt[3]{x}+\sqrt[3]{x^2})}{(1-\sqrt[3]{x})} = 3 \Rightarrow$ они одного порядка малости, но не эквивалентны $3\neq1$.

Опр-ть порядок малости (и выделить её главную часть.)

$f(x) = \sqrt[3]{1-\sqrt{x}}$ при $x\to1$ $\qquad \sin x = x + o(x)$

↑ гл. часть

$\lim\limits_{x\to1}\dfrac{\sqrt[3]{1-\sqrt{x}}}{(x-1)^k}\cdot\dfrac{\sqrt[3]{1+\sqrt{x}}}{\sqrt[3]{1+\sqrt{x}}} = \lim\limits_{x\to1}\dfrac{\sqrt[3]{1-x}}{(x-1)^k\sqrt[3]{1+\sqrt{x}}} = -\dfrac{1}{\sqrt[3]{2}}$, при $k=\dfrac{1}{3}$

$\to k(x-1)^k$

$\Rightarrow k=\dfrac{1}{3}$ – порядок малости.

$\sqrt[3]{1-\sqrt{x}} = -\dfrac{1}{\sqrt[3]{2}}\cdot\sqrt[3]{x-1} + o(\sqrt[3]{x-1})$

$f(x) = \sqrt[3]{1-\sqrt{x}} = C\cdot(x-1)^k + o((x-1)^k)$ – теорема!

Домашняя работа.

1) $x \to 1$.

$$\lim_{x\to 1}\frac{(1-x)}{(1+x)(x-1)^k} = -\lim_{x\to 1}\frac{(x-1)}{(1+x)(x-1)^k} \to k=1.$$

$$\lim_{x\to 1}\frac{1-\sqrt{x}}{(x-1)^k} = \lim_{x\to 1}\frac{(1-x)}{(1+\sqrt{x})(x-1)^k} = -\lim_{x\to 1}\frac{x-1}{(1+\sqrt{x})(x-1)^k} \to k=1.$$

=> функции одного порядка.

Проверка: $$\lim_{x\to 1}\frac{(1-x)}{(1+x)(1-\sqrt{x})} = \lim_{x\to 1}\frac{(1-\sqrt{x})(1+\sqrt{x})}{(1+x)(1-\sqrt{x})} = 1.$$

~ функции одного порядка малости + эквивалентные.

2) $x \to \frac{\pi}{2}$ ф-и $\sec x - tg x$ и $\pi - 2x$

$$\lim_{x\to\frac{\pi}{2}}\frac{\frac{1}{\cos x} - tg x}{\pi - 2x} = \lim_{x\to\frac{\pi}{2}}\frac{\frac{1}{\cos x} - \frac{\sin x}{\cos x}}{\pi - 2x} = \lim_{x\to\frac{\pi}{2}}\frac{1-\sin x}{\cos x\cdot(\pi-2x)} = \left\{\frac{0}{0}\right\} =$$

$$= \left\{\begin{array}{l} x-\frac{\pi}{2}=t \\ x=t+\frac{\pi}{2}\end{array}\right\} = \lim_{t\to 0}\frac{1-\sin(\frac{\pi}{2}+t)}{\cos(t+\frac{\pi}{2})(\pi-2t-\pi)} = \lim_{t\to 0}\frac{1-\cos t}{+\sin t\cdot 2t} =$$

$$= \lim_{t\to 0}\frac{(1-\cos t)\cdot t}{\sin t\cdot 2t^2} = \lim_{t\to 0}\frac{\overbrace{1-\cos t}^{\sim\frac{t^2}{2}}}{2t^2} = \lim_{t\to 0}\frac{t^2}{4t^2} = \frac{1}{4}.$$

~ одного порядка малости, но не эквивалентны!

3) $f(x) = \sqrt[3]{1+\sqrt[3]{x}} - 1$.

$$\lim_{x\to 0}\frac{\sqrt[3]{1+\sqrt[3]{x}} - 1}{x^k} = \frac{\cancel{1}+\sqrt[3]{x}-\cancel{1}}{x^k\left(\sqrt[3]{(1+\sqrt[3]{x})^2} + \sqrt[3]{1+\sqrt[3]{x}} + 1\right)} \Rightarrow k = \frac{1}{3}.$$

4) $f(x) = \sqrt{1+x^2}\, tg\frac{\pi x}{2}$

$$\lim_{x\to 0}\frac{\sqrt{1+x^2}\, tg\frac{\pi x}{2}}{x^k} = \lim_{x\to 0}\frac{x\sqrt{\frac{1}{x^2}+1}\, tg\frac{\pi x}{2}}{x^k} \Rightarrow k=1.$$

5) $f(x) = e^x - \cos x$

$$\lim_{x\to 0}\frac{e^x - \cos x}{x^k} = \lim_{x\to 0}\frac{e^x - 1 + (\cos x + 1)}{x^k} = \lim_{x\to 0}\frac{(e^x-1)\left(1+\frac{\cos x+1}{e^x-1}\right)}{x^k} =$$

$$= \lim_{x\to 0}\frac{x\left(1+\frac{\cos x+1}{e^x-1}\right)}{x^k} \Rightarrow k=1$$

6) $f(x) = \cos x - \sqrt[3]{\cos x}$

$$\lim_{x\to 0}\frac{\cos x-\sqrt[3]{\cos x}}{x^k}=\lim_{x\to 0}\frac{\cos x-1+\sqrt[3]{\cos x}+1}{x^k}=$$

$$=\lim_{x\to 0}\frac{(\cos x-1)\left(1+\frac{\sqrt[3]{\cos x}+1}{\cos x-1}\right)}{x^k}=\lim_{x\to 0}\frac{(1-\cos x)\left(-1-\frac{\sqrt[3]{\cos x}+1}{\cos x-1}\right)}{x^k}=$$

$$=\lim_{x\to 0}\frac{x^2\left(-1-\frac{\sqrt[3]{\cos x}+1}{\cos x-1}\right)}{2x^k}\Rightarrow k=2.$$

7) $x\to+\infty$; Док-ть, что $\sqrt{x+\sqrt{x+\sqrt{x}}}\sim\sqrt{x}$

$$\lim_{x\to+\infty}\frac{\sqrt{x+\sqrt{x+\sqrt{x}}}}{\sqrt{x}}=\lim_{x\to+\infty}\sqrt{1+\frac{\sqrt{x+\sqrt{x}}}{x}}=\lim_{x\to+\infty}\sqrt{1+\sqrt{\frac{1}{x}+\frac{\sqrt{x}}{x^2}}}=$$

$$=\lim_{x\to+\infty}\sqrt{1+\sqrt{\underbrace{\frac{1}{x}}_{0}}+\sqrt{\underbrace{\frac{1}{x^3}}_{0}}}=1\Rightarrow$$ из определения $\sqrt{x+\sqrt{x+\sqrt{x}}}\sim\sqrt{x}$, что и т.д.

8) $f(x)=tg\,x-\sin x$ при $x\to 0$

$$\lim_{x\to 0}\frac{\frac{\sin x}{\cos x}-\sin x}{x^k}=\lim_{x\to 0}\frac{\sin x-\sin x\cos x}{\cos x\cdot x^k}=\lim_{x\to 0}\frac{\sin x(1-\cos x)}{\cos x\cdot x^k}=$$

$$=\lim_{x\to 0}\frac{x\cdot x^2}{2\cos x\cdot x^k}\Rightarrow k=3\Rightarrow\lim_{x\to 0}\frac{x^3}{2\cos x\cdot x^3}=\frac{1}{2}$$

Гл. часть: $\frac{1}{2}\cdot x^3$.

9) $f(x)=\ln x$ при $x\to 1$.

$$\lim_{x\to 1}\frac{\ln x}{(x-1)^k}=\left\{\begin{matrix}x-1=t\\x=t+1\end{matrix}\right\}=\lim_{t\to 0}\frac{\ln(1+t)}{t^k}=\frac{t}{t^k}\Rightarrow k=1.$$

$$\lim_{t\to 0}\frac{t}{t}=1.$$

Гл. часть: $1\cdot(x-1)$

10) $f(x)=\frac{x+1}{x^4+1}$ при $x\to+\infty$

x^4+1 имеет более высокий порядок роста, чем $x+1$.

$$\lim_{x\to+\infty}\frac{x+1}{x^4+1}=0$$

$$\lim_{x\to+\infty}\frac{x\left(1+\frac{1}{x}\right)}{x^4\left(1+\frac{1}{x^4}\right)\cdot x^k}=\lim_{x\to+\infty}\frac{1\left(1+\frac{1}{x}\right)}{x^3\left(1+\frac{1}{x^4}\right)\cdot x^k}\to k=-3$$

Гл. часть: $1\cdot\frac{1}{x^3}$

11) $f(x) = \sqrt{1+\sqrt{1+\sqrt{x}}}$ при $x \to +\infty$ (порядок роста)

1) Если $\lim\limits_{x\to\infty} \frac{f(x)}{g(x)} = k$, где k – ненулевая константа, то функции имеют одинаковый порядок роста. Если $k=1$, то функции называют ***эквивалентными*** на бесконечности.

2) Если $\lim\limits_{x\to\infty} \frac{f(x)}{g(x)} = \infty$, то функция $f(x)$ более высокого порядка роста, чем $g(x)$.

3) Если $\lim\limits_{x\to\infty} \frac{f(x)}{g(x)} = 0$, то функция $g(x)$ более высокого порядка роста, чем $f(x)$.

***! Примечание**: при $x \to -\infty$ суть выкладок не меняется.*

$$\lim_{x\to+\infty} \frac{\sqrt{1+\sqrt{1+\sqrt{x}}}}{x^k} = \lim_{x\to+\infty} \sqrt{\frac{1}{x^{2k}} + \sqrt{\frac{1}{x^{4k}} + \sqrt{\frac{x}{x^{8k}}}}} \to k = \frac{1}{8}. \quad \cancel{\lim_{x\to+\infty} \sqrt{\frac{1}{x^{2k}} + \sqrt{\frac{1}{x^{4k}}+}}}$$

$$\to \lim_{x\to+\infty} \sqrt{\frac{1}{x^{2k}} + \sqrt{\frac{1}{x^{4k}} + \sqrt{\frac{x}{x^{8k}}}}} = 1.$$

Гл. часть: $1 \cdot x^{\frac{1}{8}} = x^{\frac{1}{8}} = \sqrt[8]{x}$

12) $\cancel{f(x) = \frac{x}{\sqrt[3]{1-x^3}}}$ при $x \to 1$.

$$\cancel{\lim_{x\to1} \frac{x}{\sqrt[3]{1-x^3}\cdot(x-1)^k} = \lim_{x\to1} \frac{x}{x\sqrt[3]{\frac{1}{x^3}-1}\cdot(x-1)^k} = \lim_{x\to1} \frac{1}{\sqrt[3]{\frac{1}{x^3}-1}\cdot(x-1)^k} =}$$

$$\cancel{= \lim_{x\to1} \frac{1}{\sqrt[3]{\left(\frac{1}{x^3}-1\right)(x-1)^{3k}}} = \lim_{x\to1} \frac{1}{\sqrt[3]{\left(\frac{1}{x^3}-1\right)x^{3k}\left(1-\frac{1}{x^{3k}}\right)^{3k}}} =}$$

$$\cancel{= \lim_{x\to1} \frac{1}{\sqrt[3]{\left(\frac{x^{3k}}{x^3} - x^{3k}\right.}}}$$

$$\cancel{\lim_{x\to1} \frac{x}{\sqrt[3]{1-x^3}\cdot(x-1)^k} = \lim_{x\to1} \frac{1\cdot(x-1)^{-k}}{\sqrt[3]{\frac{1}{x^3}-1}\cdot} = \lim_{x\to1} \frac{(x-1)^{-\frac{1}{3}}}{\sqrt[3]{\frac{1}{x^3}-1}} =}$$

$$\cancel{= \lim_{x\to1} \frac{1}{\sqrt[3]{\left(\frac{1}{x^3}-1\right)(x-1)}}}$$

$$\cancel{\lim_{x\to1} \frac{x}{\sqrt[3]{1-x^3}} = \left\{\begin{matrix} x-1=t \\ x=t+1 \end{matrix}\right\} = \lim_{t\to0} \frac{t+1}{\sqrt[3]{1-(t+1)^3}} = \lim_{t\to0} \frac{t+1}{\sqrt[3]{(t^3+3t^2+3t+1)+1}} =}$$

$$\cancel{= \lim_{t\to0} \frac{t+1}{\sqrt[3]{1-t^3-3t^2-3t-1}} = \lim_{t\to0} \frac{t+1}{\sqrt[3]{-t^3-3t^2-3t}} = \lim_{t\to0} \frac{t+1}{\sqrt[3]{t^3\left(-1-\frac{3}{t}-\frac{3}{t^2}\right)}} =}$$

$$\cancel{= \lim_{t\to0} \frac{t\left(1+\frac{1}{t}\right)}{t\sqrt[3]{-1-\frac{3}{t}-\frac{3}{t^2}}} =}$$

$$\cancel{\lim_{x\to+\infty} \frac{\sqrt[3]{1-x^3}}{x(x-1)^k} = \lim_{x\to\infty} \frac{-\sqrt[3]{x^3-1}}{x(x-1)^k} = \lim_{x\to\infty} \frac{-(x^3-1)^{\frac{1}{3}}}{x(x-1)^k} = \lim_{x\to+\infty} \frac{-x\left(1-\frac{1}{x^3}\right)^{\frac{1}{3}}}{x(x-1)^k} =}$$

$$\cancel{= \lim_{x\to+\infty} \frac{\left(\frac{1}{x^3}-1\right)^{\frac{1}{3}}}{(x-1)^k} = \lim_{x\to+\infty}}$$

12) $f(x) = \frac{x}{\sqrt[3]{1-x^3}}$ при $x \to 1$.

$\lim\limits_{x\to1} \frac{x}{\sqrt[3]{1-x^3}\,(x-1)^k} = \lim\limits_{x\to1} \frac{1}{\sqrt[3]{\frac{1}{x^3}-1}\,(x-1)^k}$

Если $k = \frac{1}{3}$ $\rightarrow \sqrt[3]{\left(\frac{1}{x^3}-1\right)(x-1)}$ - от неопределённости не избавиться.

Если $k = -\frac{1}{3}$

$\lim\limits_{x\to1} \sqrt[3]{\frac{(x-1)}{\frac{1}{x^3}-1}} = \lim\limits_{x\to1} \sqrt[3]{\frac{x\left(1-\frac{1}{x}\right)}{\left(\frac{1}{x}-1\right)\left(\frac{1}{x^2}+\frac{1}{x}+1\right)}} = -\frac{1}{\sqrt[3]{3}} \Rightarrow k = -\frac{1}{3}$.

Гл. часть: $-\frac{1}{\sqrt[3]{3}} \cdot \frac{1}{\sqrt[3]{(x-1)}}$

15.10.2020.

678, 679, 688, 690, 694

Опр. 2.4. Пусть $f: (E \subset \mathbb{R}) \to \mathbb{R}$ - функция действительного переменного и $x_0 \in E$ - предельная точка множества E.

1) Будем говорить, что $f(x)$ разрывна в точке x_0, если она не является непрерывной в этой точке.

2) Если $f(x)$ разрывна в точке x_0 и если существуют конечные значения $f(x_0+0)$ и $f(x_0-0)$, то говорят, что $f(x)$ имеет в точке x_0 разрыв первого рода (простой разрыв). В противном случае разрыв называется разрывом второго рода.

Возможны две разновидности разрывов первого рода: либо

$$f(x_0+0) \neq f(x_0-0)$$

(в этом случае говорят о разрыве типа конечного скачка, а разность $f(x_0+0) - f(x_0-0)$ называют скачком функции $f(x)$ в точке x_0),

либо:

$$f(x_0+0) = f(x_0-0) \neq f(x_0)$$

(в этом случае говорят об устранимом разрыве: значение функции $f(x)$ в точке x_0 можно изменить, полагая его равным пределу $\lim\limits_{x\to x_0} f(x)$)

Для разрывов 2-го рода возможны случаи: один из пределов $f(x-0)$ или $f(x+0)$ равен бесконечности (в этом случае говорят о бесконечном разрыве) либо один из этих пределов не существует.

678. а) $f_1(x) = \left|\frac{\sin x}{|x|}\right|$, если $x \neq 0$ и $f_1(0) = 1$.

$\lim\limits_{x\to 0} f_1(x) = 1 = f_1(0)$

$\lim\limits_{x\to +0} \left|\frac{\sin x}{x}\right| = 1$.

$\lim\limits_{x\to -0} \left|-\frac{\sin x}{x}\right| = 1$ $\Rightarrow$: $x = 0$ - точка непрерывности.

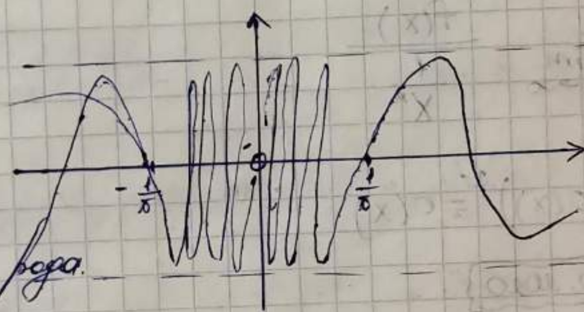
~ функция непрерывна.

б) $f_2(x) = \frac{\sin x}{|x|}$; $f_2(0) = 1$.

$\lim\limits_{x\to +0} \frac{\sin x}{|x|} = 1$.

$\lim\limits_{x\to -0} \frac{\sin x}{-x} = -1$ $\rightarrow$ $x = 0$ - точка разрыва 1-ого рода.

678. $f(x) = \begin{cases} \sin\frac{1}{x}, & x \neq 0 \\ f(0) - \text{произвольно.} \end{cases}$



$\lim\limits_{x\to 0} \sin\frac{1}{x}$ ~ несуществует.

~ точка разрыва второго рода.

688. $y = \frac{1 + x}{1 + x^3}$

$x = -1$. (возможная точка разрыва.)

$\lim\limits_{x\to -1} \frac{1 + x}{1 + x^3} = \lim\limits_{x\to -1} \frac{1+x}{(1+x)(1 - x + x^2)} = \frac{1}{3}$

~ разрыв первого рода. (второго типа.)

т.е можно устранить разрыв. $f(x) = \begin{cases} \frac{1+x}{1+x^3}, & x \neq -1 \\ \frac{1}{3}, & x = -1. \end{cases}$

690. $y = \frac{\frac{1}{x} - \frac{1}{x+1}}{\frac{1}{x-1} - \frac{1}{x}} = \frac{x - 1}{x + 1}$

$\lim\limits_{x\to 1} \frac{x - 1}{x + 1} = 0$

$\lim\limits_{x\to -1} \frac{x - 1}{x + 1} = \infty$ - при $x = -1$ разрыв второго рода.

$\lim\limits_{x\to 0} \frac{x - 1}{x + 1} = -1$

(12) $f(x)$ - б.б. при $x \to a$.

$$\lim_{x\to a} \frac{f(x)}{\frac{1}{(x-a)^k}} = C$$

$k>0$

$$\lim_{x\to 1} \frac{\frac{x}{\sqrt[3]{1-x^3}}}{\frac{1}{(x-1)^k}} = \lim_{x\to 1} \frac{x(x-1)^k}{\sqrt[3]{1-x}\,\sqrt[3]{1+x+x^2}} = -\lim_{x\to 1} \frac{x(x-1)^{k-\frac{1}{3}}}{\sqrt[3]{1+x+x^2}} =$$

$$= -\frac{1}{\sqrt[3]{3}} \quad \text{при } k = \frac{1}{3}$$

Гл. часть: $C \cdot \frac{1}{(x-1)^k} = -\frac{1}{\sqrt[3]{3}} \cdot \frac{1}{\sqrt[3]{(x-1)}}$

$$(1+2x+x^2)(1+x) =$$
$$= 1+x+2x+2x^2+x^2+x^3 =$$
$$=$$

$$f(x) = \frac{1}{x^2} + \frac{2}{x}$$

$$\lim_{x\to\infty} \frac{f(x)}{\frac{1}{x^k}}$$

$$(o(x))^2 = o(x).$$

20.10.2020

Дифференцирование. Произведения и частного.

При вычислении производных суммы, произведения и частного функций пользуются следующими правилами:

$$(u \pm v)' = u' \pm v', \qquad (cu)' = cu'$$

$$(uv)' = u'v + uv'$$

$$(uvw)' = u'vw + uv'w + uvw'$$

$$\left(\frac{u}{v}\right)' = \frac{u'v - v'u}{v^2}$$

где $u(x)$, $v(x)$ - дифференцируемые функции, c - постоянная.

Формула дифференцирования сложной функции имеет вид:

$$(F(u(x)))' = F'_u \cdot u'$$

В частности, если $u = u(x)$, то:

1) $(u^n)' = n u^{n-1} \cdot u'$
2) $(a^u)' = a^u \ln a \cdot u'$
3) $(e^u)' = e^u \cdot u'$
4) $(\log_a u)' = \frac{1}{u \ln a} \cdot u'$
5) $(\ln u)' = \frac{1}{u} \cdot u'$
6) $(\sin u)' = \cos u \cdot u'$
7) $(\cos u)' = -\sin u \cdot u'$
8) $(\operatorname{tg} u)' = \frac{1}{\cos^2 u} \cdot u'$
9) $(\operatorname{ctg} u)' = -\frac{1}{\sin^2 u} \cdot u'$
10) $(\arcsin u)' = \frac{1}{\sqrt{1-u^2}} \cdot u'$
11) $(\arccos u)' = -\frac{1}{\sqrt{1-u^2}} \cdot u'$
12) $(\operatorname{arctg} u)' = \frac{1}{1+u^2} \cdot u'$
13) $(\operatorname{arcctg} u)' = -\frac{1}{1+u^2} \cdot u'$

$(\sqrt{x})' = \frac{1}{2\sqrt{x}}$

$\left(\frac{1}{x}\right)' = -\frac{1}{x^2}$

$(x^x)' = x^x(\ln x + 1)$

$(\operatorname{sh} x)' = \operatorname{ch} x$

$(\operatorname{ch} x)' = \operatorname{sh} x$

$(\operatorname{th} x)' = \operatorname{sech}^2 x = \frac{1}{\operatorname{ch}^2 x}$

$(\operatorname{cth} x)' = -\operatorname{csch}^2 x = -\frac{1}{\operatorname{sh}^2 x}$

873, 845, 847, 881, 885, 887, 891.

845. $y' = \left(\frac{2x}{1-x^2}\right)' = \cancel{\frac{(2x)' - (1-x^2)'}{(1-x^2)^2}} = \cancel{\frac{2+2x}{(1-x^2)^2}} =$

$x_1 + x_2 = 6$

$x_1 \cdot x_2 = -3$

847. $y' = \frac{x}{(1-x)^2(1+x)^3} = \frac{x'(1-x)^2(1+x)^3 - ((1-x)^2(1+x)^3)' \cdot x}{(1-x)^4(1+x)^6} =$

$= \cancel{(1-x)^2(1+x)^3} = x \cdot (1-x)^{-2}(1+x)^{-3} =$

$= x' \cdot (1-x)^{-2} \cdot (1+x)^{-3} + ((1-x)^{-2})' \cdot x \cdot (1+x)^{-3} + ((1+x)^{-3})' \cdot x \cdot (1-x)^{-2} =$

$= \frac{1}{(1-x)^2(1+x)^3} + \frac{x \cdot (-2) \cdot (-1) \cdot (1-x)^{-3}}{(1+x)^3} + \frac{1 \cdot (-4) \cdot 1 \cdot (1+x)^{-4} \cdot x}{(1-x)^2} =$

$= \frac{1}{(1-x)^2(1+x)^3} + \frac{2x}{(1-x)^3(1+x)^3} - \frac{3x}{(1-x)^2(1+x)^4} = \frac{(1-x)(1+x) + 2x(1+x) - 3x(1-x)}{(1-x)^3(1+x)^4} =$

$= \cancel{\frac{1+x-x-x^2+2x+2x^2-4+4x}{(1-x)^3(1+x)^4}} = \cancel{\frac{x^2+6x-3}{(1-x)^3(1+x)^4}} =$

$= \frac{1-x^2+2x+2x^2-3x+3x^2}{(1-x)^3(1+x)^4} = \frac{4x^2-x+1}{(1-x)^3(1+x)^4}$

851. $y' = \left(x + x^{\frac{1}{2}} + x^{\frac{1}{3}}\right)' = 1 + \frac{1}{2\sqrt{x}} + \frac{1}{3\sqrt[3]{x^2}}$

853. $y = \sqrt[3]{x^2} - \frac{2}{\sqrt{x}}$

$y' = \frac{2}{3\sqrt[3]{x}} + \frac{1}{\sqrt{x^3}}$

859.

$$y = \frac{1}{\sqrt{1+x^2}\,(x+\sqrt{1+x^2})} =$$

$$y' = \left(\frac{1}{u}\right)' = -\frac{1}{u^2}\cdot u' = -\frac{1\cdot\left((1+x^2)^{\frac{1}{2}}\cdot\left(x+(1+x^2)^{\frac{1}{2}}\right)\right)'}{(1+x^2)(x+\sqrt{1+x^2})^2} =$$

?

$$= -\frac{\frac{1\cdot 2x}{2\sqrt{1+x^2}}\cdot(x+\sqrt{1+x^2}) + \left(1+\frac{1\cdot 2x}{2\sqrt{1+x^2}}\right)\cdot\sqrt{1+x^2}}{(1+x^2)(x+\sqrt{1+x^2})^2} =$$

$$= -\frac{\frac{x(x+\sqrt{1+x^2})}{\sqrt{1+x^2}} + \sqrt{1+x^2} + x}{(1+x^2)(x+\sqrt{1+x^2})^2} = -\frac{x(x+\sqrt{1+x^2}) + (1+x^2) + x}{(1+x^2)(x+\sqrt{1+x^2})^2} =$$

$$= \underbrace{-\frac{x^2+x\sqrt{1+x^2}+1+x^2+x}{(1+x^2)(x+\sqrt{1+x^2})^2} = -\frac{2x^2+x(\sqrt{1+x^2}+1)+1}{(1+x^2)(x+\sqrt{1+x^2})^2}}_{?} = -\frac{1}{(x^2+1)^{\frac{3}{2}}}$$

861.

$$y = \sqrt[3]{1+\sqrt[3]{1+\sqrt[3]{x}}}$$

$$y' = \left(\left(1+(1+x^{\frac{1}{3}})^{\frac{1}{3}}\right)^{\frac{1}{3}}\right)' = \frac{1\cdot\left(1+(1+\sqrt[3]{x})^{\frac{1}{3}}\right)'}{3\sqrt[3]{(1+\sqrt[3]{1+\sqrt[3]{x}})^2}} =$$

$$= \cancel{\frac{1}{3\sqrt[3]{(1+\sqrt[3]{1+\sqrt[3]{x}})^2}}\cdot} = \frac{1\cdot}{3\sqrt[3]{(1+\sqrt[3]{1+\sqrt[3]{x}})^2}}\cdot\frac{1\cdot(1+\sqrt[3]{x})'}{3\sqrt[3]{(1+\sqrt[3]{x})^2}} =$$

$$= \frac{1}{27\sqrt[3]{(1+\sqrt[3]{1+\sqrt[3]{x}})^2}\cdot\sqrt[3]{(1+\sqrt[3]{x})^2}\cdot\sqrt[3]{x^2}} =$$

$$= \frac{1}{27\sqrt[3]{(1+\sqrt[3]{1+\sqrt[3]{x}})^2\cdot(1+\sqrt[3]{x})^2\cdot x^2}}$$

867.

$$y = \frac{\sin^2 x}{\sin x^2}$$

$$y' = \frac{(\sin^2 x)'\sin x^2 - (\sin x^2)'\cdot\sin^2 x}{\sin^2 x^2} = \frac{2\cos x\cdot\sin x\cdot\sin x^2 - 2x\cdot\cos x^2\cdot\sin^2 x}{\sin^2 x^2} =$$

$$= \frac{2\sin x\cdot\cos x\cdot\sin x^2 - 2x\cdot\sin^2 x\cos x^2}{\sin^2 x^2} = \frac{2\sin x(\cos x\cdot\sin x^2 - x\cdot\sin x\cdot\cos x^2)}{\sin^2 x^2}$$

873. $y = 4\sqrt[3]{ctg^2 x} + \sqrt[3]{ctg^8 x}$

$$y' = \left(4\cdot ctg^{\frac{2}{3}}x + ctg^{\frac{8}{3}}x\right)' = \frac{4\cdot 2}{3\sqrt[3]{ctg\, x}}\cdot\left(-\frac{1}{\sin^2 x}\right) + \frac{8}{3}\cdot\sqrt[3]{ctg^5 x}\,($$

$$= -\frac{8}{3}\cdot\frac{1}{\sin^2 x}\left(\frac{1}{\sqrt[3]{ctg\, x}} + \sqrt[3]{ctg^5 x}\right) = -\frac{8}{3}\cdot\frac{1}{\sin^4 x}\cdot\frac{1}{\sqrt[3]{ctg\, x}}$$

875. $y =$

$y' =$

$\cdot 3$

$= \frac{1}{\cos^2 x}$

877. y

$= -$

885. $y =$

$y' =$

$= 0$

884. $y =$

$y' =$

3.1 (**858.**)

$y' = ($

$= \frac{1}{3}$

$= \frac{2x}{1}$

3.2. (**860**)

y

$= \frac{1}{2}$

3.3.

$y' = ($

875.
$$y=\sin[\cos^2(tg^3x)]$$
$$y'=\cos[\cos^2(tg^3x)]\cdot(\cos^2(tg^3x))'=\cos[\cos^2(tg^3x)]\cdot 2\cos(tg^3x)(-\sin(tg^3x))\cdot$$
$$\cdot 3tg^2x\cdot\frac{1}{\cos^2x}$$
$$=\frac{1}{\cos^2x}\cdot 3tg^2x\cdot(-\sin(tg^3x))\cdot 2\cos(tg^3x)\cdot\cos(\cos^2(tg^3x))$$

877.
$$y=2^{tg\frac{1}{x}}=2^{tg\frac{1}{x}}\cdot\ln a\cdot(tg\tfrac{1}{x})'=2^{tg\frac{1}{x}}\cdot\ln 2\cdot(-1)\cdot x^{-2}\cdot\frac{1}{\cos^2(\frac{1}{x})}=$$
$$=-\ln 2\cdot 2^{tg\frac{1}{x}}\cdot x^{-2}\cdot\frac{1}{\cos^2(\frac{1}{x})}$$

885.
$$y=x^{a^a}+a^{x^a}+a^{a^x}\qquad(a>0)$$
$$y'=a^a\cdot x^{a^a-1}+a^{x^a}\cdot\ln a\cdot a\cdot x^{a-1}+(a^x)'\cdot\ln a\cdot a^{ax}=$$
$$=a^a\cdot x^{a^a-1}+a^{x^a}\cdot\ln a\cdot a\cdot x^{a-1}+\ln a\cdot a^x\cdot a^{a^x}\cdot\ln a$$

887.
$$y=\ln(\ln(\ln x))$$
$$y'=\frac{1}{\ln(\ln x)}\cdot\frac{1}{\ln x}\cdot\frac{1}{x}$$

Домашняя работа.

3.1 (858.)
$$y'=\left(\sqrt[3]{\frac{1+x^3}{1-x^3}}\right)'=\frac{1\cdot\sqrt[3]{(1-x^3)^2}}{3\sqrt[3]{(1+x^3)^2}}\cdot\left(\frac{1+x^3}{1-x^3}\right)'=\frac{1}{3}\sqrt[3]{\frac{(1-x^3)^2}{(1+x^3)^2}}\cdot\frac{(1+x^3)'(1-x^3)-(1-x^3)'(1+x^3)}{(1-x^3)^2}=$$
$$=\frac{1}{3}\sqrt[3]{\frac{(1-x^3)^2}{(1+x^3)^2}}\cdot\frac{3x^2(1-x^3)+3x^2(1+x^3)}{(1-x^3)^2}=\frac{1}{3}\sqrt[3]{\frac{(1-x^3)^2}{(1+x^3)^2}}\cdot\frac{3x^2-\cancel{3x^5}+3x^2+\cancel{3x^5}}{(1-x^3)^2}=$$
$$=\frac{2x^2}{1}\cdot\sqrt[3]{\frac{(1-x^3)^2}{(1+x^3)^2\cdot(1-x^3)^6}}=2x^2\cdot\sqrt[3]{\frac{1}{(1+x^3)^2\cdot(1-x^3)^4}}$$

3.2. (860)
$$y'=\left(\sqrt{x+\sqrt{x+\sqrt{x}}}\right)'=\frac{1\cdot(x+\sqrt{x+\sqrt{x}})'}{2\sqrt{x+\sqrt{x+\sqrt{x}}}}=\frac{1+\frac{1\cdot(x+\sqrt{x})'}{2\sqrt{x+\sqrt{x}}}}{2\sqrt{x+\sqrt{x+\sqrt{x}}}}=$$
$$=\frac{1+\frac{1+\frac{1}{2\sqrt{x}}}{2\sqrt{x+\sqrt{x}}}}{2\sqrt{x+\sqrt{x+\sqrt{x}}}}=\frac{2\sqrt{x+\sqrt{x}}+1+\frac{1}{2\sqrt{x}}}{4\sqrt{x+\sqrt{x+\sqrt{x}}}\sqrt{x+\sqrt{x}}}=\frac{4\sqrt{x}\cdot\sqrt{x+\sqrt{x}}+2\sqrt{x+\sqrt{x}}+2\sqrt{x}+1}{8\sqrt{x}\cdot\sqrt{x+\sqrt{x}}\sqrt{x+\sqrt{x+\sqrt{x}}}}$$

3.3.
$$y'=\left(\sin(\cos^2x)\cdot\cos(\sin^2x)\right)'=$$

$$= \left(\sin(\cos^2 x)\right)' \cos(\sin^2 x) + \left(\cos(\sin^2 x)\right)' \cdot \sin(\cos^2 x) =$$

$$= \cos(\cos^2 x) \cdot (-2)\sin x \cos x \cdot \cos(\sin^2 x) - \sin(\sin^2 x) \cdot 2\sin x \cos x \cdot \sin\ldots$$

$$= -2\sin x\cos x \cdot \cos(\sin^2 x) \cdot \cos(\cos^2 x) - 2\sin x \cos x \cdot \sin(\sin^2 x)\sin(\cos\ldots$$

$$= -2\sin x \cos x\left(\cos(\sin^2 x) \cdot \cos(\cos^2 x) + \sin(\sin^2 x) \cdot \sin(\cos^2 x)\right)$$

3.4. (866.)

$$y' = \left(\sin[\sin(\sin x)]\right)' = \cos(\sin(\sin x)) \cdot \left(\sin(\sin x)\right)' =$$

$$= \cos(\sin(\sin x)) \cdot \cos(\sin x) \cdot (\sin x)' = \cos(\sin(\sin x)) \cdot \cos(\sin x) \cdot \cos x$$

3.5 (868.)

$$y' = \left(\frac{\cos x}{2\sin^2 x}\right)' = \frac{(\cos x)' \cdot 2\sin^2 x - 2(\sin^2 x)' \cdot \cos x}{4\sin^4 x} =$$

$$= \frac{-2\sin^3 x - 2 \cdot 2\sin x \cdot \cos x \cdot \cos x}{4\sin^4 x} = \frac{-2\sin^3 x - 4\sin x \cdot \cos^2 x}{4\sin^4 x} =$$

$$= \frac{-2\sin x(\sin^2 x + 2\cos^2 x)}{4\sin^4 x} = -\frac{\sin^2 x + 2\cos^2 x}{2\sin^3 x} = -\frac{1 + \cos^2 x}{2\sin^3 x}$$

3.6. (870)

$$y' = \left(\frac{\sin x - x\cos x}{\cos x + x\sin x}\right)' = \frac{(\sin x - x\cos x)'(\cos x + x\sin x) - (\cos x + x\sin x)'(\sin x - x\cos x)}{(\cos x + x\sin x)^2}$$

$$= \frac{(\cos x - \cos x + x\sin x)(\cos x + x\sin x) - (-\sin x + \sin x + x\cos x)(\sin x - x\cos x)}{(\cos x + x\sin x)^2} =$$

$$= \frac{x \cdot \cos x\sin x + x^2\sin^2 x - \sin x\cos x + x\cos^2 x - \sin^2 x + x\cos x\sin x - x\cos x\sin x + x^2\cos^2 x}{(\cos x + x\sin x)^2}$$

$$= \frac{x \cdot \cos x \cdot \sin x + x^2 - \sin x\cos x + x\cos^2 x - \sin^2 x}{(\cos x + x\sin x)^2} =$$

$$= \frac{x\sin x\cos x + x^2\sin^2 x - x\sin x\cos x + x^2\cos^2 x}{(\cos x + x\sin x)^2} = \frac{x^2}{(\cos x + x\sin x)^2}$$

3.7. (872.)

$$y = tg\, x - \frac{1}{3} tg^3 x + \frac{1}{5} tg^5 x$$

$$y' = \frac{1}{\cos^2 x} - \frac{tg^2 x}{\cos^2 x} + \frac{tg^4 x}{\cos^2 x} = \frac{1 - tg^2 x + tg^4 x}{\cos^2 x}$$

3.8. (874.)

$$y = \sec^2\frac{x}{a} + \operatorname{cosec}^2\frac{x}{a}$$

$$y' = \left(1+tg^2\left(\frac{x}{a}\right)\right)' + \left(1+ctg^2\left(\frac{x}{a}\right)\right)' = \frac{2tg\frac{x}{a}}{a\cos^2\frac{x}{a}} - \frac{2\cdot ctg\frac{x}{a}}{a\sin^2\frac{x}{a}} =$$

$$= \frac{2\sin\frac{x}{a}}{a\cos^3\frac{x}{a}} - \frac{2\cos\frac{x}{a}}{a\sin^3\frac{x}{a}} = \frac{2\sin^4\frac{x}{a} - 2\cos^4\frac{x}{a}}{a\cos^3\frac{x}{a}\sin^3\frac{x}{a}} =$$

$$= \frac{2}{a}\cdot\frac{\sin^4\frac{x}{a} - \cos^4\frac{x}{a}}{\sin^3\frac{x}{a}\cos^3\frac{x}{a}} = \frac{-16\cos\frac{2x}{a}}{a\sin^3\frac{2x}{a}}$$

3.9. $y = e^{-\cos^2 x^2}$

$$y' = \left(e^{-\cos^2 x^2}\right)' = e^{-\cos^2 x^2}\cdot(-1)\cdot(\cos^2 x^2)' = -e^{-\cos^2 x^2}\cdot 4x\sin x^2\cos x^2$$

3.10. $y = 2^{tg^3 5x}$

$$y' = 2^{tg^3 5x}\cdot\ln 2\cdot(tg^3 5x)' = 2^{tg^3 5x}\cdot\ln 2\cdot\frac{15\,tg^2 5x}{\cos^2 5x}$$

3.11. $y = e^x\left(1+ctg\frac{x}{2}\right)$

$$y' = (e^x)'\left(1+ctg\frac{x}{2}\right) + e^x\left(1+ctg\frac{x}{2}\right)' =$$

$$= e^x\left(1+ctg\frac{x}{2}\right) + e^x\left(\frac{1}{2}\cdot\left(-\frac{1}{\sin^2\frac{x}{2}}\right)\right) = e^x\left(1+ctg\frac{x}{2}\right) - \frac{e^x}{2\sin^2\frac{x}{2}} =$$

$$= e^x\left(1 + \frac{\cos\frac{x}{2}}{\sin\frac{x}{2}} - \frac{1}{2\sin^2\frac{x}{2}}\right) = e^x\left(\frac{2\sin^2\frac{x}{2} + 2\cos\frac{x}{2}\sin\frac{x}{2} - 1}{2\sin^2\frac{x}{2}}\right) =$$

$$= e^x\left(\frac{2\sin^2\frac{x}{2} + \sin x - 1}{2\sin^2\frac{x}{2}}\right) = e^x\left(\frac{1-\cos x + \sin x - 1}{2\sin^2\frac{x}{2}}\right) =$$

$$= e^x\cdot\frac{\sin x - \cos x}{2\sin^2\frac{x}{2}}$$

3.13. (886)

$$y = lg^3 x^2$$

$$y' = 3(lg\,x^2)'\cdot lg^2 x^2 = \frac{3\cdot 2x\;lg^2 x^2}{x^2\ln 10} = \frac{6\,lg^2 x^2}{x\ln 10} = \frac{6}{x}\,lg\,e\cdot lg^2 x^2$$

3.14.

$$y = \ln(\ln^2(\ln^3 x))$$

$$y' = \frac{1}{\ln^2(\ln^3 x)}\cdot(\ln^2(\ln^3 x))' = \frac{2\cdot\ln(\ln^3 x)}{\ln^3 x\cdot\ln^2(\ln^3 x)}\cdot(\ln^3 x)' =$$

$$= \frac{2\cdot 3\cancel{\ln^2 x}}{\ln^{\cancel{3}} x\cdot\ln(\ln^3 x)\cdot x} = \frac{6}{x\cdot\ln x\cdot\ln(\ln^3 x)}$$

3.15
(1890.) $y = \frac{1}{4} \ln \frac{x^2-1}{x^2+1}$

$$y' = \frac{1}{4}\left(\ln \frac{x^2-1}{x^2+1}\right)' = \frac{1}{4} \cdot \frac{1\cdot(x^2+1)}{x^2-1} \cdot \left(\frac{x^2-1}{x^2+1}\right)' =$$

$$= \frac{(x^2+1)}{4(x^2-1)} \cdot \frac{(x^2-1)'(x^2+1) - (x^2+1)'(x^2-1)}{(x^2+1)^2} =$$

$$= \frac{2x(x^2+1) - 2x(x^2-1)}{4(x^2-1)(x^2+1)} = \frac{\cancel{2x^3}+2x-\cancel{2x^3}+2x}{4(x^2-1)(x^2+1)} = \frac{x}{x^4-1}$$

3.16.
(904.) $y = \ln\left(\sqrt{\frac{1+\sin x}{1-\sin x}}\right)$

$$y' = \frac{1\cdot\sqrt{1-\sin x}}{\sqrt{1+\sin x}} \cdot \left(\left(\frac{1+\sin x}{1-\sin x}\right)^{\frac{1}{2}}\right)' = \frac{\sqrt{1-\sin x}}{\sqrt{1+\sin x}} \cdot \frac{1\cdot\sqrt{1-\sin x}}{2\sqrt{1+\sin x}} \cdot \left(\frac{1+\sin x}{1-\sin x}\right)' =$$

$$= \frac{(1-\sin x)}{2(1+\sin x)} \cdot \frac{(1+\sin x)'(1-\sin x) - (1-\sin x)'(1+\sin x)}{(1-\sin x)^2} =$$

$$= \frac{\cos x(1-\sin x) + \cos x(1+\sin x)}{2(1+\sin x)(1-\sin x)} = \frac{\cos x - \cancel{\cos x \sin x} + \cos x + \cancel{\cos x \sin x}}{2(1-\sin^2 x)} =$$

$$= \frac{2\cos x}{2\cdot\cos^2 x} = \frac{1}{\cos x}$$

?

3.17.
(1896.) $y = x\ln(x+\sqrt{1+x^2}) - \sqrt{1+x^2} =$

$$y' = x'\cdot\ln(x+\sqrt{1+x^2}) + x\cdot(\ln(x+\sqrt{1+x^2}))' - ((1+x^2)^{\frac{1}{2}})' =$$

$$= \ln(x+\sqrt{1+x^2}) + \frac{x}{(x+\sqrt{1+x^2})} \cdot (x+\sqrt{1+x^2})' - \frac{x}{\sqrt{1+x^2}} =$$

$$= \ln(x+\sqrt{1+x^2}) + \frac{x\cdot\left(1+\frac{x}{\sqrt{1+x^2}}\right)}{(x+\sqrt{1+x^2})} - \frac{x}{\sqrt{1+x^2}} =$$

$$= \ln(x+\sqrt{1+x^2}) + \frac{x\cdot\cancel{(x+\sqrt{1+x^2})}}{\cancel{(x+\sqrt{1+x^2})}\sqrt{1+x^2}} - \frac{x}{\sqrt{1+x^2}} = \ln(x+\sqrt{1+x^2})$$

3.12.
(882) $y = e^{ax}\,\frac{a\sin bx - b\cos bx}{a^2+b^2}$

$$y' = (e^{ax})' \cdot \frac{a\sin bx - b\cos bx}{a^2+b^2} + e^{ax}\cdot\left(\frac{a\sin bx - b\cos bx}{a^2+b^2}\right)' =$$

$$= a\cdot e^{ax}\cdot\frac{a\sin bx - b\cos bx}{a^2+b^2} + e^{ax}\cdot\frac{ab\cos bx + b^2\sin bx}{(a^2+b^2)^2} =$$

$$= e^{ax}\left(\frac{a^2\sin bx - ab\cos bx}{a^2+b^2} + \frac{ab\cos bx + b^2\sin bx}{(a^2+b^2)^2}\right) =$$

$$= e^{ax}\left(\frac{(a^2\sin bx - ab\cos bx)(a^2+b^2) + ab\cos bx + b^2\sin bx}{(a^2+b^2)^2}\right) =$$

$$= e^{ax}\,\frac{a^4\sin bx - a^3b\cos bx + a^2b^2\sin bx - ab^3\cos bx + ab\cos bx + b^2\sin bx}{(a^2+b^2)^2} =$$

$$= e^{ax}\,\frac{a^4\sin bx + a^2b^2\sin bx + b^2\sin bx - a^3b\cos bx - ab^3\cos bx + ab\cos bx}{(a^2+b^2)^2} =$$

$$= e^{ax}\,\frac{\sin bx\,(a^4+a^2b^2+b^2) - \cos bx\,(a^3b - ab^3 + ab)}{(a^2+b^2)^2} =$$

$$= e^{ax}\,\frac{\sin bx\cdot(a^4+a^2b^2+b^2) - a\cos bx\,(a^3 - ab^2 + a)}{(a^2+b^2)^2}$$

?

22.10.2020

1) $y = \arcsin\frac{x}{2}$

$y' = \frac{1}{2\sqrt{1-\frac{x^2}{4}}}$

2) $y = \operatorname{arctg}\frac{x^2}{a}$

$y' = \frac{1}{\left(1+\frac{x^4}{a^2}\right)} \cdot \frac{2x}{a} = \frac{2x}{a+\frac{x^4}{a}}$

3) $y = \arcsin(\sin x)$

$y' = \frac{1}{\sqrt{1-\sin^2 x}} \cdot (\sin x)' = \frac{\cos x}{\sqrt{\cos^2 x}} = \frac{\cos x}{|\cos x|} = \operatorname{sgn}(\cos x)$

4) $y = \frac{1}{\arccos^2(x^2)}$

$y' = \frac{1'(\arccos^2(x^2)) - (\arccos^2(x^2))'}{\arccos^4(x^2)} = \frac{-\left(2\arccos(x^2)\cdot\left(-\frac{1\cdot 2x}{\sqrt{1-x^4}}\right)\right)}{\arccos^4(x^2)} =$

$= \frac{+\,2\arccos(x^2)\cdot 2x}{\arccos^4(x^2)\sqrt{1-x^4}} = \frac{4x}{\arccos^3(x^2)\sqrt{1-x^4}}$

5) $y = x\cdot\arcsin\sqrt{\frac{x}{1+x}} + \operatorname{arctg}\sqrt{x} - \sqrt{x}$

$y' = \left(x'\arcsin\sqrt{\frac{x}{1+x}} + x\cdot\left(\arcsin\sqrt{\frac{x}{1+x}}\right)'\right) + \frac{1}{1+x}\cdot(\sqrt{x})' - \frac{1}{2\sqrt{x}} =$

$= \arcsin\sqrt{\frac{x}{1+x}} + \frac{x}{\sqrt{1-\frac{x}{1+x}}}\cdot\left(\frac{\sqrt{x}}{\sqrt{1+x}}\right)' + \frac{1}{(1+x)2\sqrt{x}} - \frac{1}{2\sqrt{x}} =$

$= \arcsin\sqrt{\frac{x}{1+x}} + \frac{x}{\sqrt{\frac{1}{1+x}}}\cdot\frac{1}{2(1+x)\sqrt{x}\cdot\sqrt{1+x}} + \frac{1}{2\sqrt{x}(1+x)} - \frac{1}{2\sqrt{x}} =$

$= \arcsin\sqrt{\frac{x}{1+x}} + \frac{x\cdot\cancel{\sqrt{1+x}}}{2(1+x)\sqrt{x}\cdot\cancel{\sqrt{1+x}}} + \frac{1}{2\sqrt{x}(1+x)} - \frac{(1+x)}{2\sqrt{x}(1+x)} =$

$= \arcsin\sqrt{\frac{x}{1+x}} + \frac{\overbrace{x+1-1-x}^{=0}}{2\sqrt{x}(1+x)} = \arcsin\sqrt{\frac{x}{1+x}}$

6) $y = \frac{1}{6}\ln\frac{(x+1)^2}{x^2-x+1} + \frac{1}{\sqrt{3}}\operatorname{arctg}\frac{2x-1}{\sqrt{3}} =$

$y' = \frac{1}{6}\cdot\frac{1\cdot(x^2-x+1)}{(x+1)^2}\cdot\left(\frac{(x+1)^2}{x^2-x+1}\right)' + \frac{1}{\sqrt{3}}\cdot\frac{1}{1+\frac{(2x-1)^2}{3}}\cdot\left(\frac{2x-1}{\sqrt{3}}\right)' =$

$= \frac{1}{6}\frac{(x^2-x+1)\cdot(-3)(x^2-1)}{(x+1)^2\,(x^2-x+1)^2} + \frac{1}{\sqrt{3}}\cdot\frac{\cancel{3}}{3+4x^2-4x+1}\cdot\frac{2}{\sqrt{3}} =$

$$= -\frac{1}{2}\frac{(x-1)}{(x+1)(x^2-x+1)} + \frac{2}{4(x^2-x+1)} = \frac{1}{2(x^2-x+1)} - \frac{(x-1)}{2(x+1)(x^2-x+1)} =$$

$$= \frac{x+1-x+1}{2(x+1)(x^2-x+1)} = \frac{1}{(x+1)(x^2-x+1)} = \frac{1}{x^3-x^2+x+x^2-x+1} = \frac{1}{x^3+1}$$

7) $y = \ln(\operatorname{ch}x) + \frac{1}{2\operatorname{ch}^2 x} = \ln(\operatorname{ch}x) + \frac{1}{2}\cdot\operatorname{ch}^{-2}x$

$$y' = \frac{1}{\operatorname{ch}x}\cdot\operatorname{sh}x - \frac{\operatorname{sh}x}{\operatorname{ch}^3x} = \frac{\operatorname{sh}x\,\operatorname{ch}^2x - \operatorname{sh}x}{\operatorname{ch}^3x} = \frac{\operatorname{sh}x\,(\operatorname{ch}^2x-1)}{\operatorname{ch}^3x} =$$

$$= \frac{\operatorname{sh}^3x}{\operatorname{ch}^3x} = \operatorname{th}^3x$$

~~8) y = arctg(th x)~~

6) $y = \underbrace{\frac{1}{6}\cdot\ln\frac{(x+1)^2}{x^2-x+1}}_{y_1} + \underbrace{\frac{1}{\sqrt{3}}\operatorname{arctg}\frac{2x-1}{\sqrt{3}}}_{y_2.}$

$$y_1 = \frac{1}{6}\cdot\ln\frac{(x+1)^2}{x^2-x+1}.$$

$$y_1 = \frac{1}{6}(2\ln|x+1| - \ln(x^2-x+1))$$

$$y_1' = \frac{1}{6}\left(\frac{2}{x+1} - \frac{2x-1}{x^2-x+1}\right) = \frac{1}{6}\left(\frac{\cancel{2x^2}-2x+2-\cancel{2x^2}-2x+x+1}{x^3+1}\right) =$$

$$= \frac{1}{6}\left(\frac{-3x+3}{x^3+1}\right) = -\frac{1}{2}\frac{(x-1)}{x^3+1}.$$

$$y_2' = \frac{1}{\sqrt{3}}\;\frac{1}{1+\frac{(2x-1)^2}{3}}\cdot\frac{2}{\sqrt{3}} = \frac{2}{3+4x^2-4x+1} = \frac{1}{2(x^2-x+1)}$$

$$\Rightarrow y' = \frac{1}{2(x^2-x+1)} - \frac{1}{2}\frac{(x-1)}{(x^3+1)} = \frac{x+1-x+1}{2(x^3+1)} = \frac{1}{x^3+1}.$$

* $y = \sqrt{\frac{1-\arcsin x}{1+\arcsin x}}$

$$\ln y = \frac{1}{2}\ln\left|\frac{1-\arcsin x}{1+\arcsin x}\right| = \frac{1}{2}(\ln|1-\arcsin x| - \ln|1+\arcsin x|)$$

$$\frac{1}{y}\cdot y' = \frac{1}{2}\left(\frac{1}{1-\arcsin x}\cdot\left(-\frac{1}{\sqrt{1-x^2}}\right) - \frac{1}{1+\arcsin x}\cdot\frac{1}{\sqrt{1-x^2}}\right)$$

$$y' = \frac{1}{2}\sqrt{\frac{1-\arcsin x}{1+\arcsin x}}\cdot\frac{(-1)}{\sqrt{1-x^2}}\left(\frac{1}{1-\arcsin x} + \frac{1}{1+\arcsin x}\right) =$$

$$= \cancel{\frac{1}{2}}\sqrt{\frac{1-\arcsin x}{1+\arcsin x}}\cdot\frac{1}{\sqrt{1-x^2}}\cdot\frac{1}{\arcsin^2x-1}$$

$y = x^x$

$\ln y = x \ln x$

$\frac{y'}{y} = x' \ln x + (\ln x)' x =$

$y' = x^x (\ln x + 1)$

$\to \quad y = e^{\ln x^x} = e^{x \ln x}$

$y' = e^{x \ln x} (\ln x + 1)$

$y' = x^x (\ln x + 1)$

10) $y = x + x^x + \underbrace{x^{x^x}}_{y_1}$

$y' = 1 + x^x(\ln x + 1) + (x^{x^x})'$

$y_1 = x^{x^x}$

$\ln y_1 = \ln(x^{x^x})$

$\frac{y_1'}{y_1} = (x^x \ln x)'$

$y_1' = y_1 \left((x^x)' \ln x + (\ln x)' x^x\right) = y_1 \left(x^x (\ln x + 1) \cancel{\ln x} + x^{x-1} \right)$

$y_1' = x^{x^x} \left(x^x (\ln x + 1) \cancel{\ln x} + x^{x-1} \right)$

$\Rightarrow y' = 1 + x^x(\ln x + 1) + x^{x^x} \left(x^x \left(x^x (\ln x + 1) \cancel{\ln x} + x^{x-1} \right)\right)$

$y' = 1 + x^x(\ln x + 1) + x^{x^x}$

$y_1' = x^{x^x + \cancel{x}} \left(\ln^2 x + \ln x + \frac{1}{x} \right)$

9) $y = (\ln x)^x : x^{\ln x} = e$

$y' = \cancel{\frac{((\ln x)^x)' x^{\ln x} - (x^{\ln x})' (\ln x)^x}{(x^{\ln x})^2}}$

$y = e^{x \ln(\ln x)} : e^{\ln^2 x} = e^{x \ln(\ln x) - (\ln x)^2}$

$y' = \frac{\ln x}{x^{\ln x}} \left(\ln(\ln x) + x \frac{1}{\ln x \cdot x} - 2 \ln x \cdot \frac{1}{x} \right.$

8) $y =$

4.1. y

4.2. y

4.3. y

4.4. $y' =$

8) $y = \operatorname{arctg}(\operatorname{th} x)$

$$y' = \frac{1}{1+\operatorname{th}^2 x} \cdot \operatorname{th}' x = \frac{1}{(1+\operatorname{th}^2 x)} \cdot \frac{1}{\operatorname{ch}^2 x} = \frac{1}{\operatorname{ch}^2 x(1+\operatorname{th}^2 x)} = \frac{1}{\operatorname{ch}^2 x + \operatorname{sh}^2 x} =$$

$$= \frac{1}{\operatorname{ch} 2x}$$

$\frac{2}{4}$

$-1{,}01$

$1{,}01$

Домашняя работа:

4.1. $$y' = \left(\ln \operatorname{tg}\left(\frac{x}{2}+\frac{\pi}{4}\right)\right)' = \frac{1}{\operatorname{tg}\left(\frac{x}{2}+\frac{\pi}{4}\right)} \cdot \frac{1}{\cos^2\left(\frac{x}{2}+\frac{\pi}{4}\right)} \cdot \frac{1}{2} =$$

$$= \frac{1-\operatorname{tg}\frac{x}{2}\cdot 1}{\operatorname{tg}\frac{x}{2}+\operatorname{tg}\frac{\pi}{4}} \cdot \frac{1}{\left(\cos\frac{x}{2}\cos\frac{\pi}{4}-\sin\frac{x}{2}\sin\frac{\pi}{4}\right)^2} \cdot \frac{1}{2} =$$

$$= \frac{1-\operatorname{tg}\frac{x}{2}}{1+\operatorname{tg}\frac{x}{2}} \cdot \frac{1}{\left(\frac{\sqrt{2}}{2}\cos\frac{x}{2}-\frac{\sqrt{2}}{2}\sin\frac{x}{2}\right)^2} \cdot \cancel{\frac{1}{2}} = \frac{1-\frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}}{1+\frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}} \cdot \frac{1}{\cancel{\sqrt{2}}\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)} =$$

$$= \frac{\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)\cdot\cos\frac{x}{2}}{\cos\frac{x}{2}\cdot\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)} \cdot \frac{1}{\cancel{\sqrt{2}}\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)^2} = \frac{1}{\cancel{\sqrt{2}}\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)} =$$

$$= \frac{1}{\cos^2\frac{x}{2}-\sin^2\frac{x}{2}} = \frac{1}{\cos x}$$

4.2. $$y' = \left(\ln\sqrt{\frac{1-\sin x}{1+\sin x}}\right)' = \sqrt{\frac{1+\sin x}{1-\sin x}} \cdot \left(\sqrt{\frac{(1-\sin x)}{(1+\sin x)}}\right)' = \frac{1}{2} \cdot \frac{\sqrt{1+\sin x}}{\sqrt{1-\sin x}} \cdot \frac{\sqrt{1+\sin x}}{\sqrt{1-\sin x}} \cdot$$

$$\cdot \left(\frac{1-\sin x}{1+\sin x}\right)' = \frac{(1+\sin x)}{2(1-\sin x)} \cdot \frac{(1-\sin x)'(1+\sin x)-(1+\sin x)'(1-\sin x)}{(1+\sin x)^2} =$$

$$= \frac{-\cos x(1+\sin x)-\cos x(1-\sin x)}{2(1-\sin^2 x)} = \frac{-2\cos x}{2\cos^2 x} = -\frac{1}{\cos x}$$

4.3. $$y' = \left(\frac{1}{4x^4}\ln\frac{1}{x}-\frac{1}{16x^4}\right)' = \left(\frac{1}{4}x^{-4}\right)'\ln\frac{1}{x} + \left(\ln\frac{1}{x}\right)'\cdot\frac{1}{4x^4} - \left(\frac{1}{16}x^{-4}\right)' =$$

$$= \frac{1}{x^5}\ln\frac{1}{x} - x\cdot\frac{1}{x^2}\cdot\frac{1}{4x^4} + \frac{1}{4x^5} = \frac{\ln\frac{1}{x}}{x^5} - \cancel{\frac{1}{4x^5}} + \cancel{\frac{1}{4x^5}} = \frac{\ln\frac{1}{x}}{x^5}$$

4.4. $$y' = \left(\ln\left[\frac{1}{x}+\ln\left(\frac{1}{x}+\ln\frac{1}{x}\right)\right]\right)' = \frac{1}{\left[\frac{1}{x}+\ln\left(\frac{1}{x}+\ln\frac{1}{x}\right)\right]} \cdot \left(\frac{1}{x}+\ln\left(\frac{1}{x}+\ln\frac{1}{x}\right)\right)' =$$

$$= \frac{1}{\left[\frac{1}{x}+\ln\left(\frac{1}{x}+\ln\frac{1}{x}\right)\right]} \cdot \left(-\frac{1}{x^2}+\frac{1}{\left(\frac{1}{x}+\ln\frac{1}{x}\right)}\cdot\left(-\frac{1}{x^2}-\frac{x}{x^2}\right)\right) =$$

$$= \frac{1}{\left[\frac{1}{x}+\ln\left(\frac{1}{x}+\ln\frac{1}{x}\right)\right]} \cdot \left(-\frac{1}{x^2}-\frac{1}{x^2\left(\frac{1}{x}+\ln\frac{1}{x}\right)}-\frac{x}{x^2\left(\frac{1}{x}+\ln\frac{1}{x}\right)}\right) =$$

$$= \frac{-\left(\frac{1}{x}+\ln\frac{1}{x}\right)-1-x}{x^2\left(\frac{1}{x}+\ln\frac{1}{x}\right)} \cdot \frac{1}{\left[\frac{1}{x}+\ln\left(\frac{1}{x}+\ln\frac{1}{x}\right)\right]} = \frac{-\frac{1}{x}-\ln\frac{1}{x}-1-x}{\left(1+x\ln\frac{1}{x}\right)\left(1+x\ln\left(\frac{1}{x}+\ln\frac{1}{x}\right)\right)} =$$

$$= \frac{-1-x\ln\frac{1}{x}-x-x^2}{x\left(1+x\ln\frac{1}{x}\right)\left(1+x\ln\left(\frac{1}{x}+\ln\frac{1}{x}\right)\right)} = -\frac{\left(x^2+x+x\ln\frac{1}{x}+1\right)}{x\left(1+x\ln\frac{1}{x}\right)\left(1+x\ln\left(\frac{1}{x}+\ln\frac{1}{x}\right)\right)}$$

4.5. $y' = \left(\ln tg\frac{x}{2} - \cos x\cdot\ln tg x\right)' = \frac{1}{tg\frac{x}{2}}\cdot\frac{1}{\cos^2\frac{x}{2}}\cdot\frac{1}{2} - \left((\cos x)'\cdot\ln tg x + (\ln tg x)'\cdot\cos x\right.$

$= \frac{\cos\frac{x}{2}}{2\sin\frac{x}{2}\cdot\cos^2\frac{x}{2}} - \left(-\sin x\ln tg x + \frac{1}{tg x}\cdot\frac{1\cdot\cos x}{\cos^2 x}\right) =$

$= \frac{1}{2\sin\frac{x}{2}\cdot\cos\frac{x}{2}} + \sin x\ln tg x - \frac{\cos^2 x}{\sin x\cdot\cos^2 x} = \frac{1}{\sin x} - \frac{1}{\sin x} + \sin x\ln tg x$

$= \sin x\cdot\ln tg x$

4.6. $y' = \left(\arcsin\left(\frac{1-x}{\sqrt2}\right)\right)' = \frac{1}{\sqrt{1-\frac{(1-x)^2}{2}}}\cdot\left(\frac{(1-x)}{\sqrt2}\right)' = \frac{1}{\sqrt{\frac{2-(1-x)^2}{2}}}\cdot\frac{(1-x)'\sqrt2 - (\sqrt2)'(1-x)}{2}$

$= \frac{\cancel{\sqrt2}\cdot(-1)\cdot\cancel{\sqrt2}}{\sqrt{2-1+2x-x^2}\cdot\cancel{2}} = -\frac{1}{\sqrt{1+2x-x^2}}$

4.7. $y' = \left(\arccos(\cos^2 x)\right)' = -\frac{1}{\sqrt{1-\cos^4 x}}\cdot(\cos^2 x)' = \frac{2\cos x\sin x}{\sqrt{1-\cos^4 x}} = \frac{2\cos x\sin x}{\sqrt{\cos^2 x+\sin^2 x-\cos^4 x}}$

$= \frac{2\cos x\sin x}{\sqrt{\cos^2 x(1-\cos^2 x)+\sin^2 x}} = \frac{2\cos x\sin x}{\sqrt{\cos^2 x\sin^2 x+\sin^2 x}} = \frac{2\cos x\sin x}{|\sin x|\sqrt{\cos^2 x+1}} =$

$= \frac{2\,sgn(\sin x)\cos x}{\sqrt{1+\cos^2 x}}$

4.8 $y' = \left(\arccos\sqrt{1-x^2}\right)' = \frac{-1}{\sqrt{1-1+x^2}}\cdot\frac{1}{2\sqrt{1-x^2}}\cdot\frac{(-2x)}{1} = \frac{+2x}{|x|\cdot2\sqrt{1-x^2}} =$

$= \frac{sgn\, x}{\sqrt{1-x^2}}$

4.9. $y' = \left(arctg\left(\frac{\sin x+\cos x}{\sin x-\cos x}\right)\right)' = \frac{1}{1+\left(\frac{\sin x+\cos x}{\sin x-\cos x}\right)^2}\cdot\left(\frac{\sin x+\cos x}{\sin x-\cos x}\right)' =$

$= \frac{1}{1+\left(\frac{\sin x+\cos x}{\sin x-\cos x}\right)^2}\cdot\frac{(\sin x+\cos x)'(\sin x-\cos x) - (\sin x-\cos x)'(\sin x+\cos x)}{(\sin x-\cos x)^2} =$

$= \frac{\cancel{(\sin x-\cos x)^2}}{(\sin x-\cos x)^2+(\sin x+\cos x)^2}\cdot\frac{(\cos x-\sin x)(\sin x-\cos x)-(\cos x+\sin x)^2}{\cancel{(\sin x-\cos x)^2}} =$

$= \frac{\cancel{\cos x\sin x} - \cos^2 x - \sin^2 x + \cancel{\cos x\sin x} - \cos^2 x - \cancel{2\cos x\sin x} - \sin^2 x}{\sin^2 x - 2\sin x\cos x + \cos^2 x + \sin^2 x + 2\sin x\cos x + \cos^2 x} =$

$= \frac{-2\cos^2 x - 2\sin^2 x}{2\sin^2 x + 2\cos^2 x} = \frac{-2(\cos^2 x+\sin^2 x)}{2(\sin^2 x+\cos^2 x)} = -1$?

4.10. $$y' = \left(\arcsin \frac{1-x^2}{1+x^2}\right)' = \frac{1}{\sqrt{1-\frac{(1-x^2)^2}{(1+x^2)^2}}} \cdot \frac{(1-x^2)'(1+x^2) - (1+x^2)'(1-x^2)}{(1+x^2)^2} =$$

$$= \frac{(1+x^2)}{\sqrt{(1+x^2)^2-(1-x^2)^2}} \cdot \frac{(-2x)(1+x^2) - 2x(1-x^2)}{(1+x^2)^2} =$$

$$= \frac{-2x - \cancel{2x^3} - 2x + \cancel{2x^3}}{\sqrt{1+2x^2+x^4-1+2x^2-x^4}\,(1+x^2)} = -\frac{4x}{2|x|\cdot(1+x^2)} = -\frac{2\,\mathrm{sgn}(x)}{1+x^2}$$

4.11. $$y' = \left(\operatorname{arctg} x + \frac{1}{3}\operatorname{arctg}(x^3)\right)' = \frac{1}{1+x^2} + \frac{1}{\cancel{3}} \cdot \frac{1\cdot \cancel{3}x^2}{1+x^6} =$$

$$= \frac{1}{1+x^2} + \frac{x^2}{1+x^6} = \frac{1+x^6+x^2(1+x^2)}{(1+x^2)(1+x^6)} = \frac{1+x^4}{1+x^6}$$

4.12. $$y' = \left(\ln\left(\arccos \frac{1}{\sqrt{x}}\right)\right)' = \frac{1}{\arccos\frac{1}{\sqrt{x}}} \cdot \left(-\frac{1}{\sqrt{1-\frac{1}{x}}}\right) \cdot \left(-\frac{1}{2\sqrt{x^3}}\right) =$$

$$= \frac{1}{\arccos\frac{1}{\sqrt{x}} \cdot \sqrt{x-1}\,2x}$$

4.13. $$y' = \left(\frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\cdot \arcsin\frac{x}{a}\right)' = \left(\left(\frac{x}{2}\right)'\sqrt{a^2-x^2} + \left((a^2-x^2)^{\frac{1}{2}}\right)'\frac{x}{2}\right) +$$

$$+ \left(\frac{a^2}{2}\cdot\frac{1}{\sqrt{1-\frac{x^2}{a^2}}}\cdot\frac{1}{a}\right) = \frac{1\sqrt{a^2-x^2}}{2} + \frac{1\cdot(-\cancel{2}x)\cdot x}{2\sqrt{a^2-x^2}\cdot\cancel{2}} + \frac{a\cdot a}{2\sqrt{a^2-x^2}} =$$

$$= \frac{\sqrt{a^2-x^2}}{2} - \frac{x^2}{2\sqrt{a^2-x^2}} + \frac{a^2}{2\sqrt{a^2-x^2}} = \frac{(a^2-x^2)-x^2+a^2}{2\sqrt{a^2-x^2}} = \frac{2(a^2-x^2)}{2\sqrt{a^2-x^2}} =$$

$$= \sqrt{a^2-x^2}$$

4.14. $$y' = \left(\frac{\arcsin x}{\sqrt{1-x^2}} + \frac{1}{2}\ln\frac{1-x}{1+x}\right)' =$$

$$= \frac{(\arcsin x)'\sqrt{1-x^2} - \left((1-x^2)^{\frac{1}{2}}\right)'\arcsin x}{1-x^2} + \frac{(1+x)}{2(1-x)} \cdot \frac{(1-x)'(1+x)-(1+x)'(1-x)}{(1+x)^2} =$$

$$= \frac{\frac{\sqrt{1-x^2}}{\sqrt{1-x^2}} + \frac{1\cdot 2x}{2\sqrt{1-x^2}}\arcsin x}{1-x^2} + \frac{-1-\cancel{x}-1+\cancel{x}}{2(1-x)(1+x)} =$$

$$= \frac{1+\frac{x\arcsin x}{\sqrt{1-x^2}}}{1-x^2} \cancel{+} - \frac{1}{1-x^2} = \frac{\cancel{2}+\cancel{\frac{x\arcsin x}{\sqrt{1-x^2}}}}{1-x^2} \cancel{=}$$

$$= \frac{\frac{x\arcsin x}{1}}{(1-x^2)^{\frac{3}{2}}}$$

$$\cancel{= 2\sqrt{1-x^2} + x\arcsin x}$$

4.15 $$y' = \left((\sin x)^{\cos x} + (\cos x)^{\sin x}\right) = \left(e^{\cos x\ln\sin x} + e^{\sin x\ln\cos x}\right)' =$$

$$= e^{\cos x\ln\sin x}\left(\frac{\cos^2 x}{\sin x} - \sin x\ln\sin x\right) + e^{\sin x\ln\cos x}\left(\cos x\ln\cos x - \frac{\sin^2 x}{\cos x}\right) =$$

$$= e^{\cos x\ln\sin x}\cdot\sin x\left(\operatorname{ctg}^2 x - \ln\sin x\right) + e^{\sin x\ln\cos x}\cdot\cos x\left(\ln\cos x - \operatorname{tg}^2 x\right) =$$

$$= (\sin x)^{1+\cos x}\cdot\left(\operatorname{ctg}^2 x - \ln\sin x\right) - (\cos x)^{1+\sin x}\cdot\left(\operatorname{tg}^2 x - \ln\cos x\right)$$

4.16. $y' = \left(\frac{ch\,x}{sh^2 x} - \ln\left(cth\frac{x}{2}\right)\right)' =$

$= \frac{ch'x\, sh^2 x - (sh^2 x)'\, ch\,x}{sh^4 x} - \frac{1}{cth\frac{x}{2}} \cdot \left(cth\frac{x}{2}\right)' =$

$= \frac{sh^3 x - 2ch^2 x\, sh\,x}{sh^4 x} + \frac{1}{cth\frac{x}{2}\, sh^2\frac{x}{2}} \cdot \frac{1}{2} =$

$= \frac{sh^2 x - 2ch^2 x}{sh^3 x} + \frac{1}{\frac{ch\frac{x}{2}\, sh^2\frac{x}{2}}{sh\frac{x}{2}} \cdot 2} = \frac{sh^2 x - 2ch^2 x}{sh^3 x} + \frac{1}{2ch\frac{x}{2}\, sh\frac{x}{2}} =$

4.17. $y' = \left(\arccos\left(\frac{1}{ch x}\right)\right)' = -\frac{1}{\sqrt{1-\frac{1}{ch^2 x}}} \cdot \frac{1'\, ch x - ch' x \cdot 1}{ch^2 x} =$

$= \frac{sh\,x \cdot \sqrt{ch^2 x}}{\sqrt{ch^2 x - 1} \cdot ch^2 x} = \frac{sign(sh\,x)}{ch\,x}$

4.18. $y' = \left((\arccos x)^2 \left(\ln^2(\arccos x) - \ln(\arccos x) + \frac{1}{2}\right)\right)' =$

$= \left((\arccos x)^2\right)' \cdot \left(\ln^2(\arccos x) - \ln(\arccos x) + \frac{1}{2}\right) + (\arccos x)^2 \cdot \left(\ln^2(\arccos x) - \ln(\arccos x)\right)'$

$= \frac{2\arccos x \cdot 1}{\sqrt{1-x^2}} \cdot \left(\ln^2(\arccos x) - \ln(\arccos x) + \frac{1}{2}\right) + (\arccos x)^2 \cdot \left(\frac{1 - 2\ln(\arccos x)}{\arccos x\, \sqrt{1-x^2}}\right) =$

$= -\frac{2\arccos x \cdot \ln^2(\arccos x) + \cancel{2\arccos x \ln(\arccos x)} - \cancel{\arccos x} + \cancel{\arccos x} - \cancel{2\arccos x \ln(\arccos x)}}{\sqrt{1-x^2}}$

?

$= -\frac{2}{\sqrt{1-x^2}} \cdot \arccos x \cdot \ln^2(\arccos x)$

4.19. $y' = \left(\frac{a^x}{1+a^{2x}} - \frac{1-a^{2x}}{1+a^{2x}} \cdot arctg\, a^{-x}\right)' =$

$= \frac{(a^x)'(1+a^{2x}) - a^x(1+a^{2x})'}{(1+a^{2x})^2} - \left(\left(\frac{1-a^{2x}}{1+a^{2x}}\right)' \cdot arctg\, a^{-x} + (arctg\, a^{-x})' \cdot \frac{1-a^{2x}}{1+a^{2x}}\right) =$

$= \frac{a^x \ln a (1+a^{2x}) - 2a^{3x} \cdot \ln a}{(1+a^{2x})^2} - \left(\frac{-4a^{2x} \ln a \cdot arctg\, a^{-x}}{(1+a^{2x})^2} + \frac{a^{-x} \ln a}{1+a^{-2x}}\right) =$

$= \frac{a^x \ln a + a^{3x} \ln a - 2a^{3x} \ln a}{(1+a^{2x})^2} + \frac{4a^{2x} \ln a \cdot arctg\, a^{-x}}{(1+a^{2x})^2} + \frac{\ln a}{a^x\left(1 + \frac{1}{a^{2x}}\right)} =$

$$= \frac{2a^{2x}\ln a + 4a^{2x}\ln a \cdot \operatorname{arctg} a^{-x}}{(1+a^{2x})^2}$$

?

4.20. $$y' = \left(x\cdot\sqrt{\frac{1-x}{1+x}}\right)' = \sqrt{\frac{1-x}{1+x}} + x\cdot\frac{1\sqrt{1+x}}{2\sqrt{1-x}}\cdot\left(\frac{1-x}{1+x}\right)' =$$

$$= \frac{\sqrt{1-x}}{\sqrt{1+x}} + \frac{x\sqrt{1+x}}{\sqrt{1-x}}\cdot\frac{(1-x)'(1+x)-(1+x)'(1-x)}{(1+x)^2} =$$

$$= \sqrt{\frac{1-x}{1+x}} + \frac{x\sqrt{1+x}}{\sqrt{1-x}}\cdot\frac{(-1-\cancel{x}-1+\cancel{x})}{(1+x)^2} =$$

$$= \frac{\sqrt{1-x}}{\sqrt{1+x}} - \frac{2x\sqrt{1+x}}{\sqrt{1-x}\,(1+x)^2} = \frac{(1-x)(1+2x+x^2)-2x(1+x)}{\sqrt{1-x^2}\,(1+x)^2} =$$

$$= \frac{1+2x+x^2-x-2x^2-x^3-2x-2x^2}{\sqrt{1-x^2}\,(1+x)^2} = \frac{-x^3-3x^2-x+1}{\sqrt{1-x^2}\,(1+x)^2} = \frac{-(x+1)(x^2+2x-1)}{\sqrt{1-x^2}\,(1+x)^2} =$$

$$= -\frac{x^2+2x-1}{\sqrt{1-x^2}\,(1+x)}$$

?

4.21. $y' = \left(\dfrac{(x+1)^3 \sqrt[4]{x-2}}{\sqrt[5]{(x-3)^2}} \right)' = \dfrac{\left((x+1)^3 (x-2)^{\frac{1}{4}}\right)' \cdot (x-3)^{\frac{2}{5}} - \left((x-3)^{\frac{2}{5}}\right)' (x+1)^3 \sqrt[4]{x-2}}{(x-3)^{\frac{4}{5}}}$

$= \dfrac{\dfrac{12(x+1)^2(x-2) + (x+1)^3}{4(x-2)^{\frac{3}{4}}} - \dfrac{2}{5} \cdot \dfrac{(x+1)^3 (x-2)^{\frac{1}{4}}}{(x-3)^{\frac{3}{5}}}}{(x-3)^{\frac{4}{5}}} =$

$= \dfrac{12(x+1)^2(x-2) + (x+1)^3}{4(x-2)^{\frac{3}{4}}(x-3)^{\frac{4}{5}}} - \dfrac{2(x+1)^3(x-2)^{\frac{1}{4}}}{5(x-3)^{-\frac{1}{5}}} =$

$= \dfrac{12(x+1)^2(x-2) + (x+1)^3}{4(x-2)^{\frac{3}{4}}(x-3)^{\frac{4}{5}}} - \dfrac{2(x+1)^3(x-2)^{\frac{1}{4}}(x-3)^{\frac{1}{5}}}{5} =$

$= \dfrac{60(x+1)^2(x-2) + 5(x+1)^3 - 8(x+1)^3(x-2)^{\frac{1}{4}}(x-2)^{\frac{3}{4}}(x-3)^{\frac{1}{5}}(x-3)^{\frac{4}{5}}}{20(x-2)^{\frac{3}{4}}(x-3)^{\frac{4}{5}}} =$

$= \dfrac{60(x+1)^2(x-2) + 5(x+1)^3 - 8(x+1)^3(x-2)(x-3)}{20(x-2)^{\frac{3}{4}}(x-3)^{\frac{4}{5}}} =$

$= \dfrac{(x+1)^2\left(60(x-2) + 5x + 5 - 8(x+1)(x^2 - 5x + 6)\right)}{20(x-2)^{\frac{3}{4}}(x-3)^{\frac{4}{5}}} =$

$= \dfrac{(x+1)^2(-8x^3 + 32x^2 + 57x - 163)}{20\sqrt[4]{(x-2)^3} \cdot \sqrt[5]{(x-3)^4}}$

Практические задания.

1.1 $y' = \left((x^3)^{\log_3 x} - \sqrt{\sin(3^x \ln x)}\right)' =$

$y_1 = (x^3)^{\log_3 x}$; $\ln y_1 = \ln (x^3)^{\log_3 x} = 3\log_3 x \cdot \ln x$

$\frac{y_1'}{y_1} = 3 \cdot \left((\log_3 x)' \ln x + (\ln x)' \log_3 x\right) =$

$\Rightarrow y_1' = 3(x^3)^{\log_3 x}\left(\frac{1 \cdot \ln x}{x \ln 3} + \frac{\log_3 x}{x}\right) =$

$= \frac{3}{x}(\log_3 x + \log_3 x) \cdot (x^3)^{\log_3 x} = \frac{6\log_3 x}{x} \cdot x^{3\log_3 x}$

$y_2' = \left(\sqrt{\sin(3^x \ln x)}\right)' = \frac{1 \cdot \cos(3^x \ln x)}{2\sqrt{\sin(3^x \ln x)}} \cdot \left(3^x \ln 3 \ln x + \frac{3^x}{x}\right) =$

$= \frac{3^x \cos(3^x \ln x)(\ln 3 \ln x + \frac{1}{x})}{2\sqrt{\sin(3^x \ln x)}}$

1.25. $y' = \left(\underbrace{\operatorname{arctg}\sqrt{3^x + x^3}}_{y_1} + \underbrace{(tg\, x)^{x^2}}_{y_2}\right)' =$

$y_1' = \frac{1}{1+3^x+x^3} \cdot \left(\sqrt{3^x + x^3}\right)' = \frac{3^x \ln 3 + 3x^2}{1+3^x+x^3} \cdot \frac{1}{2\sqrt{3^x+x^3}} = \frac{3^x \ln 3 + 3x^2}{2\sqrt{3^x+x^3}\,(1+3^x+x^3)}$

$y_2' = \left(e^{\ln (tg\, x)^{x^2}}\right)' = \left(e^{x^2 \ln tg\, x}\right)' = e^{x^2 \ln tg x} \cdot \left(2x\, tg x + \frac{x^2}{\cos^2 x}\right) =$

$= (tg x)^{x^2}\left(2x \cdot tg x + \frac{x^2}{\cos^2 x} \cdot \frac{1}{tg x}\right)$

$y' = \frac{3^x \ln 3 + 3x^2}{2\sqrt{3^x+x^3}\,(1+3^x+x^3)} + (tg x)^{x^2}\left(2x \cdot tg x + \frac{x^2}{\cos^2 x} \cdot \frac{1}{tg x}\right)$

1.14 $y' = \left(\underbrace{x^{\arcsin^3 x}}_{y_1} + \underbrace{\log_3 \cos 2^{\sqrt{x}}}_{y_2}\right)'$

$(y_1)' = \left(e^{\ln x^{\arcsin^3 x}}\right)' = \left(e^{\arcsin^3 x \ln x}\right)' = e^{\ln x^{\arcsin^3 x}} \cdot \left((\arcsin^3 x)' \ln x + (\ln x)' \arcsin^3 x\right)$

$= x^{\arcsin^3 x}\left(\frac{3\arcsin^2 x \ln x}{\sqrt{1-x^2}} + \frac{\arcsin^3 x}{x}\right)$

$y_2' = \cancel{\cos 2^{\sqrt{x}} \cdot \ln 3 \cdot (\cos 2^{\sqrt{x}})'} = \cancel{\cos 2^{\sqrt{x}} \cdot \ln 3 \cdot \sin 2^{\sqrt{x}}}\ \frac{1 \cdot (-\sin 2^{\sqrt{x}}) \cdot 2^{\sqrt{x}} \cdot \ln 2}{\cos(2^{\sqrt{x}}) \ln 3 \cdot 2\sqrt{x}}$

$y' = y_1' + y_2' =$

$= x^{\arcsin^3 x}\left(\frac{3\arcsin^2 x \ln x}{\sqrt{1-x^2}} + \frac{\arcsin^3 x}{x}\right) + \frac{(-\sin 2^{\sqrt{x}}) \cdot 2^{\sqrt{x}} \cdot \ln 2 \cdot \ln x}{\cos(2^{\sqrt{x}}) \ln 3 \cdot 2\sqrt{x}}$

1.16. $y' = (\underbrace{\log_2 \sin(2^x)}_{y_1} + \underbrace{(\sin x)^x}_{y_2})$

$$y_1' = \frac{1\cdot\cos 2^x \cdot \cancel{x}2^x\ln 2}{\sin(2^x)\cdot\ln 2} = ctg\, 2^x \cdot 2^x$$

$$y_2' = (e^{x\ln\sin x})' = (\sin x)^x\cdot\left(\ln\sin x + \frac{1\cdot\cos x}{\sin x}\cdot x\right) =$$

$$= (\sin x)^x \cdot (\ln\sin x + x\, ctg\, x)$$

$$y' = ctg\, 2^x \cdot 2^x + (\sin x)^x\cdot(\ln \sin x + x\, ctg\, x)$$

1.20 $y' = (\ln\cos(3^x) + \arcsin(e^{-2x}) + (\sqrt x)^x)'$

$$y_1' = \frac{-1\cdot\sin 3^x\cdot 3^x\ln 3}{\cos 3^x} = -tg\, 3^x \cdot 3^x\ln 3$$

$$y_2' = \frac{-2\cdot e^{-2x}}{\sqrt{1-e^{-4x}}} = \frac{-2}{e^{2x}\sqrt{1-e^{-4x}}}$$

$$y_3' = e^{x\ln\sqrt x} = (\sqrt x)^x\left(\ln\sqrt x + \frac{1}{\sqrt x}\cdot\frac{1\cdot x}{2\sqrt x}\right) = (\sqrt x)^x\left(\ln\sqrt x + \frac12\right)$$

$$y' = (\sqrt x)^x\left(\ln\sqrt x + \frac12\right) - tg\, 3^x \cdot 3^x \ln 3 - \frac{2}{e^{2x}\sqrt{1-e^{-4x}}}$$

1.21. $y = x^{tg^2 x} + \log_2 \arccos\sqrt x$

$$y_1' = e^{tg^2x\ln x} = x^{tg^2x}\left(\frac{2tg\, x \ln x}{\cos^2 x} + \frac{tg^2 x}{x}\right)$$

$$y_2' = \frac{1}{\arccos\sqrt x\cdot\ln 2}\cdot\left(-\frac{1}{\sqrt{1-x}}\right)\cdot\frac{1}{2\sqrt x} = -\frac{1}{2\sqrt x\cdot\sqrt{1-x}\cdot\ln 2\cdot\arccos\sqrt x}$$

$$y' = y_1'+y_2' = x^{tg^2x}\left(\frac{2tg\,x\ln x}{\cos^2x} + \frac{tg^2x}{x}\right) - \frac{1}{2\sqrt x\cdot\sqrt{1-x}\cdot\ln2\cdot\arccos\sqrt x}$$

1.22. $y = \underbrace{\log_2\arccos\sqrt{2^x+x^2}}_{y_1} + \underbrace{x^{\ln x}}_{y_2}$

$$y_1' = \frac{1}{\arccos\sqrt{2^x+x^2}\cdot\ln 2}\cdot\frac{1\cdot 2^x\ln 2 \cdot 2x}{2\sqrt{2^x+x^2}} = \frac{\cancel{2}x\cdot2^x}{\cancel{2}\sqrt{2^x+x^2}\cdot\arccos\sqrt{2^x+x^2}}$$

$$y_2' = (e^{\ln x\cdot\ln x})' = x^{\ln x}(\ln x'\cdot\ln x + \ln x'\cdot \ln x) =$$

$$= x^{\ln x}\cdot\frac{2\ln x}{x} = \frac{2\ln x\cdot x^{\ln x}}{x}$$

$$y' = y_1'+y_2' = \frac{x\cdot 2^x}{\sqrt{2^x+x^2}\arccos\sqrt{2^x+x^2}} + \frac{2\ln x\cdot x^{\ln x}}{x}$$

1.23. $y = (\sqrt{x})^{ctg(2x)} + \log_2 \sin(\cos e^{-2x})$

y_1 ... y_2

$y_1' = \left(e^{ctg2x \cdot \ln\sqrt{x}}\right)' = (\sqrt{x})^{ctg2x}\left((ctg2x)'\ln\sqrt{x} + (\ln\sqrt{x})' ctg2x\right) =$

$= (\sqrt{x})^{ctg2x} \cdot \left(\frac{-\ln\sqrt{x} \cdot 2}{\sin^2 2x} + \frac{1 \cdot 1 \cdot ctg2x}{\sqrt{x} \cdot 2\sqrt{x}}\right) = ?$

$= (\sqrt{x})^{ctg2x} \cdot \left(\frac{ctg2x}{2x} - \frac{2\ln\sqrt{x}}{\sin^2 2x}\right)$

$y_2' = \frac{1 \cdot \cos(\cos e^{-2x}) \cdot (-\sin e^{-2x}) \cdot (-2)}{\ln 2 \cdot \sin(\cos e^{-2x}) \cdot e^{2x}} =$

$= \frac{2\cos(\cos e^{-2x}) \cdot \sin e^{-2x}}{e^{2x} \ln 2 \cdot \sin(\cos e^{-2x})}$

$y' = y_1' + y_2' = (\sqrt{x})^{ctg2x}\left(\frac{ctg2x}{2x} - \right.$

Производное от функции, заданой параметрически.

$$\begin{cases} x = x(t) \\ y = y(t) \end{cases}$$

$t \in T$

$$\boxed{y_x' = \frac{y_t'}{x_t'}}$$

1) $\begin{cases} x = \sin^2 t \\ y = \cos^2 t \end{cases}$

$y_x' = -1 = \frac{2\sin t \cdot \cos t}{-2\cos t\,\sin t} = \cancel{(\cos^2 t)_t'} =$

$= \frac{(\cos^2 t)_t'}{(\sin^2 t)_t'} = \frac{2\cos t(-\sin t)}{2\sin t \cdot \cos t} = -1$

2) $\begin{cases} x = \arcsin\frac{t}{\sqrt{1+t^2}} \\ y = \arccos\frac{1}{\sqrt{1+t^2}} \end{cases}$

$x_t' = \frac{1}{\sqrt{1-\frac{t^2}{1+t^2}}} \cdot \frac{\sqrt{1+t^2} - \frac{1 \cdot 2t}{2\sqrt{1+t^2}} t}{1+t^2} = \frac{((1+t^2)-t^2) \cdot \sqrt{1+t^2}}{1+t^2 \cdot \sqrt{1+t^2}} = \frac{t^2 - t^2 + 1}{1+t^2} = \frac{1}{1+t^2}$

$y_t' = -\frac{1}{\sqrt{1-\frac{1}{1+t^2}}} \cdot \left(-\frac{1 \cdot 2t}{(1+t)^2}\right) = \frac{2t}{\sqrt{\frac{1+t^2-1}{1+t^2}} \cdot (1+t^2)^2} = \frac{2t \cdot \sqrt{1+t^2}}{\sqrt{t^2} \cdot (1+t)^2} =$

$= \frac{2t}{|t|}$

$y_x' = \frac{y_t'}{x_t'} = \frac{2t \cdot \sqrt{1+t^2}}{\sqrt{t^2}\,(1+t)^2}$

$$y'_t = -\frac{1}{\sqrt{1-\frac{1}{1+t^2}}} \cdot \left(-\frac{1}{2} \cdot \frac{1}{\sqrt{(1+t^2)^3}} \cdot 2t\right) = \frac{1}{\sqrt{1-\frac{1}{1+t^2}}} \cdot \frac{t}{(1+t^2)\cdot\sqrt{1+t^2}} =$$

$$= \frac{t}{\sqrt{1+t^2-1}} = \operatorname{sign}(t)$$

~~1039, 1041, 1042, 1043, 1044, 1045~~

Домашняя работа.

1039. $\begin{cases} x = \sqrt[3]{1-\sqrt{t}} \\ y = \sqrt{1-\sqrt[3]{t}} \end{cases}$

$$x'_t = \left((1-\sqrt{t})^{\frac{1}{3}}\right)' = \frac{1\cdot(-1)}{3\sqrt[3]{(1-\sqrt{t})^2}\cdot 2\sqrt{t}} = -\frac{1}{6\sqrt{t}\cdot\sqrt[3]{(1-\sqrt{t})^2}}$$

$$y'_t = \frac{1\cdot(-1)\cdot 1}{2\sqrt{1-\sqrt[3]{t}}\cdot 3\sqrt[3]{t^2}} = -\frac{1}{6\sqrt[3]{t^2}\cdot\sqrt{1-\sqrt[3]{t}}}$$

$$y'_x = \frac{y'_t}{x'_t} = \frac{6\sqrt{t}\cdot\sqrt[3]{(1-\sqrt{t})^2}}{6\sqrt[3]{t^2}\cdot\sqrt{1-\sqrt[3]{t}}} = \sqrt[6]{\frac{t^3(1-\sqrt{t})^4}{t^4(1-\sqrt[3]{t})^3}} = \sqrt[6]{\frac{(1-\sqrt{t})^4}{t(1-\sqrt[3]{t})^3}}$$

1041. $\begin{cases} x = a\cos t \\ y = b\sin t \end{cases}$ $\quad x'_t = -a\sin t$

$$y'_t = b\cos t$$

$$y'_x = -\frac{b}{a}\operatorname{ctg} t \quad (0 < |t| < \pi)$$

1042. $\begin{cases} x = a\operatorname{ch} t \\ y = b\operatorname{sh} t \end{cases}$ $\quad x'_t = a\operatorname{sh} t$

$$y'_t = b\operatorname{ch} t$$

$$y'_x = \frac{y'_t}{x'_t} = \frac{b}{a}\operatorname{cth} t \quad (|t| > 0)$$

1043. $\begin{cases} x = a\cos^3 t \\ y = a\sin^3 t \end{cases}$

$$x'_t = -3a\cos^2 t\sin t$$

$$y'_t = 3a\sin^2 t\cos t$$

$$y'_x = \frac{3a\sin^2 t\cos t}{-3a\cos^2 t\sin t} = -\frac{a}{a}\operatorname{tg} t = -\operatorname{tg} t$$

1044. $\begin{cases} x = a(t-\sin t) = at - a\sin t \\ y = a(1-\cos t) = a - a\cos t \end{cases}$

$$x'_t = a - a\cos t = a(1-\cos t) = 2a\sin^2\frac{t}{2}$$

$$y'_t = a\sin t$$

$$y'_x = \frac{\sin t}{1-\cos t} = \operatorname{ctg}\frac{t}{2} = \frac{2\sin\frac{t}{2}\cos\frac{t}{2}}{2\sin^2\frac{t}{2}}$$

1045. $\begin{cases} x = e^{2t}\cos^2 t \\ y = e^{2t}\sin^2 t \end{cases}$

$$x'_t = (e^{2t})'\cos^2 t + (\cos^2 t)' e^{2t} =$$
$$= 2e^{2t}\cos^2 t - 2\cos t\sin t \cdot e^{2t} =$$
$$= 2e^{2t}\cos t(\cos t - \sin t)$$

$3_{,}^{15}$

$$y'_t = 2e^{2t}\sin^2 t + 2\sin t\cos t \cdot e^{2t} = 2e^{2t}\sin t(\sin t + \cos t)$$

$$y'_x = \frac{y'_t}{x'_t} = \frac{2e^{2t}\sin t(\sin t + \cos t)}{2e^{2t}\cos t(\cos t - \sin t)} = tg\,t \cdot \frac{\sin t + \cos t}{\cos t - \sin t}.$$

$$\lim_{x\to 0}\left(\frac{1+x}{(1-x)}\right)^{\frac{1}{x}} = \lim_{x\to 0}\frac{(1+x)^{\frac{1}{x}}}{(1-x)^{\frac{1}{x}}} = e\lim_{x\to 0}\frac{1}{\left(\left(\underbrace{(1+(-x))^{\frac{1}{-x}}}_{e}\right)^{-x}\right)^{\frac{1}{x}}} =$$

$$= e\lim_{x\to 0}\frac{1}{e^{-1}} = e^{2}$$

$$(1-x)^{\frac{1}{x}}$$

$$\lim_{x\to\frac{\pi}{6}}\frac{\sin(x-\frac{\pi}{6})}{\frac{\sqrt3}{2}-\cos x} = \left\{\begin{aligned} x-\tfrac{\pi}{6} &= t \\ x &= \tfrac{\pi}{6}+t \\ t &\to 0 \end{aligned}\right\} \textcircled{=} \lim_{t\to 0}\frac{\sin(\frac{\pi}{6}+t-\frac{\pi}{6})}{\frac{\sqrt3}{2}-\cos(t+\frac{\pi}{6})} =$$

$$= \frac{\sin t}{\frac{\sqrt3}{2}-(\cos t\cos\frac{\pi}{6}-\sin\frac{\pi}{6}\sin t)} = \lim_{t\to 0}\frac{\sin t}{\frac{\sqrt3}{2}-\cos t\cdot\frac{\sqrt3}{2}+\sin\frac{\pi}{6}\cdot\sin t} =$$

$$= \lim_{t\to 0}\frac{\overset{1}{\frac{\sin t}{t}}\cdot t}{(\frac{\sqrt3}{2}-\cos t\cdot\frac{\sqrt3}{2}+\frac{1}{2}\sin t)\cdot t} = \lim_{t\to 0}\frac{t}{\frac{\sqrt3}{2}(1-\cos t)+\frac{1}{2}\sin t} =$$

$$= \lim_{t\to 0}\frac{t}{\frac{\sqrt3}{2}\sin^2\frac{t}{2}+\frac{1}{2}\sin t} = \lim_{t\to 0}\frac{2t}{\sqrt3\sin^2\frac{t}{2}+\sin t} =$$

$$\textcircled{=} \lim_{t\to 0}\frac{\sin t}{\cos\frac{\pi}{6}-\cos(t+\frac{\pi}{6})} = \lim_{t\to 0}\frac{\sin t}{-2\sin(\frac{t}{2}+\frac{\pi}{6})\sin(-\frac{t}{2})} =$$

$$= \lim_{t\to 0}\frac{\sin t}{2\sin(\frac{t}{2}+\frac{\pi}{6})\sin\frac{t}{2}} = \frac{t\cdot 2}{2t\cdot\sin(\frac{t}{2}+\frac{\pi}{6})} = 2$$

Вычисление производных
от неявно заданных функций.

1051, 1052, 1053, 1054 (в.) 1048 - 1054 ; Д/з: 104

1051.

$$\sqrt{x}+\sqrt{y}=\sqrt{a}$$

$$\frac{1}{2\sqrt{x}}+\frac{1}{2\sqrt{y}}\cdot y'_x=0$$

$$\frac{y'_x}{\sqrt{y}}=-\frac{1}{\sqrt{x}} \Rightarrow y'_x=-\frac{\sqrt{y}}{\sqrt{x}}$$

1052.

$$x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}$$

$$\frac{2}{3\sqrt[3]{x}}+\frac{2}{3\sqrt[3]{y}}\cdot y'=0$$

$$\frac{2}{3\sqrt[3]{y}}\cdot y'=-\frac{2}{3\sqrt[3]{x}} \Rightarrow y'_x=-\frac{\sqrt[3]{y}}{\sqrt[3]{x}}$$

1053. $\operatorname{arctg}\frac{y}{x}=\ln\sqrt{x^2+y^2}$

$$\frac{1}{1+\frac{y^2}{x^2}}\cdot y'=\frac{1}{\sqrt{x^2+y^2}}\cdot(x^2+y^2)'$$

$$\frac{x^2}{x^2+y^2}\left(\frac{y}{x}\right)'=\frac{2x+2y\cdot y'}{\sqrt{x^2+y^2}}$$

$$\frac{x^2}{x^2+y^2}\cdot\frac{(y'x-x'y)}{x^2}=\frac{2x+2y\cdot y'}{\sqrt{x^2+y^2}}$$

$$\frac{x^2}{x^2+y^2}\cdot\frac{(y'x-y)}{x^2}=\frac{2x+2y\cdot y'}{\sqrt{x^2+y^2}}$$

$$\frac{y'x-y}{x^2+y^2}=\frac{2x+2y\cdot y'}{\sqrt{x^2+y^2}}$$

$$(y'x-y)=(2x+2y\cdot y')\cdot\sqrt{x^2+y^2}$$

$$\frac{x^2}{x^2+y^2}\cdot\left(\frac{y}{x}\right)'=\frac{1}{\sqrt{x^2+y^2}}\cdot\frac{1\cdot(x^2+y^2)'}{2\sqrt{x^2+y^2}}$$

$$\frac{y'x-y}{(x^2+y^2)}=\frac{1\cdot 2x+2y\cdot y'}{2(x^2+y^2)}$$

$$y'x-y=x+y\cdot y'$$

$$y'x-y\cdot y'=x+y \Rightarrow y'=\frac{x+y}{x-y}$$

1054. a) $r = a\varphi$; $\frac{y}{\sin\varphi} = a \cdot \arccos\left(\frac{x\sin\varphi}{y}\right)$

$x = r\cos\varphi$
$y = r\sin\varphi$
$r = \frac{y}{\sin\varphi}$
$\cos\varphi = \frac{x}{r}$
$\varphi = \arccos\left(\frac{x}{r}\right)$
$\varphi = \arccos\left(\frac{x\sin\varphi}{y}\right)$
$r = \sqrt{x^2+y^2}$
$\varphi = \operatorname{arctg}\frac{y}{x}$

$\left(\frac{y}{\sin\varphi}\right)' = a \cdot \arccos\left(\frac{x\sin\varphi}{y}\right)'$

$\sqrt{x^2+y^2} = a \cdot \operatorname{arctg}\frac{y}{x}$

$\frac{1\cdot(x^2+y^2)'}{2\sqrt{x^2+y^2}} = a \cdot \frac{1}{1+\frac{y^2}{x^2}} \cdot \left(\frac{y}{x}\right)'$

$\frac{2x+2y\cdot y'}{2\sqrt{x^2+y^2}} = \frac{a\cdot \cancel{x^2}}{x^2+y^2} \cdot \frac{y'x - x'y}{\cancel{x^2}}$

$\frac{x+y'\cdot y}{\sqrt{x^2+y^2}} = \frac{a\cdot(y'x - x'y)}{x^2+y^2}$

Домашняя работа.

1048. $x^2+2xy-y^2 = 2x$

$2x + 2(xy)' - 2y\cdot y' = 2$

$2x + 2(x'y + y'x) - 2y\cdot y' = 2 \quad /:2$

$x + y + y'x - y'\cdot y = 1.$

$y'(x-y) = 1 - x - y \quad \rightarrow \quad y'_x = \frac{1-x-y}{x-y}$

1049. $y^2 = 2px$

$2y\cdot y' = 2p. \rightarrow y' = \frac{2p}{2y} = \frac{p}{y}$

1050. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 0.$

$\frac{2x}{a^2} + \frac{2y\cdot y'}{b^2} = 0.$; $\cancel{\frac{2y\cdot y'}{b^2} = 1 - \frac{2x}{a^2}}$

$y' = \cancel{\frac{b^2\left(1-\frac{2x}{a^2}\right)}{2y}} = \cancel{\frac{b^2 - \frac{2xb^2}{a^2}}{2y}} = \cancel{\frac{a^2b^2 - 2xb^2}{2yb^2}}$

$y' = \frac{-2x\cdot b^2}{a^2\cdot 2y} = -\frac{b^2x}{a^2y}$

1054.

а) $r = a\varphi$

$y = r\sin\varphi \qquad y'_\varphi = (r\sin\varphi)' \qquad y'_x = \dfrac{y'_\varphi}{x'_\varphi}$

$x = r\cos\varphi \qquad x'_\varphi = (r\cos\varphi)'$

$r = \sqrt{x^2+y^2} \qquad \varphi = \operatorname{arctg}\left(\dfrac{y}{x}\right)$

$\sqrt{x^2+y^2} = a\cdot\operatorname{arctg}\left(\dfrac{y}{x}\right)$

$\dfrac{1\cdot(x^2+y^2)'}{2\sqrt{x^2+y^2}} = a\cdot\dfrac{1}{1+\frac{y^2}{x^2}}\cdot\left(\dfrac{y}{x}\right)'$

$\dfrac{2x+2y\cdot y'}{2\sqrt{x^2+y^2}} = \dfrac{a\cdot x^2}{x^2+y^2}\cdot\dfrac{(y'x - x'y)}{x^2}$

$\dfrac{x+y\cdot y'}{\sqrt{x^2+y^2}} = \dfrac{a(y'x-y)}{x^2+y^2}$

$\dfrac{dy}{dx} = \dfrac{r\cos\varphi}{r\sin\varphi}$

$\dfrac{dy}{dx} = \dfrac{y'_\varphi}{x'_\varphi} = \dfrac{(r(\varphi)\sin\varphi)'}{(r(\varphi)\cos\varphi)'} = \dfrac{r'(\varphi)\sin\varphi + r(\varphi)\cos\varphi}{r'(\varphi)\cos\varphi - r(\varphi)\sin\varphi}$

а) $r = a\varphi$

$$y' = \frac{(a\varphi)'\sin\varphi + a\varphi\cos\varphi}{(a\varphi)'\cos\varphi - a\varphi\sin\varphi} = \frac{a\sin\varphi + a\varphi\cos\varphi}{a\cos\varphi - a\varphi\sin\varphi} = \frac{\sin\varphi + \varphi\cos\varphi}{\cos\varphi - \varphi\sin\varphi} =$$

$$= \operatorname{tg}\varphi\,\frac{(1+\varphi\operatorname{ctg}\varphi)}{(1-\varphi\operatorname{tg}\varphi)} = \frac{\operatorname{tg}\varphi + \varphi}{1-\operatorname{tg}\varphi\cdot\varphi} = \operatorname{tg}(\varphi + \operatorname{arctg}\varphi)$$

б) $r = a(1+\cos\varphi) = a + a\cos\varphi$

$$y'_x = \frac{-a\sin^2\varphi + a(1+\cos\varphi)\cos\varphi}{-a\sin\varphi\cos\varphi - a(1+\cos\varphi)\sin\varphi} = \frac{-\sin^2\varphi + \cos^2\varphi + \cos\varphi}{-\sin\varphi\cos\varphi - \sin\varphi - \sin\varphi\cos\varphi} =$$

$$= \frac{-\sin^2\varphi + 2\cos^2\frac{\varphi}{2}\cdot\cos\varphi}{-2\sin\varphi\cos\varphi - \sin\varphi} = \frac{\sin^2\varphi - 2\cos^2\frac{\varphi}{2}\left(\cos^2\frac{\varphi}{2} - \sin^2\frac{\varphi}{2}\right)}{\sin 2\varphi + \sin\varphi} =$$

$$= \frac{\sin^2\varphi - 2\cos^2\frac{\varphi}{2} + 2\cos\frac{\varphi}{2}\sin\frac{\varphi}{2}\cdot\cos\frac{\varphi}{2}\sin\frac{\varphi}{2}}{2\sin\frac{\varphi}{2}\sin\frac{3\varphi}{2}} =$$

$$= -\frac{\cos 2\varphi + \cos\varphi}{\sin 2\varphi + \sin\varphi} = -\frac{2\cos\frac{\varphi}{2}\cos\frac{3\varphi}{2}}{2\sin\frac{3\varphi}{2}\cos\frac{\varphi}{2}} = -\operatorname{ctg}\frac{3\varphi}{2}$$

в) $r = a\cdot e^{m\varphi}$

$$y' = \frac{m\cdot a\cdot e^{m\varphi}\sin\varphi + a\cdot e^{m\varphi}\cos\varphi}{m\cdot a\cdot e^{m\varphi}\cos\varphi - a\cdot e^{m\varphi}\sin\varphi} = \frac{m\sin\varphi + \cos\varphi}{m\cos\varphi - \sin\varphi} = \frac{\sin\varphi + \frac{1}{m}\cos\varphi}{\cos\varphi - \frac{1}{m}\sin\varphi} =$$

$= tg\left(\varphi + arctg\,\frac{\ell}{m}\right)$

Домашняя работа.

5.7. $y^3 - 3y + 2ax = 0$

$3y^2 \cdot y' - 3y' + 2a = 0$

$y'(3y^2 - 3) = -2a$

$\Rightarrow \quad y' = \frac{2a}{3(1-y^2)}$

5.8. $y^2 - 2xy + b^2 = 0$

$2y \cdot y' - 2(x'y + y'x) = 0$

$2y \cdot y' - 2y - 2y'x = 0 \quad /:2$

$y \cdot y' - y - y'x = 0$

$y'(y-x) = y \quad \longrightarrow \quad y'_x = \frac{y}{y-x}$

5.9. $\sin(xy) + \cos(xy) = tg(x+y)$

$\cos(xy) \cdot (xy)' - \sin(xy) \cdot (xy)' = \frac{1 \cdot (x+y)'}{\cos^2(x+y)}$

$(xy)'(\cos xy - \sin xy) = \frac{1+y'}{\cos^2(x+y)}$

$(xy' + y)(\cos xy - \sin xy) = \frac{1+y'}{\cos^2(x+y)}$

$\left(y'x\cos(xy) - y'x\sin(xy) + y(\cos xy) - y\sin(xy)\right)\cos^2(x+y) = 1 + y'$

$\underline{y'x\cos(xy)\cos^2(x+y)} - \underline{y'x\sin(xy)\cos^2(x+y)} + y\cos(xy)\cos^2(x+y) - y\sin(xy)\cos^2(x+y) \underline{- y'} - 1 = 0$

$y'\left(x\cos(xy)\cos^2(x+y) - x\sin(xy)\cos^2(x+y) - 1\right) = -\left(y\cos(xy)\cos^2(x+y) - y\sin(xy)\cos^2(x+y) - 1\right)$

$\Rightarrow y' = -\frac{y\cos^2(x+y)\left(\cos(xy) - \sin(xy)\right) - 1}{x\cos^2(x+y)\left(\cos(xy) - \sin(xy)\right) - 1}$

5.10. $x - y = \arcsin x - \arcsin y$

$1 - y' = \frac{1}{\sqrt{1-x^2}} - \frac{1 \cdot y'}{\sqrt{1-y^2}}$

$1 - \frac{1}{\sqrt{1-x^2}} = y'\left(1 - \frac{1}{\sqrt{1-y^2}}\right)$; $\quad \frac{\sqrt{1-x^2} - 1}{\sqrt{1-x^2}} = y'\left(\frac{\sqrt{1-y^2} - 1}{\sqrt{1-y^2}}\right)$

$\Rightarrow y' = \frac{\sqrt{1-y^2}\left(\sqrt{1-x^2} - 1\right)}{\sqrt{1-x^2}\left(\sqrt{1-y^2} - 1\right)} = \frac{\sqrt{1-y^2}\left(1 - \sqrt{1-x^2}\right)}{\sqrt{1-x^2}\left(1 - \sqrt{1-y^2}\right)}$

5.11. $r = a\cdot e^{m\varphi}$

$$r = \sqrt{x^2+y^2}\ ;\ \varphi = \operatorname{arctg}\frac{y}{x}$$

$$\frac{dy}{dx} = \frac{y_\varphi'}{x_\varphi'} = \frac{(r(\varphi)\sin\varphi)'}{(r(\varphi)\cos\varphi)'} = \frac{r'(\varphi)\cdot\sin\varphi + r(\varphi)\cos\varphi}{r'(\varphi)\cdot\cos\varphi - r(\varphi)\sin\varphi}$$

$$y' = \frac{m\cdot a\cdot e^{m\varphi}\cdot\sin\varphi + a\cdot e^{m\varphi}\cos\varphi}{m\cdot a\cdot e^{m\varphi}\cos\varphi - a\cdot e^{m\varphi}\sin\varphi} =$$

$$= \frac{m\sin\varphi+\cos\varphi}{m\cos\varphi-\sin\varphi} = \frac{\sin\varphi+\frac{1}{m}\cos\varphi}{\cos\varphi-\frac{1}{m}\sin\varphi} = \frac{\operatorname{tg}\varphi+\frac{1}{m}}{1-\frac{1}{m}\operatorname{tg}\varphi} =$$

$$= \operatorname{tg}\left(\varphi+\operatorname{arctg}\frac{1}{m}\right)$$

Демидович:

965. $$y = \left(\frac{(\ln x)^x}{x^{\ln x}}\right)' = \frac{((\ln x)^x)'(x^{\ln x}) - (x^{\ln x})'(\ln x)^x}{x^{2\ln x}} =$$

$$=$$

$$((\ln x)^x)' = \left(e^{x\ln(\ln x)}\right)' = (\ln x)^x\cdot\left(\ln(\ln(x)) + \frac{x}{\ln x}\cdot\frac{1}{x}\right) =$$

$$= (\ln x)^x\cdot\left(\ln(\ln x) + \frac{1}{\ln x}\right) = (\ln x)^{x-1} + (\ln x)^x\cdot\ln(\ln x)$$

$$(x^{\ln x})' = \left(e^{\ln x\cdot\ln x}\right)' = x^{\ln x}\cdot\left(\frac{\ln x}{x}+\frac{\ln x}{x}\right) = \frac{2\ln x}{x}\cdot x^{\ln x} =$$

$$= 2\ln x\cdot x^{\ln x-1}.$$

$$y' = \frac{\left((\ln x)^{x-1} + (\ln x)^x\cdot\ln(\ln x)\right)\cdot x^{\ln x} - 2\ln x\cdot x^{\ln x-1}\cdot(\ln x)^x}{x^{2\ln x}} =$$

$$y' = \left(\frac{e^{x\ln(\ln x)}}{e^{\ln^2 x}}\right)' = \left(e^{x\ln(\ln x)-\ln^2 x}\right)' =$$

$$= e^{x\ln(\ln x)-\ln^2 x}\cdot\left(x\ln(\ln x)-\ln^2 x\right)' =$$

$$= e^{x\ln(\ln x)-\ln^2 x}\cdot\left(\ln(\ln x)+\frac{1}{\ln x}-\frac{2\ln x}{x}\right) =$$

$$= \frac{(\ln x)^x}{x^{\ln x}}\cdot\left(\ln(\ln x)+\frac{1}{\ln x}-\frac{2\ln x}{x}\right) =$$

$$= \frac{(\ln x)^{x-1}}{x^{\ln x}}\left(\ln x\cdot\ln(\ln x)+1-\frac{2\ln^2 x}{x}\right) =$$

$$= \frac{(\ln x)^{x-1}}{x^{\ln x+1}}\left(x\ln x\cdot\ln(\ln x)+x-2\ln^2 x\right)$$

966. $y=\log_x e = \frac{\ln e}{\ln x} = \frac{1}{\ln x}$

$y' = -\frac{1}{\ln^2 x}\cdot(\ln x)' = -\frac{1}{x\ln^2 x} = -\frac{\ln^2 e}{x\ln^2 x} = -\frac{1}{x}\cdot(\log_x e)^2$

967. $y=\ln(\operatorname{ch} x) + \frac{1}{2\operatorname{ch}^2 x}$

$y' = \frac{1}{\operatorname{ch} x}\cdot\operatorname{sh} x + \frac{1}{2}\cdot\left(-\frac{1}{\operatorname{ch}^4 x}\right)\cdot(\operatorname{ch}^2 x)' = \operatorname{th} x - \frac{2\operatorname{ch} x\operatorname{sh} x}{2\operatorname{ch}^4 x} =$

$= \operatorname{th} x - \frac{\operatorname{sh} x}{\operatorname{ch}^3 x} = \frac{\operatorname{sh} x}{\operatorname{ch} x} - \frac{\operatorname{sh} x}{\operatorname{ch}^3 x} = \frac{\operatorname{sh} x\cdot\operatorname{ch}^2 x - \operatorname{sh} x}{\operatorname{ch}^3 x} = \frac{\operatorname{sh} x(\operatorname{ch}^2 x - 1)}{\operatorname{ch}^3 x} =$

$= \operatorname{th}^3 x$

28.10.2020

a) $r = a\varphi$

$\sqrt{x^2+y^2} = a\cdot\operatorname{arctg}\frac{y}{x}$

$\frac{1}{2\sqrt{x^2+y^2}}\cdot(2x+2y\cdot y') = a\cdot\frac{1}{1+\frac{y^2}{x^2}}\cdot\frac{y'x-y}{x^2}$

$\frac{x+yy'}{\sqrt{x^2+y^2}} = a\frac{y'x-y}{x^2+y^2}$

$r(x+yy') = a(y'x-y)$

$rx+ryy' = ay'x - ay$

$y'(ry-ax) = -ay-rx)$

$y' = \frac{ay+rx}{ax-ry} = \frac{a\cdot\frac{y}{x}+r}{a-r\cdot\frac{y}{x}} = \frac{a\operatorname{tg}\varphi + a\varphi}{a-a\varphi\cdot\operatorname{tg}\varphi} = \frac{\operatorname{tg}\varphi+\varphi}{1-\varphi\operatorname{tg}\varphi} =$

$= \operatorname{tg}(\varphi + \operatorname{arctg}\varphi)$

Повторное дифференцирование.

1117.

$$y = x\sqrt{1+x^2}$$

$$y' = \sqrt{1+x^2} + \frac{x\cdot 2x}{2\sqrt{1+x^2}} \ ! = \sqrt{1+x^2} + \frac{2x^2}{\sqrt{1+x^2}} = \sqrt{1+x^2} + \frac{x^2}{\sqrt{1+x^2}}$$

$$= \frac{1+x^2+2x^2}{\sqrt{1+x^2}} = \frac{1+3x^2}{\sqrt{1+x^2}}$$

$$y'' = \frac{(1+3x^2)'\sqrt{1+x^2} - (\sqrt{1+x^2})'(1+3x^2)}{1+x^2} =$$

$$= \frac{6x\sqrt{1+x^2} - \frac{(1+3x^2)\cdot 2x}{2\sqrt{1+x^2}}}{1+x^2} = \frac{6x(1+x^2) - x(1+3x^2)}{(1+x^2)^{\frac{3}{2}}} =$$

$$= \frac{6x+6x^3-x-3x^3}{(1+x^2)^{\frac{3}{2}}} = \frac{3x^3+5x}{(1+x^2)^{\frac{3}{2}}} = x(3x$$

$$y'' = \frac{3x+2x^3}{\sqrt{1+x^2}(1+x^2)}$$

1118.

$$y = \ln f(x)$$

$$y' = \frac{f'(x)}{f(x)}$$

$$y'' = \frac{f''(x)\cdot f(x) - (f'(x))^2}{f^2(x)}$$

1120. $y = e^{\sin x}\cos(\sin x)$

$$y(0) = 1\cdot\cos(0) = 1.$$

$$y' = +e^{\sin x}\cos x\cdot\cos(\sin x) - \sin(\sin x)\cdot\cos x\cdot e^{\sin x} =$$

$$= e^{\sin x}\cos x\cdot(+\cos(\sin x) - \sin(\sin x)) =$$

$$\cancel{y'(0) = 1\cdot(+1) = +1}$$

$$= \underbrace{e^{\sin x}\cos(\sin x)}_{y(x)}\cdot\cos x\,(1-\operatorname{tg}(\sin x)) = y(x)\cdot\cos x\,(1-\operatorname{tg}(\sin x)) =$$

$$y'' = y'(x)\cdot\cos x\,(1-\operatorname{tg}(\sin x)) - y(x)\sin x(1-\operatorname{tg}(\sin x)) - \frac{y(x)\cos^2 x}{\cos^2(\sin x)}$$

$$y''(0) = 1 - 0 - 1 = 0$$

1122.
$$y = \ln\frac{u}{v} \quad ; \quad u = \varphi(x) \qquad v = \psi(x)$$

$$y = \ln|u| - \ln|v|$$

$$y' = \frac{1}{u}\cdot u' - \frac{1}{v}\cdot v' = \frac{u'}{u} - \frac{v'}{v}$$

$$y'' = \frac{u''u - (u')^2}{u^2} - \frac{v''v - (v')^2}{v^2}$$

Домашняя работа.

7.1. $y = \ln(\sin x)$

$$y' = \frac{\cos x}{\sin x} = \operatorname{ctg} x \rightarrow y'' = -\frac{1}{\ }$$

7.2. $y = (1+x^2)\operatorname{arctg} x$

$$y' = 2x \operatorname{arctg} x + \frac{1(1+x^2)}{1+x^2} = 2x\operatorname{arctg} x + 1.$$

$$y'' = 2\left(\operatorname{arctg} x + \frac{x}{1+x^2}\right)$$

7.3. $y = \cos^2 2x$

$$y' = -2\cos 2x \sin 2x \cdot 2 = -4\cos 2x \sin 2x = -2\cdot 2\sin 2x\cos 2x = -2\sin 4x$$

$$y'' = -8\cos 4x$$

7.4. $y = \operatorname{arctg} 2x$

$$y' = \frac{2}{1+4x^2} \rightarrow y'' = \frac{2'(1+4x^2) - (1+4x^2)'\cdot 2}{(1+4x^2)^2} = \frac{-2\cdot 8x}{(1+4x^2)^2} =$$

$$= \frac{-16x}{(1+4x^2)^2}$$

7.5 $y = \ln(1+x^2)$

$$y' = \frac{2x}{1+x^2} \rightarrow y'' = \frac{2(1+x^2) - 2x(1+x^2)'}{(1+x^2)^2} = \frac{2+2x^2-2x\cdot 2x}{(1+x^2)^2} =$$

$$= \frac{2-2x^2}{(1+x^2)^2} = \frac{2(1-x^2)}{(1+x^2)^2}$$

7.6. $y = \arccos 2x$

$$y' = \frac{-2}{\sqrt{1-4x^2}} \rightarrow y'' = -2(1-4x^2)^{-\frac{1}{2}'} = \frac{1\cdot(-8x)}{\sqrt{(1-4x^2)^3}} = -\frac{8x}{(1-4x^2)^{\frac{3}{2}}}$$

7.7. $y = \arcsin\frac{x}{2}$

$$y' = \frac{1\cdot\frac{1}{2}}{\sqrt{1-\frac{x^2}{4}}} = \frac{\sqrt{4}\cdot\frac{1}{2}}{\sqrt{4-x^2}} = \frac{2\cdot\frac{1}{2}}{\sqrt{4-x^2}}$$

$$y'' = \frac{-2\cdot\left(\left(4-x^2\right)^{\frac{1}{2}}\right)'\cdot\frac{1}{2}}{4-x^2} = \frac{-2\cdot\frac{1\cdot(-2x)}{2\sqrt{4-x^2}}\cdot\frac{1}{2}}{4-x^2} =$$

$$= \frac{2x\cdot\frac{1}{2}}{(4-x^2)^{\frac{3}{2}}} = \frac{x}{(4-x^2)^{\frac{3}{2}}}$$

7.8. $y = \sqrt{1+x^2}$

$$y' = \frac{2x}{2\sqrt{1+x^2}} = \frac{x}{\sqrt{1+x^2}}$$

$$y'' = \frac{\sqrt{1+x^2} - \frac{1\cdot 2x\cdot x}{2\sqrt{1+x^2}}}{1+x^2} = \frac{1+x^2-x^2}{(1+x^2)^{\frac{3}{2}}} = \frac{1}{(1+x^2)^{\frac{3}{2}}}$$

7.9. $y = e^{2x^2}$

$$y' = e^{2x^2}\cdot 4x$$

$$y'' = 4x\cdot e^{2x^2}\cdot 4x + 4\cdot e^{2x^2} = 16x^2\cdot e^{2x^2} + 4\cdot e^{2x^2} =$$

$$= 4e^{2x^2}(4x^2+1)$$

7.10. $y = e^{\cos 2x}$

$$y' = e^{\cos 2x}\cdot(-\sin 2x)\cdot 2 = -2\sin 2x\cdot e^{\cos 2x}$$

$$y'' = -2\left(2\cos 2x\cdot e^{\cos 2x} - 2\sin 2x\cdot e^{\cos 2x}\cdot\sin 2x\right) =$$

$$= 4e^{\cos 2x}\left(\sin^2 2x - \cos 2x\right)$$

29.10.2020

$$y = u^v = e^{\ln u^v}$$

$u = u(x)$

$v = v(x)$

$$y' = \left(e^{v\ln u}\right)' = e^{v\ln u}\left(v'\ln u + (\ln u)'v\right) = u^v\left(v'\ln u + \frac{v\cdot u'}{u}\right) =$$

$$= e^{\ln u^v}\left(v'\ln u + \frac{v\cdot u'}{u}\right)$$

$$y'' = \left(e^{\ln u^v}\left(v'\cdot\ln u + \frac{v\cdot u'}{u}\right)\right)' = \left(e^{\ln u^v}\right)'\cdot\left(v'\cdot\ln u + \frac{v\cdot u'}{u}\right) +$$

$+ \left(v' \ln u + \frac{v \cdot u'}{u}\right)' \cdot e^{\ln u^v} =$

$$= u^v \left(v' \ln u + \frac{v \cdot u'}{u}\right)^2 + u^v \left(v'' \ln u + \frac{v' \cdot u'}{u} + \frac{v' u - u' v}{u^2} \cdot u' + \frac{v}{u} \cdot u''\right) =$$

$$= u^v \left(v' \ln u + \frac{v \cdot u'}{u}\right)^2 + u^v \left(v'' \ln u + \frac{2v' \cdot u'}{u} + \frac{v(u'' u - (u')^2)}{u^2}\right) =$$

$$= u^v \left(\left(v' \ln u + \frac{v \cdot u'}{u}\right)^2 + v'' \ln u + \frac{2v' \cdot u'}{u} + v \cdot \frac{(u'' \cdot u - (u')^2)}{u^2}\right)$$

$y = f\left(\frac{1}{x}\right)$

$y = \frac{1}{x}$

$\cancel{y' = -\frac{1}{x^2}}$; $\cancel{y'' = -1\left(\frac{1' x^2 - (x^2)' 1}{x^4}\right) = -\frac{(-2x)}{x^4} = \frac{2}{x^3}}$

$\cancel{y''' = 2 \cdot \left(\frac{1}{x^3}\right)' = -2 \frac{1 \cdot 3x^2}{x^6} = -\frac{6}{x^4}}$

$y' = f'\left(\frac{1}{x}\right) \cdot \left(\frac{1}{x}\right)' = -\frac{1}{x^2} \cdot f'\left(\frac{1}{x}\right)$

$y'' = f''\left(\frac{1}{x}\right) \cdot \left(-\frac{1}{x^2}\right) \cdot \left(-\frac{1}{x^2}\right) + f'\left(\frac{1}{x}\right) \frac{2x}{x^4} =$

$= \frac{1}{x^4} f''\left(\frac{1}{x}\right) + \frac{2}{x^3} f'\left(\frac{1}{x}\right)$

$$y''' = -\frac{1 \cdot 4x^3}{x^8} f''\left(\frac{1}{x}\right) + \left(-\frac{1}{x^2}\right) \cdot f'''\left(\frac{1}{x}\right) \cdot \frac{1}{x^4} - \frac{6}{x^4} f'\left(\frac{1}{x}\right) - \frac{2}{x^3} \frac{1}{x^2} f''\left(\frac{1}{x}\right) =$$

$$= -\frac{1}{x^6} f'''\left(\frac{1}{x}\right) - \frac{6}{x^5} f''\left(\frac{1}{x}\right) - \frac{6}{x^4} f'\left(\frac{1}{x}\right)$$

Производные от параметрически заданных.

$y'_x = \frac{y'_t}{x'_t}$

$y''_{xx} = \frac{(y'_x)'_t}{x'_t}$; $y'''_{xxx} = \frac{(y''_{xx})'_t}{x'_t}$

1140. $x = 2t - t^2$; $y = 3t - t^3$

$x'_t = 2 - 2t$; $y'_t = 3 - 3t^2$

$y'_x = \frac{y'_t}{x'_t} = \frac{3(1-t^2)}{2(1-t)} = \frac{3(1-t)(1+t)}{2(1-t)} = \frac{3}{2}(1+t)$

$y''(t) = \frac{3}{2}$

$\Rightarrow y''_{xx} = \frac{3}{4(1-t)}$

$\Rightarrow y$

$y''_{xx} = \frac{(y'_x)'_t}{x'_t} = \frac{3}{4(1-t)}$

$y'''_{xxx} = \frac{(y''_{xx})'_t}{x'_t} = \frac{3}{8(1-t)^3}$

<u>1042.</u> $x = a(t - \sin t)$, $y = a(1 - \cos t)$

$x'_t = (at - a\sin t)' = a - a\cos t$

$y'_t = (a - a\cos t)' = a\sin t$

$y'_x = \frac{y'_t}{x'_t} = \frac{\cancel{a}\sin t}{\cancel{a}(1-\cos t)} = \frac{\cancel{2}\sin\frac{t}{2}\cos\frac{t}{2}}{\cancel{2}\sin^2\frac{t}{2}} = \frac{\cos\frac{t}{2}}{\sin\frac{t}{2}} = \operatorname{ctg}\frac{t}{2}$

$y''_{xx} = \frac{(y'_x)'_t}{x'_t} = -\frac{1}{2\sin^2\frac{t}{2}\cdot a\cdot 2\sin^2\frac{t}{2}} = -\frac{1}{4a\sin^4\frac{t}{2}}$

$\cancel{y'''_{xxx} = \frac{(y''_{xx})'_t}{x'_t} = -\frac{1}{4a}\cdot\sin^{-5}\frac{t}{2}\cdot\cos\frac{t}{2}\cdot\frac{1}{2}\cdot\frac{1}{2a\sin^2\frac{t}{2}} =}$

$\cancel{= \frac{\cos\frac{t}{2}}{a^2\sin^7\frac{t}{2}}}$

$(y''_{xx})'_t = -\frac{1}{4a}\left(\sin^{-4}\frac{t}{2}\right)' = \frac{-1\cdot(-4)}{4a}\cdot\sin^{-5}\frac{t}{2}\,\cos\frac{t}{2}\cdot\frac{1}{2} = -\frac{-\cos\frac{t}{2}\cdot(-4)}{8a\sin^5\frac{t}{2}}$

$x'_t = 2a\sin^2\frac{t}{2}$

$y'''_{xxx} = \frac{\cos\frac{t}{2}}{4a^2\sin^7\frac{t}{2}}$

Домашняя работа.

Пусть $u = u(x)$ и $v = v(x)$

7.11. $y = \sin(uv)$

$$y' = \cos(uv) \cdot (uv)' = \cos(uv) \cdot (u'v + v'u)$$

$$y'' = -\sin(uv) \cdot (u'v + v'u)^2 + \cos(uv)\left(u'v + v'u\right)' =$$

$$= -\sin(uv) \cdot (u'v + v'u)^2 + \cos(uv)\left(u''v + v'u' + v''u + v'u'\right) =$$

$$= \cos(uv)\left(u''v + v''u + 2v'u'\right) - \sin(uv)\left(u^2v + v'u\right)^2$$

7.12. $y = \sqrt{u^2 + v^2}$

$$y' = \frac{1 \cdot (2u \cdot u' + 2v \cdot v')}{2\sqrt{u^2 + v^2}} = \frac{u \cdot u' + v \cdot v'}{\sqrt{u^2 + v^2}}$$

$$y'' = \frac{(u \cdot u' + v \cdot v')'\sqrt{u^2 + v^2} - \left(\sqrt{u^2 + v^2}\right)'(u \cdot u' + v \cdot v')}{u^2 + v^2} =$$

$$= \frac{(u' \cdot u' + u \cdot u'' + v' \cdot v' + v \cdot v'')\sqrt{u^2 + v^2} - \frac{(u \cdot u' + v \cdot v')^2}{\sqrt{u^2 + v^2}}}{u^2 + v^2} =$$

$$= \frac{(u^2 + v^2)\left(u'^2 + v'^2 + u \cdot u'' + v \cdot v''\right) - (u \cdot u' + v \cdot v')^2}{(u^2 + v^2)^{\frac{3}{2}}} =$$

$$= \frac{(u^2 + v^2)(uu'' + vv'') + (u'v - uv')^2}{(u^2 + v^2)^{\frac{3}{2}}}$$

7.13 $\begin{cases} x = a\cos t \\ y = a\sin t \end{cases}$ $\Rightarrow$ $x'_t = -a\sin t$, $y'_t = a\cos t$

$$y'_x = -\frac{a\cos t}{a\sin t} = -\operatorname{ctg} t$$

$$(y'_{xx})'_t = +\frac{1}{\sin^2 t} \Rightarrow y''_{xx} = -\frac{1}{a\sin^3 t}$$

$$(y''_{xx})'_t = -\frac{1}{a}\sin^{-3} t = +3\frac{1}{a}\sin^{-4} t \cdot \cos t = \frac{3\cos t}{a\sin^4 t}$$

$$\Rightarrow y'''_{xxx} = -\frac{3\cos t}{a^2 \sin^5 t}$$

7.14 $\begin{cases} x = e^t\cos t \\ y = e^t\sin t \end{cases}$

$$x'_t = e^t\cos t - \sin t \cdot e^t = e^t(\cos t - \sin t)$$?

$$y'_t = e^t\sin t + \cos t \cdot e^t = e^t(\cos t + \sin t)$$

$$y'_x = \frac{\cos t + \sin t}{\cos t - \sin t} = \frac{1 + \operatorname{tg} t}{1 - \operatorname{tg} t}$$

$$(y'_x)'_t = \frac{(\cos t+\sin t)'(\cos t-\sin t) - (\cos t-\sin t)'(\cos t+\sin t)}{(\cos t-\sin t)^2} =$$

$$= \frac{(-\sin t+\cos t)^2 - (-\sin t-\cos t)(\cos t+\sin t)}{(\cos t-\sin t)^2} =$$

$$= \frac{(\cos t-\sin t)^2+(\cos t+\sin t)^2}{(\cos t-\sin t)^2}$$

$$y''_{xx} = \frac{(\cos t-\sin t)^2+(\cos t+\sin t)^2}{(\cos t-\sin t)^3\cdot e^t}$$

7.14 $\begin{cases} x = e^t\cos t \\ y = e^t \sin t \end{cases} \to \begin{array}{l} x'_t = e^t\cos t - \sin t\cdot e^t = e^t(\cos t-\sin t) \\ y'_t = e^t\sin t + \cos t\cdot e^t = e^t(\cos t+\sin t) \end{array}$

$$y'_x = \frac{y'_t}{x'_t} = \frac{\cos t+\sin t}{\cos t-\sin t} = \frac{\cos t\cdot \quad + \sin t\cdot \quad}{\cos t\cdot \quad} =$$

$$= \frac{\cos t\cdot\sin\frac{\pi}{4} + \sin t\cdot\cos\frac{\pi}{4}}{\cos t\cdot\cos\frac{\pi}{4} - \sin t\cdot\sin\frac{\pi}{4}} = \frac{\sin\left(t+\frac{\pi}{4}\right)}{\cos\left(t+\frac{\pi}{4}\right)} = \operatorname{tg}\left(t+\frac{\pi}{4}\right)$$

$$(y'_x)'_t = \frac{1}{\cos^2\left(t+\frac{\pi}{4}\right)}$$

$$(y''_{xx}) = \frac{1}{\cos^2\left(t+\frac{\pi}{4}\right)\cdot e^t(\cos t-\sin t)} = \frac{e^{-t}}{\sqrt{2}\cos^3\left(t+\frac{\pi}{4}\right)}$$

$$\frac{(y''_{xx})'_t}{x'_t} = \frac{(e^{-t})'\cdot\sqrt{2}\cos^3\left(t+\frac{\pi}{4}\right) - \sqrt{2}\left(\cos^3\left(t+\frac{\pi}{4}\right)\right)'\cdot e^{-t}}{2\cos^6\left(t+\frac{\pi}{4}\right)\cdot\sqrt{2}\cdot e^t\cos\left(t+\frac{\pi}{4}\right)} =$$

$$= \frac{-e^{-t}\cdot\cancel{\sqrt{2}}\cos^3\left(t+\frac{\pi}{4}\right) + \cancel{\sqrt{2}}\cdot 3\cos^2\left(t+\frac{\pi}{4}\right)\sin\left(t+\frac{\pi}{4}\right)\cdot e^{-t}}{2\cancel{\sqrt{2}}\cdot e^t\cos^7\left(t+\frac{\pi}{4}\right)} =$$

$$= \frac{e^{-t}\cdot\cos^2\left(t+\frac{\pi}{4}\right)\left(2\sin\left(t+\frac{\pi}{4}\right) - \cos\left(t+\frac{\pi}{4}\right)\right)}{2\cdot e^t\cos^7\left(t+\frac{\pi}{4}\right)} =$$

$$= \frac{e^{-2t}(2\sin t+\cos t)}{\sqrt{2}\cos^5\left(t+\frac{\pi}{4}\right)}$$

3.11.2020

Дифференциал

1) $f(x) = x^3 - 2x + 1$

$x_0 = 1.$

$$\Delta f(x_0) = f(1+\Delta x) - f(1) = (1+\Delta x)^3 - 2(1+\Delta x) + 1 - (1-2+1) =$$

$$= 1 + 3\Delta x + 3(\Delta x)^2 + (\Delta x)^3 - 2 - 2\Delta x + 1 = (\Delta x)^3 + 3(\Delta x)^2 + \Delta x$$

$df(1) = (3x^2-2)\cdot \Delta x \,\Big|_{x=1} = \Delta x$

1) $\Delta x = 1.$

$\Delta f(1) = 5$ $\qquad df(1) = 1.$

2) $\Delta x = 0,1$

$\Delta f(1) = 0,131$ $\qquad df(1) = 0,1.$

3) $\Delta x = 0,01$

$\Delta f(1) = 0,01 + 3\cdot 0,0001 + 0,000001 =$ $\qquad df(1) = 0,01.$

$= 0,010301$

(2) ~~$y = arctg(2tgx)$~~

$y = \ln|x + \sqrt{x^2+a^2}|$

$dy = \frac{1}{x+\sqrt{x^2+a^2}}\cdot\left(1 + \frac{1\cdot \cancel{2}x}{\cancel{2}\sqrt{x^2+a^2}}\right)\cdot dx = \frac{\cancel{\sqrt{x^2+a^2}+x}}{\cancel{x+\sqrt{x^2+a^2}}\cdot\sqrt{x^2+a^2}}\cdot dx = dx\cdot\frac{1}{\sqrt{x^2+a^2}}$

(4) $y = \frac{1}{a}\,arctg\left(\frac{x}{a}\right)$

$dy = \frac{1}{a}\cdot\frac{1}{1+\frac{x^2}{a^2}}\cdot\frac{1}{a}\cdot dx = \frac{1\cdot a^2}{a^2\cdot(a^2+x^2)}\cdot dx = \frac{1\cdot dx}{a^2+x^2}$

(5) $y = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right|$ $\quad$ или $\quad y = \frac{1}{2a}\left(\ln|x-a| - \ln|x+a|\right)$

$dy = \frac{1}{2a}\cdot\frac{(x+a)}{(x-a)}\cdot\frac{(x-a)'(x+a)-(x+a)'(x-a)}{(x+a)^2}\,dx =$

$= \frac{(x+a)(-a(x+a)-a(x-a))}{2a(x-a)(x+a)^2}\,dx = \frac{(x+a)(-ax-\cancel{a^2}-ax+\cancel{a^2})}{2a(x-a)(x+a)^2}\,dx =$

$= \frac{-\cancel{2a}x(x+a)}{\cancel{2a}(x-a)(x+a)^2}\cdot dx = \frac{-1}{x^2-a^2}\cdot dx$

(6) $d(\sqrt{a^2+t^2}) = \frac{\cancel{2}t}{\cancel{2}\sqrt{a^2+t^2}}\cdot dt = \frac{t}{\sqrt{a^2+t^2}}\cdot dt$

(7) $d(\sin\varphi - \varphi\cos\varphi) = \left(\cos\varphi - (\cos\varphi - \varphi\sin\varphi)\right)\cdot d\varphi = \varphi\sin\varphi\cdot d\varphi$

(8) $y = \frac{1}{\sqrt{(u^2+v^2)^3}}$

$dy = \frac{-1}{2\sqrt[3]{(u^2+v^2)^3}}\cdot d(u^2+v^2) = -\frac{1}{2}\frac{2u\,du + 2v\,dv}{\sqrt[2]{(u^2+v^2)^3}} = \frac{u\,du + v\,dv}{\sqrt{(u^2+v^2)^3}}$

(9) $y = \operatorname{arctg}\frac{u}{v}$

$$dy = \frac{1}{1+\frac{u^2}{v^2}} \cdot d\left(\frac{u}{v}\right) = \frac{v^2}{v^2+u^2} \cdot \frac{du \cdot v - dv \cdot u}{v^2} = \frac{v\,du - u\,dv}{v^2+u^2}$$

(10) $$\frac{d}{d(x^3)}(x^3 - 2x^6 - x^9) - \frac{d}{d(x^3)}\left(x^3 - 2(x^3)^2 - (x^3)^3\right) =$$

$$= 1 - 4x^3 - 3x^6 \quad \text{или} \quad \frac{(3x^2 - 12x^5 - 9x^8)\,dx}{3x^2\,dx} = \ldots$$

$x^3 = t$

$$\frac{d}{dt}(t - 2t^2 - t^3) = 1 - 4t - 3t^2 \Rightarrow \frac{d}{d(x^3)} = 1 - 4x^3 - 3x^6$$

(11) $$\frac{d(\sin x)}{d(\cos x)} = \frac{\cos x \cdot dx}{-\sin x \cdot dx} = -\operatorname{ctg} x$$

(12) $\sqrt[3]{1,02} \rightarrow f(x) = \sqrt[3]{x}$; $x_0 = 1$.

$$f'(x) = \frac{1}{3\sqrt[3]{x^2}}$$

$$\begin{array}{r|l} -2000 & 300 \\ 1800 & 0,0066 \end{array}$$

$\Delta x = 1,02 - 1 = 0,02$

$$f(1,02) \approx f(1) + f'(1) \cdot 0,02 = 1 + \frac{0,02}{3} \approx 1 + 0,0066 \approx 1,0067$$

(13) $\sin 151^\circ$

$f(x) = \sin x$ $\qquad x_0 = 150^\circ = \frac{5}{6}\pi$ рад.

$\Delta x = 1^\circ = \frac{\pi}{180}$ рад.

$$\sin 151^\circ \approx \sin\frac{5}{6}\pi + \cos\frac{5}{6}\pi \cdot \frac{\pi}{180} = 0,5 - \frac{\sqrt{3}}{2} \cdot \frac{\pi}{180} \approx 0,48$$

Домашняя работа.

6.1. $y = \sin^3 2x$

$$dy = 3\sin^2 2x \cdot 2\cos 2x \cdot dx = 6\sin^2 2x \cdot \cos 2x \cdot dx = 3\sin 2x \cdot \sin 4x \cdot dx$$

6.2 $y = \ln(\sin\sqrt{x})$

$$dy = \frac{1 \cdot \cos\sqrt{x}}{\sin\sqrt{x}} \cdot \frac{1}{2\sqrt{x}} \cdot dx = \frac{\operatorname{ctg}\sqrt{x}}{2\sqrt{x}} \cdot dx$$

6.3 $y = e^{-\frac{1}{\cos x}}$

$$dy = e^{-\frac{1}{\cos x}} \cdot \frac{1\cdot(-\sin x)}{\cos^2 x}\cancel{dx} = -\frac{tg\, x}{\cos x} \cdot e^{-\frac{1}{\cos x}} \cdot dx$$

6.4 $y = 2^{-x^2}$

$$dy = 2^{-x^2} \ln 2 \cdot -2x \cdot dx = -2x \cdot 2^{-x^2} \ln 2 \cdot dx$$

6.5 $y = x \ln x$

$$dy = (\ln x + 1)\, dx$$

6.6. $y = \arcsin \sqrt{x}$

$$dy = \frac{1}{\sqrt{1-x}} \cdot \frac{1}{2\sqrt{x}} \cdot dx = \frac{dx}{2\sqrt{x(1-x)}}$$

6.7 $y = x^3 + x\sqrt{x}$

$$dy = \left(3x^2 + \left(\sqrt{x} + \frac{x}{2\sqrt{x}}\right)\right) \cdot dx = \left(3x^2 + \frac{3}{2}\sqrt{x}\right) dx$$

6.8. $y = arctg \sqrt{x^2+1}$

$$dy = \frac{1}{(2+x^2)} \cdot \frac{1 \cdot 2x}{2\sqrt{x^2+1}} \cdot dx = \frac{x \cdot dx}{(2+x^2)\sqrt{x^2+1}}$$

6.9 $$d(t^2 \sin\sqrt{t}) = \left(2t \sin\sqrt{t} + t^2 \cdot \cos\sqrt{t} \cdot \frac{1}{2\sqrt{t}}\right) dt =$$

$$= t\left(2\sin\sqrt{t} + \frac{t}{2\sqrt{t}} \cdot \cos\sqrt{t}\right) dt = \frac{t}{2}\left(4\sin\sqrt{t} + \sqrt{t} \cdot \cos\sqrt{t}\right) dt$$

6.10 $$d\left(\frac{u+1}{\sqrt{u+1}}\right) = \frac{(u+1)'\sqrt{u+1} - (\sqrt{u+1})'(u+1)}{u+1} \cdot du =$$

$$= \frac{\sqrt{u+1} - \frac{1}{\sqrt{u+1}}(u+1)}{u+1} \cdot du = \qquad u+1-u$$

$$d\left(\frac{u+1}{\sqrt{u+1}}\right) = d\left((u+1)^{\frac{1}{2}}\right) = \frac{1 \cdot du}{2\sqrt{u+1}} \qquad ?$$

6.11 $d\left(\frac{\cos\varphi}{1-\sin\varphi}\right) = \frac{\cos'\varphi(1-\sin\varphi)-(1-\sin\varphi)'\cos\varphi}{(1-\sin\varphi)^2}\cdot d\varphi =$

$= \frac{-\sin\varphi(1-\sin\varphi)+\cos^2\varphi}{(1-\sin\varphi)^2}\cdot d\varphi =$

$= \frac{-\sin\varphi+\sin^2\varphi+\cos^2\varphi}{(1-\sin\varphi)^2} = \frac{1\cdot d\varphi}{1-\sin\varphi}$

6.12 $d\left(\frac{s^2}{\arcsin s}\right) = \frac{2s\arcsin(s)-\frac{1\cdot s^2}{\sqrt{1-s^2}}}{\arcsin^2 s}\cdot ds =$

$= \frac{s}{\arcsin^2 s}\left(2\arcsin(s)-\frac{s}{\sqrt{1-s^2}}\right)\cdot ds$

6.13 $y = \frac{u}{v^2}$

$dy = \frac{du\,v^2 - 2v\,dv\cdot u}{v^4} = \frac{v\,du-2u\,dv}{v^3}$

6.14 $y = \ln\sqrt{u^2+v^2}$

$dy = \frac{1}{\sqrt{u^2+v^2}}\cdot\frac{1\cdot 2u\,du+2v\,dv}{2\sqrt{u^2+v^2}} = \frac{u\,du+v\,dv}{u^2+v^2}$

5.11.2020

Правило Лопиталя

Действует только для неопределённостей $\left(\frac{0}{0}\right)$ и $\left(\frac{\infty}{\infty}\right)$

Правило: 1) Пусть $f(x)$ и $g(x)$ - непрерывные ф-и в окрестности т. a, где $a \in \overline{\mathbb{R}}$

2) $\exists.\ f'(x),\ g'(x)$ в окрестности точки a, причём $g'(x) \neq 0$

3) $\lim\limits_{x\to a} f(x) = \lim\limits_{x\to a} g(x) = 0$ (или ∞)

4) $\lim\limits_{x\to a}\frac{f'(x)}{g'(x)} = A$ - существует

Тогда, $\lim\limits_{x\to a}\frac{f(x)}{g(x)} = \lim\limits_{x\to a}\frac{f'(x)}{g'(x)}$

Аналогично раскрываются неопр-и вида $\frac{\infty}{\infty}$

① $\lim\limits_{x\to 0} \dfrac{\sin ax}{\sin bx} = \left(\dfrac{0}{0}\right) = \lim\limits_{x\to 0} \dfrac{a\cos ax}{b\cos bx} = \dfrac{a}{b}$

② $\lim\limits_{x\to 0} \dfrac{\operatorname{ch}x - \cos x}{x^2} = \left(\dfrac{0}{0}\right) = \lim\limits_{x\to 0} \dfrac{\operatorname{sh}x + \sin x}{2x} = \left(\dfrac{0}{0}\right) = \lim\limits_{x\to 0} \dfrac{\operatorname{ch}x + \cos x}{2} = 1$

③ $\lim\limits_{x\to 0} \dfrac{3\operatorname{tg}4x - 12\operatorname{tg}x}{3\sin 4x - 12\sin x} = \left(\dfrac{0}{0}\right) = \lim\limits_{x\to 0} \dfrac{\dfrac{3\cdot 4}{\cos^2 4x} - \dfrac{12}{\cos^2 x}}{12\cos 4x - 12\cos x} =$

$= \lim\limits_{x\to 0} \dfrac{\dfrac{\cos^2 x - \cos^2 4x}{\cos^2 4x \cdot \cos^2 x}}{\cos 4x - \cos x} = \lim\limits_{x\to 0} \dfrac{\cos^2 x - \cos^2 4x}{\cos^2 4x \cdot \cos^2 x\,(\cos 4x - \cos x)} =$

$= -2$

④ $\lim\limits_{x\to \frac{\pi}{2}} \dfrac{\operatorname{tg}3x}{\operatorname{tg}x} = \left(\dfrac{\infty}{\infty}\right) = \lim\limits_{x\to \frac{\pi}{2}} \dfrac{\dfrac{3}{\cos^2 3x}}{\dfrac{1}{\cos^2 x}} = 3\lim\limits_{x\to \frac{\pi}{2}} \dfrac{\cos^2 x}{\cos^2 3x} = \left(\dfrac{0}{0}\right) =$

$= \cancel{3}\lim\limits_{x\to \frac{\pi}{2}} \dfrac{-2\cos x \sin x}{-2\cos 3x \cdot \sin 3x \cdot \cancel{3}} = \lim\limits_{x\to \frac{\pi}{2}} \dfrac{\sin 2x}{\sin 6x} = \left(\dfrac{0}{0}\right) =$

$= \lim\limits_{x\to \frac{\pi}{2}} \dfrac{2\cos 2x}{6\cos 6x} = \dfrac{-2}{-6} = \dfrac{1}{3}$

$\lim\limits_{x\to \frac{\pi}{2}} \dfrac{\operatorname{tg}3x}{\operatorname{tg}x} = \left(\dfrac{\infty}{\infty}\right) = \lim\limits_{x\to \frac{\pi}{2}} \dfrac{\operatorname{ctg}x}{\operatorname{ctg}3x} = \left(\dfrac{0}{0}\right) =$

$= \lim\limits_{x\to \frac{\pi}{2}} \dfrac{+\dfrac{1}{\sin^2 x}}{+\dfrac{1}{\sin^2 3x}\cdot 3} = \dfrac{1}{3}$

⑤ $\lim\limits_{x\to 0} \dfrac{x\operatorname{ctg}x - 1}{x^2} = \lim\limits_{x\to 0} \dfrac{\dfrac{x\cos x}{\sin x} - 1}{x^2} = \lim\limits_{x\to 0} \dfrac{x\cos x - \sin x}{\sin x \cdot x^2} = \left(\dfrac{0}{0}\right) =$

$= \lim\limits_{x\to 0} \dfrac{\cancel{\cos x} - x\sin x - \cancel{\cos x}}{2x\sin x + x^2\cos x} = \lim\limits_{x\to 0} \dfrac{-x\sin x}{x(2\sin x + x\cos x)} = \lim\limits_{x\to 0} \dfrac{-\sin x}{2\sin x + x\cos x} =$

$= \left(\dfrac{0}{0}\right) = -\lim\limits_{x\to 0} \dfrac{\sin x}{2\sin x + x\cos x} = -\lim\limits_{x\to 0} \dfrac{\cos x}{2\cos x + \cos x - \sin x \cdot x} =$

$= -\lim\limits_{x\to 0} \dfrac{\cos x}{3\cos x - x\cdot\sin x} = -\dfrac{1}{3}$

⑥ $\lim\limits_{x\to 0} \dfrac{1 - \cos x^2}{x^2 \sin x^2} = \left(\dfrac{0}{0}\right) = \left\{\begin{matrix}\text{замена} \\ x^2 = t\end{matrix}\right\} =$

$= \lim\limits_{t\to 0} \dfrac{1 - \cos t}{t\sin t} = \left(\dfrac{0}{0}\right) = \lim\limits_{t\to 0} \dfrac{\sin t}{\sin t + t\cdot\cos t} = \left(\dfrac{0}{0}\right) = \cancel{\lim\limits_{t\to 0} \dfrac{+\cos t}{\cos t + \cos t - t\sin t}}$

$= \cancel{\lim\limits_{t\to 0} \dfrac{\cos t}{t\sin t}} = \lim\limits_{t\to 0} \dfrac{\cos t}{\cos t + \cos t - t\sin t} = \dfrac{1}{2}$

(7) $\lim\limits_{x\to1}\dfrac{x^x - x}{\ln x - x + 1} = \left(\dfrac{0}{0}\right) = \lim\limits_{x\to1}\dfrac{x^x(\ln x+1)-1}{\frac{1}{x}-1} = \left(\dfrac{0}{0}\right) =$

$= \lim\limits_{x\to1}\dfrac{(x^x)'(\ln x+1)+x^x\cdot\frac{1}{x}}{-\frac{1}{x^2}} =$

$\left\{ \begin{array}{l} x^x = e^{\ln x^x} = e^{x\ln x} \\ \Rightarrow (x^x)' = e^{\ln x^x}\cdot\left(\ln x + \frac{x}{x}\right) = x^x(\ln x+1) \end{array} \right\}$

$= \lim\limits_{x\to1}\dfrac{x^x(\ln x+1)^2+x^x\cdot\frac{1}{x}}{-\frac{1}{x^2}} = \dfrac{1+1}{-1} = -2$

(8) $\lim\limits_{x\to0}\dfrac{\ln(\sin ax)}{\ln(\sin bx)} = \left(\dfrac{\infty}{\infty}\right) = \lim\limits_{x\to0}\dfrac{\frac{1\cdot a\cdot\cos ax}{\sin ax}}{\frac{1\cdot b\cdot\cos bx}{\sin bx}} =$

$= \lim\limits_{x\to0}\dfrac{a\cdot\cos ax\cdot\sin bx}{\sin ax\cdot b\cdot\cos bx} = \lim\limits_{x\to0}\dfrac{a\,ctg\,ax}{b\,ctg\,bx} = \left(\dfrac{\infty}{\infty}\right) = \lim\limits_{x\to0}\dfrac{a\,tg\,bx}{b\,tg\,ax} =$

$= \left(\dfrac{0}{0}\right) = \dfrac{a}{b}\lim\limits_{x\to0}\dfrac{b\cdot\cos^2 ax}{\cos^2 bx\cdot a} = \dfrac{a\cdot b}{b\cdot a} = 1$

(9) $\lim\limits_{x\to+\infty}\dfrac{x^n}{e^{ax}}\ \Big|\ (a>0, n>0)\Big| = \left(\dfrac{\infty}{\infty}\right) = \lim\limits_{x\to+\infty}\dfrac{n!}{a^n\cdot e^{ax}} = 0$

Если n раз применить правило Лопиталя при $n>0$ и $a>0$

(10) $\lim\limits_{x\to1-0}(\ln x\cdot\ln(1-x)) = (0\cdot\infty) =$

$= \lim\limits_{x\to1-0}\dfrac{\ln(1-x)}{\frac{1}{\ln x}} = \left(\dfrac{\infty}{\infty}\right) =$

$= \lim\limits_{x\to1-0}\dfrac{\frac{1}{1-x}\cdot(-1)}{-\frac{1}{\ln^2 x}\cdot\frac{1}{x}} = \lim\limits_{x\to1-0}\dfrac{x\cdot\ln^2 x}{1-x} = \left(\dfrac{0}{0}\right) = \lim\limits_{x\to1-0}\dfrac{\ln^2 x + \ldots\ln x}{\ldots -1} =$

$= 0$

(11) $\lim\limits_{x\to\frac{\pi}{4}}(tg\,x)^{tg\,2x} = (1^\infty) = \lim\limits_{x\to\frac{\pi}{4}} e^{tg\,2x\,\ln(tg\,x)} = e^{\lim\limits_{x\to\frac{\pi}{4}}(tg\,2x)\ln(tg\,x)} =$

$\lim\limits_{x\to\frac{\pi}{4}} tg\,2x\cdot\ln(tg\,x) = (\infty\cdot0) = \lim\limits_{x\to\frac{\pi}{4}}\dfrac{\ln(tg\,x)}{ctg\,2x} = \left(\dfrac{0}{0}\right) =$?

$= \lim\limits_{x\to\frac{\pi}{4}}\dfrac{\frac{1}{tg\,x\cdot\cos^2 x}}{-\frac{2}{\sin^2 2x}} = -\lim\limits_{x\to\frac{\pi}{4}}\dfrac{\sin^2 2x\cdot\cancel{\cos x}}{2\,\cancel{tg\,x}\,\sin x\cdot\cancel{\cos^2 x}} = \cancel{-\dfrac{1}{\sqrt{2}}} = \cancel{-\dfrac{\sqrt{2}}{2}}$

$= \lim\limits_{x\to\frac{\pi}{4}}\sin 2x$

$$\cancel{= e^{-\frac{1}{\sqrt{2}}} = \frac{1}{e^{\frac{1}{\sqrt{2}}}}}$$

$$= \frac{1}{e}$$

12) $$\lim_{x\to+0}\left(\ln\frac{1}{x}\right)^x = \lim_{x\to+0} e^{x\ln(-\ln x)} = e^{\lim\limits_{x\to+0} x\ln(-\ln x)} =$$

$$\lim_{x\to+0} x\ln(-\ln x) = (0\cdot\infty) = \lim_{x\to+0}\frac{\ln(-\ln x)}{\frac{1}{x}} = \left(\frac{\infty}{\infty}\right) =$$

$$= \lim_{x\to+0}\frac{\frac{1}{-\ln x}\cdot\left(-\frac{1}{x}\right)}{-\frac{1}{x^2}} = -\lim_{x\to+0}\frac{x^2}{\ln x\cdot x} = -\lim_{x\to+0}\frac{x}{\ln x} = 0$$

$$= e^0 = 1.$$

~~9.1~~ | ~~9.2~~ | ~~9.3~~ | ~~9.4~~ | 9.5 | ~~9.6~~ | ~~9.7~~ | ~~9.8~~ | ~~9.9~~ | ~~9.10~~

Домашняя работа.

9.1 $$\lim_{x\to0}\frac{\operatorname{tg}x - x}{x-\sin x} = \left(\frac{0}{0}\right) = \lim_{x\to0}\frac{\frac{1}{\cos^2 x} - 1}{1-\cos x} = \lim_{x\to0}\frac{(1-\cos x)(1+\cos x)}{\cos^2 x\,(1-\cos x)} = 2$$

9.2 $$\lim_{x\to0}\frac{x(e^x+1) - 2(e^x-1)}{x^3} = \left(\frac{0}{0}\right) = \lim_{x\to0}\frac{x\cdot e^x + x - 2\cdot e^x + 2}{x^3} =$$

$$= \lim_{x\to0}\frac{e^x + x\cdot e^x + 1 - 2\cdot e^x}{3x^2} = \lim_{x\to0}\frac{x\cdot e^x + 1 - e^x}{3x^2} = \lim_{x\to0}\frac{\cancel{e^x} + x\cdot e^x - \cancel{e^x}}{6x} = \frac{1}{6}$$

9.3 $$\lim_{x\to0}\frac{\ln(\cos ax)}{\ln(\cos bx)} = \left(\frac{0}{0}\right) = \lim_{x\to0}\frac{\frac{+\sin ax\cdot a}{\cos ax}}{+\frac{1\sin bx}{\cos bx}\cdot b} = \lim_{x\to0}\frac{\sin ax\cdot a\cdot\cos bx}{\cos ax\cdot\sin bx\cdot b} =$$

$$= \lim_{x\to0}\frac{a\cdot\operatorname{ctg}bx}{b\operatorname{ctg}ax} = \lim_{x\to0}\frac{a\operatorname{tg}ax}{b\operatorname{tg}bx} = \frac{a}{b}\lim_{x\to0}\frac{a\cdot\cos^2 bx}{\cos^2 ax\cdot b} = \left(\frac{a}{b}\right)^2$$

9.4 $$\lim_{x\to0}\frac{\cos(\sin x) - \cos x}{x^4} = \left(\frac{0}{0}\right) = \lim_{x\to0}\frac{-\sin(\sin x)\cdot\cos x + \sin x}{4x^3} =$$

$$= -\lim_{x\to0}\frac{\cos(\sin x)\cdot\cos^2 x - \sin x\cdot\sin(\sin x) + \cos x}{12x^2} =$$

$$= \lim_{x\to0}\left(\frac{\cos(\sin x)\cdot\cos^4(x) - \sin(\sin x)\cdot\cos^2 x\cdot\sin x - \sin(\sin x)\cdot\cos x\cdot\sin 2x + 2\cos(\sin x)\cos 2x - }{24}\right.$$

$$\left.\frac{-\cos(\sin x)\cdot\sin^2 x + 2\cos(\sin x)\cos^2 x - \sin(\sin x)\sin x - \cos x}{}\right) = \frac{-0-0+2-0+2}{24} =$$

$$= \frac{4}{24} = \frac{1}{6}$$

9.5. $\cancel{\lim\limits_{x\to+\infty} x^2 e^{-0{,}01x} = \lim\limits_{x\to+\infty} \frac{x^2\cdot e^{-0{,}01}\cdot x^2}{x} = \left(\frac{\infty}{\infty}\right) =}$

$= \lim\limits_{x\to+\infty} \frac{x^4\cdot e^{-0{,}01}}{x} \cancel{= \lim\limits_{x\to+\infty} \frac{e^{-0{,}01}\cdot 4x^3}{1}} = \lim\limits_{x\to+\infty} \frac{4x^3}{e^{0{,}01}} = +\infty$

9.6 $\lim\limits_{x\to+0} x^{\varepsilon}\ln x = (0\cdot\infty) = \lim\limits_{x\to+0} \ln x^{x^{\varepsilon}} = \lim\limits_{x\to+0} e^{\ln(\ln x^{x^{\varepsilon}})} =$

$= \lim\limits_{x\to+0} \frac{\ln x}{\frac{1}{x^{\varepsilon}}} = \lim\limits_{x\to+0} \frac{\frac{1}{x}}{-\frac{1}{x^{2\varepsilon}}\cdot\varepsilon\cdot x^{\varepsilon-1}} = -\lim\limits_{x\to0} \frac{x^{2\varepsilon}}{x\cdot x^{\varepsilon-1}\cdot\varepsilon} =$

$= -\lim\limits_{x\to+0} \frac{x^{2\varepsilon}}{x^{\varepsilon}\cdot\varepsilon} = -\lim\limits_{x\to+0} \frac{x^{\varepsilon}}{\varepsilon} = 0$

9.7 $\lim\limits_{x\to1} x^{\frac{1}{1-x}} = \lim\limits_{x\to1} e^{\ln x^{\frac{1}{1-x}}} = e^{\lim\limits_{x\to1}\left(\frac{1}{1-x}\ln x\right)} \circled{=}$

$\lim\limits_{x\to1} \frac{\ln x}{1-x} = \left(\frac{0}{0}\right) = \lim\limits_{x\to1} \frac{\frac{1}{x}}{-1} = -1$

$= e^{-1}$

9.8. $\lim\limits_{x\to+\infty} \sqrt[x]{x} = \lim\limits_{x\to+\infty} x^{\frac{1}{x}} = \lim\limits_{x\to+\infty} e^{\ln x^{\frac{1}{x}}} = e^{\lim\limits_{x\to+\infty}\frac{\ln x}{x}} = e^0 = 1$

$\lim\limits_{x\to+\infty} \frac{\ln x}{x} = \left(\frac{\infty}{\infty}\right) = \lim\limits_{x\to+\infty} \frac{1}{x} = 0$

9.9. $\lim\limits_{x\to1} (2-x)^{\operatorname{tg}\frac{\pi x}{2}} = \lim\limits_{x\to1} e^{\operatorname{tg}\frac{\pi x}{2}\ln(2-x)} = e^{\lim\limits_{x\to1}\left(\operatorname{tg}\frac{\pi x}{2}\ln(2-x)\right)} =$

$= e^{\frac{2}{\pi}}$

$\lim\limits_{x\to1} \frac{\ln(2-x)}{\operatorname{ctg}\frac{\pi x}{2}} = \left(\frac{0}{0}\right) = \lim\limits_{x\to1} \frac{\frac{-1}{2-x}}{-\frac{\pi}{\sin^2\frac{\pi x}{2}\cdot 2}} = \lim\limits_{x\to1} \frac{\sin^2\frac{\pi x}{2}\cdot 2}{\pi(2-x)} = \frac{2}{\pi}$

9.10. $\lim\limits_{x\to0} (\operatorname{ctg} x)^{\sin x} = \lim\limits_{x\to0} e^{\ln(\operatorname{ctg} x)^{\sin x}} = e^{\lim\limits_{x\to0}(\sin x\ln(\operatorname{ctg} x))} =$

$= e^0 = 1$

$\lim\limits_{x\to0} \sin x\cdot\ln(\operatorname{ctg} x) = \lim\limits_{x\to0} \frac{\ln(\operatorname{ctg} x)}{\frac{1}{\sin x}} = \lim\limits_{x\to0} \frac{\frac{1}{\operatorname{ctg} x}\cdot\cancel{\left(-\frac{1}{\sin^2 x}\right)}}{\cancel{\left(-\frac{1}{\sin^2 x}\right)}\cdot\cos x} =$

$= \lim\limits_{x\to0} \frac{1\cdot\sin x}{\cos^2 x} = 0$

10.11.2020

1) $d^2 y$, $y=\sqrt{1+x^2}$

$dy = y' \cdot dx = \frac{2x \cdot dx}{2\sqrt{1+x^2}} = \frac{x \cdot dx}{\sqrt{1+x^2}}$

$y'' = \left(\frac{x}{\sqrt{1+x^2}}\right)' = \frac{\sqrt{1+x^2} - \frac{2x \cdot x}{2\sqrt{1+x^2}}}{1+x^2} = \frac{1+x^2-x^2}{(1+x^2)^{\frac{3}{2}}} = \frac{1}{(1+x^2)^{\frac{3}{2}}}$

$d^2 y = \left(\frac{1}{(1+x^2)^{\frac{3}{2}}}\right) \cdot dx^2$

1130 (а, б) $y=e^x$

а) $y' = e^x$; $y'' = e^x$

$d^2 y = e^x dx^2$

б) x - промежуточный аргумент. $\rightarrow y = e^x \quad x(t)$

$d^2 y = e^x \cdot dx^2 + e^x \cdot \underbrace{x''_t \cdot dt^2}_{d^2 x} = e^x(dx^2 + d^2 x)$

$dy = y' \cdot dx$

$dx = x'(t) \cdot dt$

$\cancel{d^2 y = y'' \cdot dx^2}$

$\cancel{d^2 y = e^x \cdot (x'(t) \cdot dt)^2 =}$

$= d^2 y = e^x \cdot dx^2$

$dy = e^x \cdot dx$

$d^2 y = d(e^x \cdot dx) =$

$= dy \cdot d^2 x^2 + e^x \cdot dx^2 =$

$= e^x (dx^2 + dx^2)$

1134. $y = u \cdot v$

$y' = u'v + uv' =$

$dy = du \cdot v + dv \cdot u$

$d^2 y = d^2 u \cdot v + du \cdot dv + d^2 v \cdot u + du \cdot dv = d^2 u \cdot v + 2 du \cdot dv + d^2 v \cdot u$

• $y = tg\,\frac{u}{v}$

$dy = \left(tg\,\frac{u}{v}\right)' \cdot d\left(\frac{u}{v}\right) = \frac{1}{\cos^2\left(\frac{u}{v}\right)} \cdot \left(\frac{u}{v}\right)' \cdot d\left(\frac{u}{v}\right) = \frac{u'v - v' \cdot u}{v^2 \cos^2\left(\frac{u}{v}\right)} \cdot d\left(\frac{u}{v}\right)$

$\frac{u}{v} = t$; $y = tg\,t$

$dy = \frac{t'}{\cos^2 t} \cdot dt$

$d^2 y = \left(\frac{t'}{\cos^2 t}\right)' \cdot dt^2 = \frac{t'' \cos^2 t + 2\cos t \cdot \sin t \cdot t' \cdot t'}{\cos^4 t} \cdot dt^2 =$

$= \frac{\left(\frac{u}{v}\right)'' \cos^2\left(\frac{u}{v}\right) + 2\cos\left(\frac{u}{v}\right) \sin\left(\frac{u}{v}\right) \cdot \left(\frac{u}{v}\right)'}{\cos^4\left(\frac{u}{v}\right)} \cdot d\left(\frac{u}{v}\right)^2$

$dy = \frac{1}{\cos^2\left(\frac{u}{v}\right)} \cdot d\left(\frac{u}{v}\right) = \frac{1}{\cos^2\left(\frac{u}{v}\right)} \cdot \frac{v \cdot du - u \cdot dv}{v^2}$

$d^2 y = d\left(\frac{1}{\cos^2\left(\frac{u}{v}\right)} \cdot \frac{v\,du - u \cdot dv}{v^2}\right) = d\left(\frac{1}{\cos^2 \frac{u}{v}}\right) \cdot \frac{v \cdot du - u \cdot dv}{v^2} + d\left(\frac{v\,du - u\,dv}{v^2}\right) \frac{1}{\cos^2 \frac{u}{v}}$

$$d\left(\frac{1}{\cos^2\frac{u}{v}}\right) = +\frac{2}{\cos^3\frac{u}{v}}\cdot\sin\left(\frac{u}{v}\right)\cdot d\left(\frac{u}{v}\right) = \frac{2\sin\frac{u}{v}}{\cos^3\frac{u}{v}}\cdot\left(\frac{v\,du - u\,dv}{v^2}\right)$$

$$d\left(\frac{v\cdot du - u\cdot dv}{v^2}\right) = \frac{d(v\cdot du - u\cdot dv)v^2 - d(v\cdot v)\cdot(v\,du - u\cdot dv)}{v^4} =$$

$$= \frac{dv\,du + v\,d^2u - dv\,du - u\,d^2v - 2v\,dv}{v^4} = \frac{v\,d^2u - 2v\,dv - u\,d^2v}{v^4}$$

$$= \frac{(\cancel{dv\,du} + v\,d^2u - \cancel{dv\,du} - u\,d^2v)v^2 - 2v\,dv(v\,du - u\cdot dv)}{v^4} =$$

$$= \frac{v(v\,d^2u - u\,d^2v) - 2dv(v\,du - u\cdot dv)}{v^3} =$$

$$d^2y = \frac{2\sin\frac{u}{v}}{\cos^3\frac{u}{v}}\left(\frac{(v\cdot du - u\,dv)^2}{v^4}\right) + \frac{1}{\cos^2\frac{u}{v}}\cdot\left(\frac{v(v\,d^2u - u\,d^2v) - 2dv(v\,du - u\,dv)}{v^3}\right)$$

- $y = \sqrt{x} = x^{\frac{1}{2}}$

$$y^{(10)} = \frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)\left(\frac{1}{2}-3\right)\left(\frac{1}{2}-4\right)\left(\frac{1}{2}-5\right)\left(\frac{1}{2}-6\right)\left(\frac{1}{2}-7\right)\left(\frac{1}{2}-8\right)\left(\frac{1}{2}-9\right)\cancel{\left(\frac{1}{2}-10\right)}\cdot\cancel{\left(\frac{1}{2}-11\right)}\cdot x^{\frac{1}{2}-10} =$$

$$(\sqrt{x})^{(10)} = \frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\ldots\left(-\frac{17}{2}\right)\cdot x^{-\frac{19}{2}} = -\frac{17!!\cdot\sqrt{x}}{2^{10}\cdot x^{10}}$$

$$(x^m)^{(n)} = m(m-1)\ldots(m-n+1)\,x^{m-n}$$

Домашняя работа.

7.15

$$y = \frac{\ln x}{x}$$

$$y' = \frac{\frac{1}{x}\cdot x - \ln x}{x^2} = \frac{1-\ln x}{x^2}$$

$$y'' = \frac{(1-\ln x)'x^2 - 2x(1-\ln x)}{x^4} = \frac{-\frac{x^2}{x} - 2x + 2\ln x}{x^4} = \frac{-x-2x+2\ln x\cdot x}{x^4} =$$

$$= \frac{2\ln x - 3x}{x^4}$$

$$\Rightarrow d^2y = \left(\frac{x2\ln x - 3x}{x^4}\right)dx^2 = \left(\frac{2\ln x - 3}{x^3}\right)dx^2$$

7.16.

$$y = x^x = e^{x\ln x}$$

$$y' = e^{x\ln x}\cdot\left(\ln x + \frac{x}{x}\right) = (\ln x + 1)\cdot x^x$$

$$y'' = (1+\ln x)'x^x + (\ln x + 1)\cdot x^x\cdot(\ln x + 1) = \frac{x^x}{x} + x^x(\ln x + 1)^2 =$$

$$= x^x\left((1+\ln x)^2+\frac{1}{x}\right)$$

$$\Rightarrow d^2y = x^x\left((1+\ln x)^2+\frac{1}{x}\right)dx^2$$

7.17 $y=\dfrac{1}{u^2+v^2}$

$$dy = d\left(\frac{1}{u^2+v^2}\right) = \frac{\overbrace{d(1)}^{=0}(u^2+v^2) - d(u^2+v^2)}{(u^2+v^2)^2} = -\frac{(2u\,du+2v\,dv)}{(u^2+v^2)^2}$$

$$d^2y = \frac{d(-2u\,du-2v\,dv)(u^2+v^2)^2 - d(u^4+2u^2v^2+v^4)\cdot(-2u\,du-2v\,dv)}{(u^2+v^2)^4} =$$

$$= \frac{\left(-2(du^2+u\,d^2u)-2(dv^2+v\,d^2v)\right)(u^2+v^2)^2-\left(4u^3du+4uv^2du+4u^2v\cdot dv+4v^3dv\right)\cdot(-2u\,du-2v\,dv)}{(u^2+v^2)^4} =$$

$$= (-2du^2-2u\,d^2u-2dv^2-2v\,d^2v)(u^2+v^2)^2-(-8u^4du^2-8u^3v\,du\,dv-8u^2v^2du^2-$$

$$-8uv^3du\cdot dv-8u^3v\,du\cdot dv-8u^2v^2dv^2-8uv^3du\cdot dv-8v^4dv^2) =$$

$$= \frac{(2du^2+2u\,d^2u+2dv^2+2v\,d^2v)(u^2+v^2)^2-(8u^4du^2+8u^3v\,du\cdot dv+8u^2v^2du^2+8uv^3du\,dv+8u^3v\,du\cdot dv+8u^2v^2dv^2+8uv^3du\cdot dv+8v^4dv^2)}{(u^2+v^2)^4} =$$

?

$$= \frac{2}{(u^2+v^2)^3}\left[4(u\,du+v\,dv)^2-(u^2+v^2)(u\,d^2u+v\,d^2v+du^2+dv^2)\right]$$

7.18. $y = \left(\frac{u}{v}\right)^2 = \frac{u^2}{v^2}$

$$dy = d\left(\frac{u^2}{v^2}\right) = \frac{d(u^2)v^2 - d(v^2)\cdot u^2}{v^4} = \frac{2u\,du\,v^2 - 2vu^2\,dv}{v^4}$$

$$d^2y = d\left(\frac{2uv^2du - 2vu^2dv}{v^4}\right) =$$

$$= \frac{d(2uv^2du - 2vu^2dv)\cdot v^4 - d(v^4)(2uv^2du - 2vu^2dv)}{v^8} =$$

$$= \frac{d(2uv(vdu - udv))\cdot \cancel{v^4} - 4\cancel{v^3}dv(2uv^2du - 2vu^2dv)}{v^8} =$$

$$= \frac{2(v^2du^2 - u^2dv^2 + uv^2d^2u - u^2vd^2v)\cdot v - 8uv^2dudv + 8u^2vdv^2}{v^5} =$$

$$= \frac{2(v^2du^2 - u^2dv^2 + uv^2d^2u - u^2vd^2v)\cdot v - v(8uvdudv - 8u^2dv^2)}{v^5} =$$

$$= \frac{2v^2du^2 - 2u^2dv^2 + 2uv^2d^2u - 2u^2vd^2v - 8uvdudv + 8u^2dv^2}{v^4} =$$

$$= \frac{2v^2du^2 + 6u^2dv^2 + 2uv^2d^2u - 2u^2vd^2v - 8uvdu\cdot dv}{v^4}$$

11.11.2020

③ $y = \sqrt{x} = x^{\frac{1}{2}}$

$$y^{(10)} = \left(x^{\frac{1}{2}}\right)^{(10)} = \frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)\left(\frac{1}{2}-3\right)\left(\frac{1}{2}-4\right)\left(\frac{1}{2}-5\right)\left(\frac{1}{2}-6\right)\left(\frac{1}{2}-7\right)\ldots$$

Ок

в процессе

④ $y = \frac{x^2}{1-x} = x^2\cdot\frac{1}{1-x} = x^2\cdot(1-x)^{-1}$

$$\left(x^2\cdot\frac{1}{(1-x)}\right)^{(8)} = \sum_{k=0}^{n} C_n^k\cdot f^{(n-k)}\cdot g^{(k)}$$

$$\begin{array}{r|l} -\,x^2 & 1-x \\ \cline{2-2} x^2 - x & -x-1 \\ \hline -\,x & \\ x-1 & \\ \hline 1 & \end{array} \quad \longrightarrow \quad y = \frac{x^2}{1-x} = -(x+1) + \frac{1}{1-x}$$

$$\left(\frac{1}{x+a}\right)^{(8)} = \frac{\cancel{8!}}{(\cancel{x+a})^9}$$

$$\left(\frac{1}{x+a}\right)' = -\frac{1}{(x+a)^2}$$

$$\left(\frac{1}{x+a}\right)'' = \frac{1\cdot 2(x+a)}{(x+a)^4} = \frac{2}{(x+a)^3}$$

$$\left(\frac{1}{x+a}\right)''' = -\frac{2\cdot 3(x+a)^2}{(x+a)^6} = -\frac{6}{(x+a)^4}$$

$$\left(\frac{1}{x+a}\right)^{(n)} = (-1)^n\frac{n!}{(x+a)^{n+1}}$$

$$\left(\frac{1}{1-x}\right)^{(8)} = \frac{8!}{(1-x)^9}$$

$$-(x+1)^{(8)} = 0$$

$x_1+x_2=3$
$x_1\cdot x_2=2$

$$\Rightarrow y^{(8)} = \frac{8!}{(1-x)^9} = -\frac{8!}{(x-1)^9}$$

(4) $y = \dfrac{1}{x^2-3x+2} = \dfrac{1}{(x-1)(x-2)} = \dfrac{A}{x-1} + \dfrac{B}{x-2} =$ $\quad \backslash x-2 \quad \backslash x-1$

$$= \frac{Ax-2A+Bx-B}{(x-1)(x-2)} = \frac{x(A+B)-2A-B}{(x-1)(x-2)} \;(=)$$

$$\begin{cases} A+B=0 & |\cdot 2 \\ -2A-B=1 \end{cases} \quad \begin{cases} 2A+2B=0 \\ -2A-B=1 \end{cases}$$

$$B=1 \to 2A+2=0 \to A=-1.$$

$$(=)\; \frac{-1}{x-1} + \frac{1}{x-2} = \frac{1}{x-2} - \frac{1}{x-1}$$

или $\quad 1 = (x-1)-(x-2)$

$$\frac{(x-1)-(x-2)}{x^2-3x+2} = \frac{1}{x-2} - \frac{1}{x-1}.$$

$$y^{(n)} = (-1)^n\cdot\frac{n!}{(x-2)^{n+1}} - \frac{(-1)^n\cdot n!}{(x-1)^{n+1}} = (-1)^n\cdot n!\left(\frac{1}{(x-2)^{n+1}} - \frac{1}{(x-1)^{n+1}}\right)$$

(5) $y = x^2\cdot e^{2x}$, $\quad y_n^{(20)}$ - ?

$$(x^2\cdot e^{2x})^{(20)} = \sum_{k=0}^{n} C_n^k f^{(n-k)}\cdot g^{(k)} = C_{20}^0\cdot x^2\cdot(e^{2x})^{(20)} +$$

$$\boxed{\begin{aligned} f(x) &= e^{2x} \\ g(x) &= x^2 \end{aligned}}$$

$$+ C_{20}^1\cdot(x^2)'\cdot(e^{2x})^{(19)} + C_{20}^2\cdot(2x)'\cdot(e^{2x})^{(18)} +$$

$$+ \underbrace{C_{20}^3\cdot 2'\cdot(e^{2x})^{(17)}}_{=0} + \ldots + 0 =$$

$$= C_{20}^0\cdot x^2\cdot(e^{2x})^{(20)} + C_{20}^1\cdot 2x\cdot(e^{2x})^{(19)} + C_{20}^2\cdot 2\cdot(e^{2x})^{(18)} =$$

$$= x^2\cdot 2^{20}\cdot e^{2x} + 40x\cdot 2^{19}\cdot e^{2x} + 380\cdot 2^{18}\cdot e^{2x} =$$

$$= e^{2x}\left(2^{20}x^2 + 2^{19}\cdot 40x + 380\cdot 2^{18}\right) = 2^{20}e^{2x}(x^2+20x+95)$$

⑥ $y = x \cdot \cos 2x$; $d^{10} y$. $f = \cos 2x$; $g = x$

$$d^n(f \cdot g) = \sum_{k=0}^{n} C_n^k \cdot d^{n-k} f \cdot d^k g$$

$$d^{10} y = C_{10}^0\, x \cdot d^{10}(\cos 2x) + C_{10}^1 \cdot d^9(\cos 2x) + 0 \ldots + 0 =$$

$$= x \cdot d^{10}(\cos 2x) + 10\, d^9(\cos 2x) = x \cdot (\cos 2x)^{(10)} \cdot dx^{10} + 10(\cos 2x)$$

$$= x \cdot 2^{10} \cdot \cos\left(2x + \frac{\pi}{2} \cdot 10\right) \cdot dx^{10} + 10 \cdot 2^9 \cdot \cos\left(\left(2x + \frac{\pi}{2} \cdot 9\right)\right) \cdot dx^9 =$$

$$= -x \cdot 2^{10} \cos(2x) \cdot dx^{10} - 10 \cdot 2^9 \sin(2x) \cdot dx^{10} =$$

$$= 2^{10}(\cancel{2x \cos 2x} - 2^{10}\left(x\cos(2x) + \underline{5 \sin(2x)}\right) dx^{10}$$

$$y^{(n)} = 2^{(n-1)} \cdot \cos\left(2x + \frac{\pi}{2} n\right)$$

⑧ $y = \cos^2 x$

$$y = \cos^2 x = \frac{1 + \cos 2x}{2} = \frac{1}{2} + \frac{\cos 2x}{2}$$

$$y^{(n)} = \frac{1}{2} \cdot (\cos 2x)^{(n)} = \frac{1}{2} \cdot 2^n \cdot \cos\left(2x + \frac{\pi}{2} \cdot n\right) = 2^{n-1} \cos\left(2x + \frac{\pi}{2} n\right)$$

⑨ $y = \sin^3 x$

$$\sin 3x = 3 \sin x - 4 \sin^3 x \rightarrow 4 \sin^3 x = 3 \sin x - \sin 3x$$

$$\sin^3 x = \frac{3}{4} \sin x - \frac{1}{4} \sin(3x)$$

$$y^{(n)} = \frac{3}{4} \sin\left(x + \frac{\pi}{2} \cdot n\right) - \frac{1}{4} \cdot 3^n \sin\left(3x + \frac{\pi}{2} \cdot n\right)$$

Домашняя работа.

8.3. $y = \frac{a}{x^m} = a \cdot x^{-m}$

$$y^{(3)} = \left(\frac{a}{1} \cdot \frac{1}{x^m}\right)^{(3)}$$

$$y' = a \cdot (-m) \cdot x^{-m-1}$$

$$y'' = a \cdot (-m) \cdot (-m-1) \cdot x^{-m-2}$$

$$y''' = a \cdot (-m)(-m-1) \cdot (-m-2) \cdot x^{-m-3} = -\frac{a m (m+1)(m+2)}{x^{m+3}}$$

8.4. $y = \frac{1}{\sqrt{1-x}}$

$$y' = +\frac{1}{(1-x)} \cdot \frac{1}{2\sqrt{1-x}} = +\frac{1}{2 \cdot (1-x)^{\frac{3}{2}}}$$

$$y'' = \cancel{+\frac{1}{2}} \cdot \frac{1}{(1-x)^3} \cdot \quad \frac{3}{4\cdot(1-x)\cdot(1-x)^{\frac{1}{2}}} = \frac{3}{4\cdot(1-x)^{\frac{3}{2}}}$$

$$y''' = \frac{15\cdot\sqrt{(1-x)^3}}{8\cdot(1-x)^5} = \frac{15\cdot(1-x)^{\frac{3}{2}}}{8\cdot(1-x)^{\frac{10}{2}}} = \frac{15}{8\cdot(1-x)^{\frac{7}{2}}}$$

$$y^{(n)} = \frac{(2n-\cancel{1})!!\ (\text{нечёт})}{2^n\,(1-x)^{n+\frac{1}{2}}}$$

$$\Rightarrow y^{(100)} = \frac{199!!}{2^{100}(1-x)^{100+\frac{1}{2}}}$$

8.5 $y = x^2\sin 2x \quad \to f(x) = \sin 2x \qquad g(x) = x^2$

$$y^{(n)} = \sum_{k=0}^{n} C_n^k f^{(n-k)} g^{(k)}$$

$$y^{(50)} = C_{50}^0 \cdot x^2 \cdot (\sin 2x)^{(50)} + C_{50}^1 \cdot 2x \cdot (\sin 2x)^{(49)} + C_{50}^2 \cdot 2 \cdot (\sin 2x)^{(48)} =$$

$$= x^2 \cdot 2^{50}\sin\left(2x + \frac{\pi\cdot 50}{2}\right) + \frac{50!}{1\cdot 49!}\cdot 2x \cdot 2^{50}\sin\left(2x + \frac{\pi\cdot 49}{2}\right) \ +$$

$$\frac{50!}{2!\cdot 48!} = \frac{49\cdot 50}{2} = 49\cdot 25 = 1225$$

$$+\ 1225\cdot 2^{49}\cdot\sin\left(2x + \frac{\pi\cdot 48}{2}\right) =$$

$$= -x^2\cdot 2^{50}\sin(2x) + 50x\cdot 2^{50}\cos(2x) + 1225\cdot 2^{49}\cdot\sin(2x) =$$

$$= 2^{50}\left(-x^2\sin(2x) + 50x\cdot\cos(2x) + \frac{1225}{2}\cdot\sin(2x)\right)$$

8.6 $y = x\cdot 2^x$; Найти $d^{10}y$

$$f(x) = 2^x \qquad g(x) = x$$

$$d^{10}y = C_{10}^0\, x\cdot d^{10}(2^x) + C_{10}^1\, dx\cdot d^9(2^x) + \ldots + 0 =$$

$$= x\cdot d^{10}(2^x) + 10\,dx\cdot d^9(2^x) = x\cdot 2^x\ln^{10}(2)\cdot dx^{10} + 10\cdot 2^x\ln^9(2)\cdot dx^{10} =$$

$$(2^x)^{(n)} = 2^x\ln^n 2 \qquad = 2^x(\ln 2)^9\cdot dx^{10}(x\ln 2 + 10) \quad ?$$

$$(2^x)^{(10)} = 2^x\ln^{10}(2)$$

8.7 $$y = \frac{1}{x(1-x)} = \frac{A}{x} + \frac{B}{1-x} = \frac{A - Ax + Bx}{x(1-x)} = \frac{x(B-A) + A}{x(1-x)}$$

$A = 1 \Rightarrow B = 1$.

$$y = \frac{1}{x(1-x)} = \frac{1}{x} + \frac{1}{1-x} = \frac{1}{x} - \frac{1}{x-1}$$

$$y^{(n)} = \frac{(-1)^n\cdot n!}{x^{n+1}} - \frac{(-1)^n\cdot n!}{(x-1)^{n+1}} = (-1)^n\cdot n!\left(\frac{1}{x^{n+1}} - \frac{1}{(x-1)^{n+1}}\right)$$

8.8. $y = \dfrac{1}{\sqrt{1-2x}}$

$$y^{(n)} = \frac{(2n-1)!! \cdot 2^n}{2^n (1-2x)^{n+\frac{1}{2}}} = \frac{(2n-1)!!}{(1-2x)^{n+\frac{1}{2}}}$$

8.9. $y = \sin^2 x = \dfrac{1-\cos 2x}{2} = \dfrac{1}{2} - \dfrac{\cos 2x}{2}$

$$y^{(n)} = -\frac{1}{2} \cdot (\cos 2x)^{(n)} = -2^{n-1} \cdot \cos\left(2x + \frac{\pi}{2} \cdot n\right)$$

8.10 $y = \cos^3 x$

$\cos 3x = 4\cos^3 x - 3\cos x \quad \to \quad 4\cos^3 x = \cos 3x + 3\cos x$

$$\cos^3 x = \frac{1}{4}\cos 3x + \frac{3}{4}\cos x$$

$$y^{(n)} = \frac{1}{4} \cdot 3^n \cos\left(3x + \frac{\pi}{2} \cdot n\right) + \frac{3}{4} \cos\left(x + \frac{\pi}{2} n\right)$$

8.12 $y = \ln u$

$$d^3 y = (\ln u)''' \cdot du^3$$

$\cancel{(\ln u)'' =}$

$\cancel{d^3 y = \frac{(-1)^2 \cdot 2!}{u^3} \cdot du^3}$

$\cancel{dy = \frac{du}{u}}$

$\cancel{d^2 y = \frac{1}{u} du}$

8.11 $y = u^2$ $\qquad d^{10}y - ?$

$$(y)^{(n)} = \sum_{k=0}^{10} (u)^{(10-k)} \quad (u)^{(k)} \sim n$$

$$d^{n}y = \sum_{k=0}^{n} d^{n-k}f \; d^{k}(g) \cdot C_n^k$$

$$d^{10}y = \underline{C_{10}^{0}\, u\, d^{10}u} + \underline{C_{10}^{1}\, du\, d^{9}u} + C_{10}^{2}\, d^2u\, d^8u + C_{10}^{3}\, d^3u\, d^7u +$$
$$+ C_{10}^{4}\, d^4u\, d^6u + C_{10}^{5}(d^5u)^2 + C_{10}^{6}\, d^6u\, d^4u + C_{10}^{7}\, d^7u\, d^3u +$$
$$+ C_{10}^{8}\, d^8u\, d^2u + \underline{C_{10}^{9}\, d^9u\, du} + \underline{C_{10}^{10}\, u\, d^{10}u} =$$
$$= 2u\,d^{10}u + 20\,du\,d^9u + 90\,d^2u\,d^8u + 240\,d^3u\,d^7u + 420\,d^4u\,d^6u +$$
$$+ 252\,(d^5u)^2$$

12.11.2020

Исследование рациональных функций.

Схема исследования ф-ии.

I. Общие вопросы

1. ООФ - область определения функции
2. Вид функции - чётная, нечётная, периодическая, общего вида
3. Точки пересечения с осями
4. Интервалы знакопостоянства.

II. Исследование с помощью пределов.

1. Поведение на границе области определения
2. Асимптоты.

III. Исследование с помощью первой производной.

1. Интервалы возрастания и убывания функции.
2. Максимумы и минимумы функции.

IV. Исследование с помощью второй производной.

1. Промежутки выпуклости графика функции (вверх и вниз).
2. Точки перегиба.

(1) $y = \dfrac{x^4}{(1+x)^3}$

I.

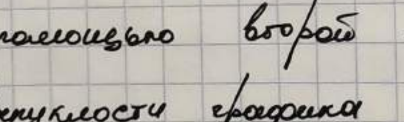
1) $D(y): x \in (-\infty; -1) \cup (-1; +\infty)$

2) $y(-x) = \dfrac{x^4}{(1-x)^3} \Rightarrow$ ф-я $y(x)$ - общего вида.

3) $x=0 \quad y=0$
$(0;0)$ - точка пересечения с Ox и Oy.

4) $y > 0$

$\dfrac{x^4}{(1+x)^3} > 0$



$y > 0$ при $x \in (-1; 0) \cup (0; +\infty)$

$y < 0$ при $x \in (-\infty; -1)$

II. 1) $\lim\limits_{x\to-\infty}\dfrac{x^4}{(1+x)^3}=-\infty$

$\lim\limits_{x\to+\infty}\dfrac{x^4}{(1+x)^3}=+\infty$

$\lim\limits_{x\to-1+0}\dfrac{x^4}{(1+x)^3}=+\infty$ $\qquad$ $\lim\limits_{x\to-1-0}\dfrac{x^4}{(1+x)^3}=-\infty$

2) Асимптоты. $x=-1$ ~ вертикальная асимптота.

$y=kx+b$ ~ наклонная асимптота.

$$k=\lim_{x\to\pm\infty}\frac{y}{x}=\lim_{x\to\pm\infty}\frac{x^4}{(1+x)^3\cdot x}=\lim_{x\to\pm\infty}\frac{x^3}{(1+x)^3}=\lim_{x\to\pm\infty}\frac{x^3}{x^3\left(\frac{1}{x}+1\right)^3}=1$$

$$b=\lim_{x\to\infty}(y-kx)=\lim_{x\to\infty}\left(\frac{x^4}{(1+x)^3}-x\right)=$$

$$=\lim_{x\to\infty}\frac{x^4-x(x+1)^3}{(1+x)^3}=\lim_{x\to\infty}\frac{x^4-x(x^3+3x^2+3x+1)}{(1+x)^3}=$$

$$=\lim_{x\to\infty}\frac{-3x^3-3x^2-x}{(1+x)^3}=-3.$$

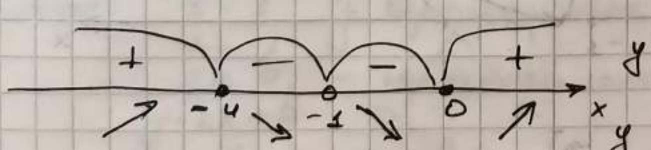
$y=x-3$ ~ наклонная асимптота.

III.

$$y=\frac{x^4}{(1+x)^3}$$

$$y'=\frac{4x^3(1+x)^3-x^4\cdot 3(1+x)^2}{(1+x)^6}=\frac{4x^3(1+x)-3x^4}{(1+x)^4}=$$

$$=\frac{x^3(4+4x-3x)}{(1+x)^4}=\frac{x^3(x+4)}{(1+x)^4}$$



$x=-4$ – точка максимума. $\quad y(-4)=-\dfrac{256}{27}$

$x=0$ – точка минимума $\quad y(0)=0$

$y=\dfrac{x^4}{(1+x)^3}$ возрастает при $x\in(-\infty;-4)\cup(0;+\infty)$

убывает при $x\in(-4;-1)\cup(-1;0)$

$$y''=\left(\frac{x^3(x+4)}{(x+1)^4}\right)'=\frac{(x^3(x+4))'(x+1)^4-((x+1)^4)'x^3(x+4)}{(x+1)^8}=$$

$=$

$$= \frac{((x^3)'(x+4) + (x+4)'x^3)\cdot(x+1)^4 - 4(x+1)^3 x^3(x+4)}{(x+1)^8} =$$

$$= \frac{(3x^2(x+4) + x^3)\cdot(x+1) - 4x^3(x+4)}{(x+1)^5} =$$

$(4x+12)(x+1) =$
$= 4x^2+4x+12x+12 =$
$= 4x^2+16x+12$

$$= \frac{(3x^3+12x^2+x^3)(x+1) - 4x^4 - 16x^3}{(x+1)^5} =$$

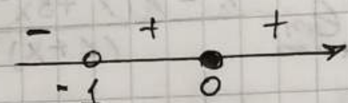
$$= \frac{x^2(3x+12+x)\cdot(x+1) - 4x^3(x+4)}{(x+1)^5} = \frac{x^2(\cancel{4x^2}+\cancel{16x}+12-\cancel{4x^2}-\cancel{16x})}{(x+1)^5} =$$

$$= \frac{12x^2}{(1+x)^5}$$

Если $y''>0$, то график ф-и выпуклый вниз (выпуклая) ∪

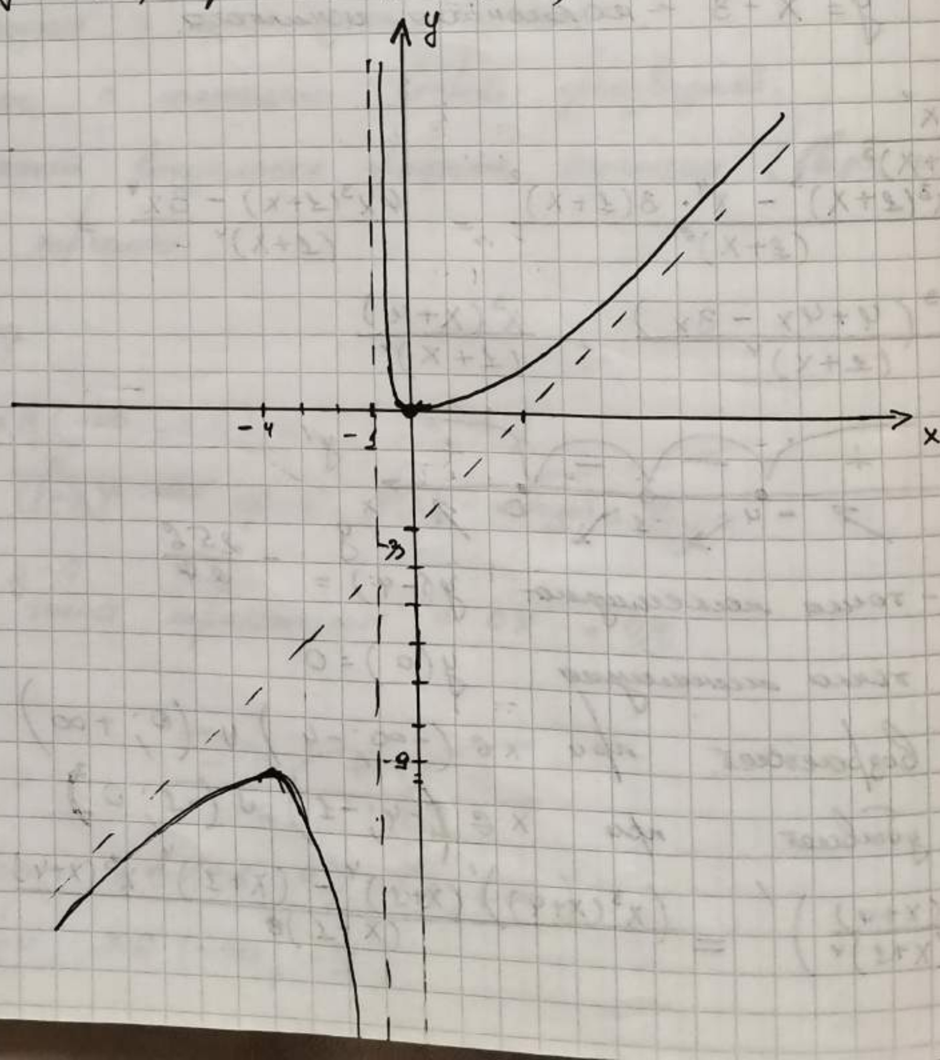
$y''<0$, то график ф-и выпуклый вверх (вогнутый) ∩

$$\frac{12x^2}{(1+x)^5} \geq 0$$



т.е. $y'' \geq 0$, при $x \in (-1; +\infty)$

$y''<0$, при $x \in (-\infty; -1)$



②

$$y = \frac{x}{(1-x^2)^2}$$

I.

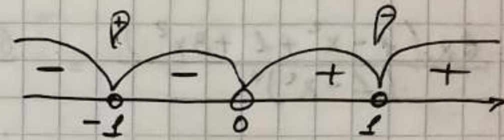
1) $D(y)$: $x \neq \pm 1 \rightarrow x \in (-\infty; -1) \cup (-1; 1) \cup (1; +\infty)$

2) $y(-x) = \frac{-x}{(1-x^2)^2} = -\left(\frac{x}{(1-x^2)^2}\right) - y(x)$ - нечётная.

3) $x = 0 \rightarrow y = 0$
$(0; 0)$ - точка пересечения с OX и OY.

4) $y > 0$

$$\frac{x}{(1-x^2)^2} = \frac{x}{((1-x)(1+x))^2} > 0$$



$y > 0$ при $x \in (0; 1) \cup (1; +\infty)$

$y < 0$ при $x \in (-\infty; -1) \cup (-1; 0)$

II.
1) $$\lim_{x \to -\infty} \frac{x}{(1-x^2)^2} = \lim_{x \to -\infty} \frac{x}{x^4\left(\frac{1}{x^2} - 1\right)^2} = \lim_{x \to -\infty} \frac{1}{x^3\left(\frac{1}{x^2} - 1\right)^2} = 0$$

$$\lim_{x \to +\infty} \frac{x}{(1-x^2)^2} = \lim_{x \to +\infty} \frac{1}{x^3\left(\frac{1}{x^2} - 1\right)^2} = 0$$

$$\lim_{x \to -1+0} \frac{x}{(1-x^2)^2} = -\infty \qquad \lim_{x \to -1-0} \frac{x}{(1-x^2)^2} = -\infty$$

$$\lim_{x \to 1+0} \frac{x}{(1-x^2)^2} = +\infty \qquad \lim_{x \to 1-0} \frac{x}{(1-x^2)^2} = +\infty$$

2) Асимптоты: $\left.\begin{matrix} x = 1 \\ x = -1 \end{matrix}\right\}$ вертикальные асимптоты
$y = 0$ - горизонтальная асимптота.
$y = kx + b$

$$k = \lim_{x \to \pm\infty} \frac{y}{x} = \lim_{x \to \pm\infty} \frac{x}{(1-x^2)^2 \cdot x} = \lim_{x \to \pm\infty} \frac{1}{(1-x^2)^2} = 0$$

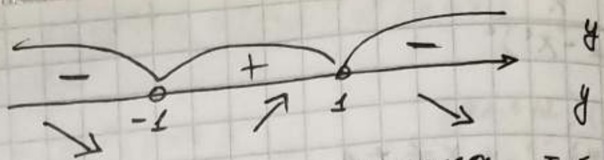
$$b = \lim_{x \to \infty} (y - \underbrace{kx}_{=0}) = 0$$

III.
$$y = \frac{x}{(1-x^2)^2}$$

$$y' = \frac{(1-x^2)^2 - x \cdot ((1-x^2)^2)'}{(1-x^2)^4} = \frac{(1-x^2)^2 + x \cdot 2(1-x^2) \cdot 2x}{(1-x^2)^4} =$$

$$= \frac{(1-x^2) + 4x^2}{(1-x^2)^3} = \frac{1+3x^2}{(1-x^2)^3}$$

$y' = \frac{1+3x^2}{(1-x^2)^3}$



Нет точек максимума и минимума, т.к. $x \neq \pm 1$ по ООФ

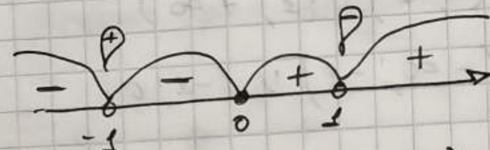
IV. $y'' = \frac{(1+3x^2)'(1-x^2)^3 - (1+3x^2)\cdot((1-x^2)^3)'}{(1-x^2)^6} =$

$= \frac{6x(1-x^2)^3 + 3(1-x^2)^2\cdot 2x\cdot(1+3x^2)}{(1-x^2)^6} = \frac{6x(1-x^2) + 6x(1+3x^2)}{(1-x^2)^4} =$

$= \frac{6x(1-x^2+1+3x^2)}{(1-x^2)^4} = \frac{6x(2+2x^2)}{(1-x^2)^4} = \frac{12x(1+x^2)}{(1-x^2)^4}$

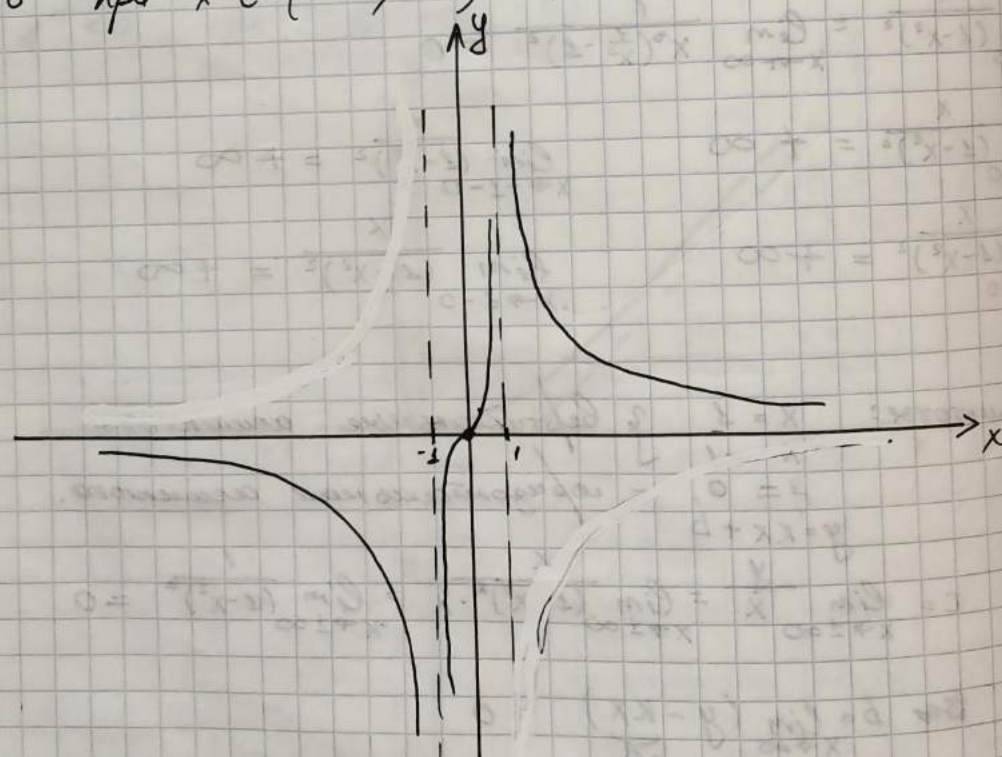
$y'' > 0$

$\frac{12x(1+x^2)}{(1-x^2)^4} \geq 0$



$y'' \geq 0$ при $x \in [0;1) \cup (1;+\infty)$ ∪   т.(0;0)-т. перегиба

$y'' < 0$ при $x \in (-\infty;-1) \cup (-1;0)$ ∩



Домашняя работа.

13.1. $y = \frac{x^2+8}{x+1}$

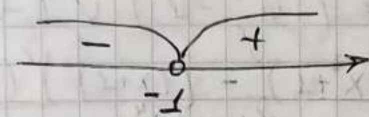
I. 1) $D(y)$: $x \in (-\infty; -1) \cup (-1; \infty)$

2) $y(-x) = \frac{x^2+8}{-x+1}$ => ф-я $y(x)$ - общего вида.

3) $x = 0 \to y = 8$ → $(0; 8)$ - т. пер. с Oy.

$y = 0 \to x^2 + 8 \neq 0$

4) $\frac{x^2+8}{x+1} > 0$



$y > 0$ при $x \in (-1; \infty)$

$y < 0$ при $x \in (-\infty; -1)$

$x_1 + x_2 = -2$
$x_1 \cdot x_2 = -8$
$x_1 = -4$
$x_2 = 2$

II. 1) $\lim\limits_{x\to-\infty} \frac{x^2+8}{x+1} = \lim\limits_{x\to-\infty} \frac{\cancel{x^2}\left(1+\frac{8}{x^2}\right)}{\cancel{x}\left(1+\frac{1}{x}\right)} = -\infty$

$\lim\limits_{x\to+\infty} \frac{x\left(1+\frac{8}{x^2}\right)}{1+\frac{1}{x}} = +\infty$

$\lim\limits_{x\to-1+0} \frac{x^2+8}{x+1} = +\infty$ $\qquad$ $\lim\limits_{x\to-1-0} \frac{x^2+8}{x+1} = -\infty$

2) Асимптоты: $x = -1$ - вертик. асимптота

$y = kx + b$ ~ наклонная

$k = \lim\limits_{x\to\pm\infty}\left(\frac{y}{x}\right) = \lim\limits_{x\to\pm\infty} \frac{x^2+8}{(x+1)x} = \lim\limits_{x\to\pm\infty} \frac{\cancel{x^2}\left(1+\frac{8}{x^2}\right)}{\cancel{x^2}\left(1+\frac{1}{x}\right)} = 1.$

$b = \lim\limits_{x\to\infty}(y - kx) = \lim\limits_{x\to\infty}\left(\frac{x^2+8}{x+1} - x\right) = \lim\limits_{x\to\infty}\left(\frac{\cancel{x^2}+8-\cancel{x^2}-x}{x+1}\right) =$

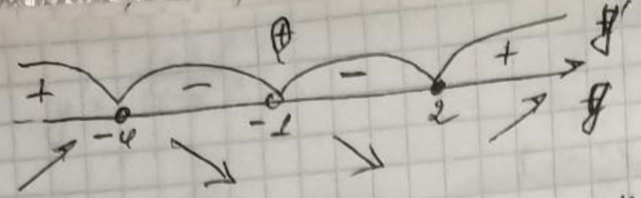
$= \lim\limits_{x\to\infty}\left(\frac{8-x}{x+1}\right) = \lim\limits_{x\to\infty} \frac{\cancel{x}\left(\frac{8}{x}-1\right)}{\cancel{x}\left(1+\frac{1}{x}\right)} = -1.$

$y = x - 1$ - наклонная асимптота.

III. $y = \frac{x^2+8}{x+1}$

$y' = \frac{(x^2+8)'(x+1) - (x+1)'(x^2+8)}{(x+1)^2} = \frac{2x(x+1) - x^2 - 8}{(x+1)^2} =$

$= \frac{2x^2 + 2x - x^2 - 8}{(x+1)^2} = \frac{x^2+2x-8}{(x+1)^2} = \frac{(x+4)(x-2)}{(x+1)^2}$



$x = -4$ – точка макс. $\quad y(-4) = \frac{16+8}{-3} = \frac{25}{-3} \approx -8\frac{1}{3}$

$x = 2$ – точка мин. $\quad y(2) = \frac{4+8}{3} = \frac{12}{3} = 4$

$y = \frac{x^2+8}{x+1}$ возр. при $x \in (-\infty; -4)$ и $(2; +\infty)$

убыв при $x \in (-4; -1) \cup (-1; 2)$

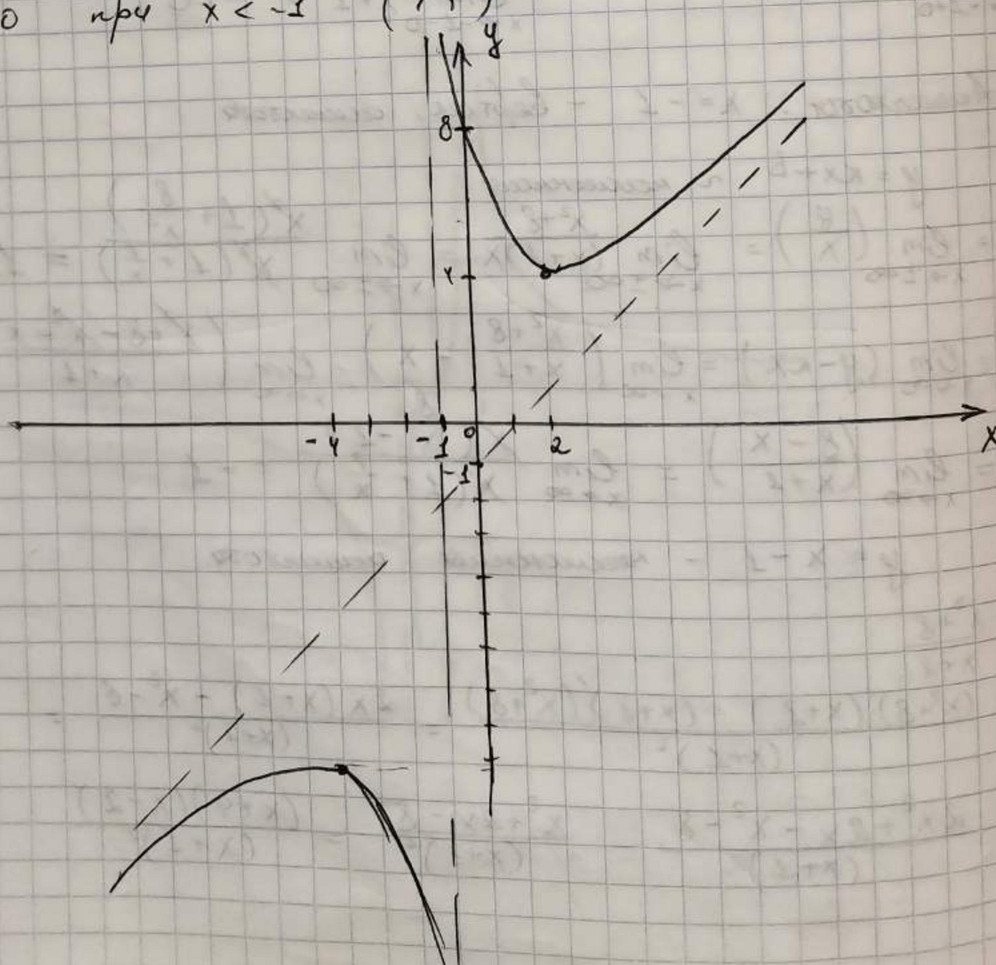
IV. $y'' = \left(\frac{x^2+2x-8}{(x+1)^2}\right)' = \frac{(x^2+2x-8)'(x+1)^2 - ((x+1)^2)'(x^2+2x-8)}{(x+1)^4} =$

$= \frac{(2x+2)(x+1)^2 - 2(x+1)(x^2+2x-8)}{(x+1)^4} = \frac{(2x+2)(x+1) - 2(x^2+2x-8)}{(x+1)^3}$

$= \frac{\cancel{2x^2} + \cancel{4x} + 2 - \cancel{2x^2} - \cancel{4x} + 16}{(x+1)^3} = \frac{18}{(x+1)^3}$

$y'' > 0$ при $x > -1$ $(\cup)$

$y'' < 0$ при $x < -1$ $(\cap)$



13.2 $y = \frac{(x+1)^2}{x^2+2x} = \frac{(x+1)^2}{x(x+2)}$

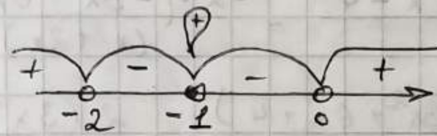
(I). 1) $D(y)$: $x \in (-\infty; -2) \cup (-2; 0) \cup (0; +\infty)$

2) $y(-x) = \frac{(-x+1)^2}{x^2-2x} = \frac{(1-x)^2}{x^2-2x}$ ~ ф-я $y(x)$ – общего вида.

3) $x \neq 0$

$y=0 \rightarrow (x+1)=0 \quad x=-1 \Rightarrow (-1; 0)$ – т. пер. с OX.

4) $\frac{(x+1)^2}{x(x+2)} > 0$



$y>0$ при $x \in (-\infty; -2) \cup (0; +\infty)$

$y<0$ при $x \in (-2; -1) \cup (-1; 0)$

(II). 1) $\lim\limits_{x\to+\infty} \frac{(x+1)^2}{x^2+2x} = \frac{\cancel{x^2}(1+\frac{1}{x})^2}{\cancel{x^2}(1+\frac{2}{x})} = 1$

$\lim\limits_{x\to-\infty} \frac{(x+1)^2}{x^2+2x} = 1$

$\lim\limits_{x\to-2+0} \frac{(x+1)^2}{x^2+2x} = \lim\limits_{x\to-2+0} \frac{(1+\frac{1}{x})^2}{(1+\frac{2}{x})} = -\infty \qquad \lim\limits_{x\to-2-0} \frac{(x+1)^2}{x^2+2x} = +\infty$

$\lim\limits_{x\to+0} \frac{(x+1)^2}{x^2+2x} = \lim\limits_{x\to+0} \frac{(1+\frac{1}{x})^2}{(1+\frac{2}{x})} = +\infty$

$\lim\limits_{x\to-0} \frac{(x+1)^2}{x^2+2x} = \lim\limits_{x\to-0} \frac{(1+\frac{1}{x})^2}{(1+\frac{2}{x})} = -\infty$

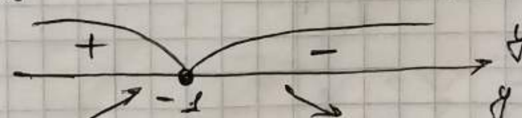
2) $y=1$ – горизонтальная асимптота.
$x=-2$ и $x=0$ – вертикальная асимптота.

$k = \lim\limits_{x\to\pm\infty}\left(\frac{y}{x}\right) = \lim\limits_{x\to\pm\infty}\left(\frac{(x+1)^2}{x^3+2x^2}\right) = 0 \Rightarrow$ нет наклонных асимптот.

(III) $y' = \frac{((x+1)^2)'(x^2+2x) - (x+1)^2(x^2+2x)'}{(x^2+2x)^2} = \frac{2(x+1)(x^2+2x) - (2x+2)(x+1)^2}{(x^2+2x)^2} =$

$= \frac{2(x+1)(x^2+2x) - 2(x+1)^3}{(x^2+2x)^2} = \frac{2(x+1)(\cancel{x^2}+\cancel{2x}-\cancel{x^2}-\cancel{2x}-1)}{(x^2+2x)^2} =$

$= \frac{-2(x+1)}{(x^2+2x)^2}$



$x=-1$ – точка макс. $\rightarrow y(-1)=0$

$y = \frac{(x+1)^2}{x^2+2x}$ возр. при $x \in (-\infty; -1)$

убыв при $x \in (-1; +\infty)$

IV.

$$y'' = -2\left(\frac{(x+1)}{(x^2+2x)^2}\right)' = -2\,\frac{(x^2+2x)^2 - (x+1)\left((x^2+2x)^2\right)'}{(x^2+2x)^4} =$$

$$= -2\cdot\frac{(x^2+2x)^2 - (x+1)\cdot 2(x^2+2x)(2x+2)}{(x^2+2x)^4} =$$

$$= -2\cdot\frac{(x^2+2x) - 4(x+1)^2}{(x^2+2x)^3} = -2\cdot\frac{x^2+2x - 4(x^2+2x+1)}{(x^2+2x)^3} =$$

$$= -2\cdot\frac{x^2+2x-4x^2-8x-4}{(x^2+2x)^3} = \frac{-2\cdot(-3x^2-6x-4)}{(x^2+2x)^3} =$$

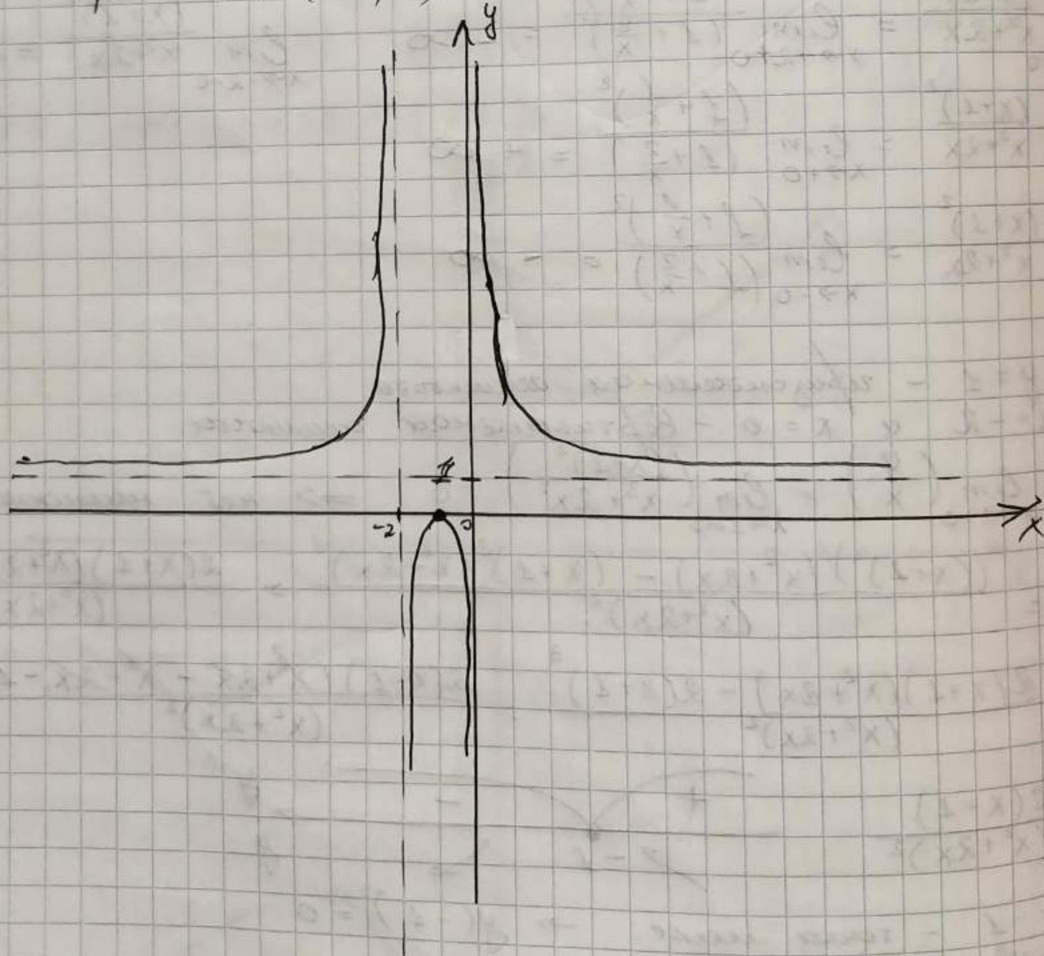
$$= \frac{6x^2+12x+8}{(x^2+2x)^3} = \frac{2(3x^2+6x+4)}{(x^2+2x)^3} = \frac{2(3x^2+6x+4)}{x^3(x+2)^3} =$$

$+ \quad -2 \quad - \quad 0 \quad + \quad \to y''$

y

$y'' > 0$ при $x \in (-\infty; -2) \cup (0; +\infty)$ $(\cup)$

$y'' < 0$ при $x \in (-2; 0)$ $(\cap)$



3. $y = \frac{(x+1)^3}{(x-1)^2}$

(I) 1) $D(y)$: $x \in (-\infty; 1) \cup (1; +\infty)$

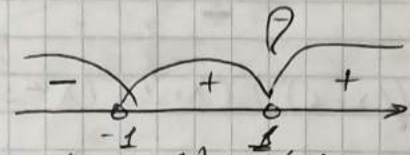
2) $y(-x) = \frac{(-x+1)^3}{(-x-1)^2}$ $\to$ $y(x)$ - общего вида.

3) $x = 0 \to y = 1.$

$y = 0 \to x = -1$

$\Rightarrow$ т. $(0; 1)$ и $(-1; 0)$ - точки пер. с Ox и Oy

4) $\frac{(x+1)^3}{(x-1)^2} > 0$



$y > 0$ при $x \in (-1; 1) \cup (1; +\infty)$
$y < 0$ при $x \in (-\infty; -1)$

(II) 1) $\lim\limits_{x \to +\infty} \frac{x^3(1+\frac{1}{x})^3}{x^2(1-\frac{1}{x})^2} = +\infty$ $\Rightarrow$ $\lim\limits_{x \to -\infty} \frac{(x+1)^3}{(x-1)^2} = -\infty$

$\lim\limits_{x \to 1+0} \frac{(x+1)^3}{(x-1)^2} = +\infty$ $\quad$ $\lim\limits_{x \to 1-0} \frac{(x+1)^3}{(x-1)^2} = +\infty$

2) Асимптоты: $x = 1$ - вертикальная асимптота.

$k = \lim\limits_{x \to \pm\infty} \frac{(x+1)^3}{(x-1)^2 \cdot x} = \lim\limits_{x \to \pm\infty} \frac{x^3(1+\frac{1}{x})^3}{x^3(1-\frac{1}{x})^2} = 1.$

$b = \lim\limits_{x \to \infty} (y - kx) = \lim\limits_{x \to \infty} \left(\frac{(x+1)^3}{(x-1)^2} - x \right) = \lim\limits_{x \to \infty} \left(\frac{(x+1)^3 - x(x-1)^2}{(x-1)^2} \right) =$

$= \lim\limits_{x \to \infty} \frac{(x+1)(x^2+2x+1) - x(x^2-2x+1)}{(x-1)^2} =$

$= \lim\limits_{x \to \infty} \frac{\cancel{x^3} + 2x^2 + \cancel{x} + x^2 + 2x + 1 - \cancel{x^3} + 2x^2 - \cancel{x}}{(x-1)^2} = \lim\limits_{x \to \infty} \frac{5x^2 + 2x + 1}{(x-1)^2} =$

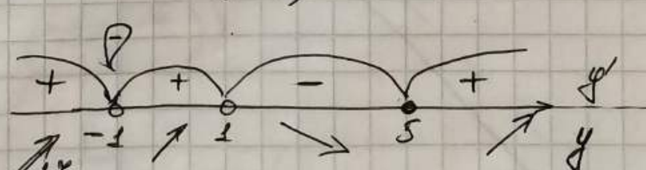
$= \lim\limits_{x \to \infty} \frac{\cancel{x^2}(5 + \frac{2}{x} + \frac{1}{x^2})}{\cancel{x^2}(1 - \frac{1}{x})^2} = 5$

$\Rightarrow y = x + 5$ - наклонная асимптота

(III) $y' = \frac{((x+1)^3)'(x-1)^2 - ((x-1)^2)'(x+1)^3}{(x-1)^4} = \frac{3(x+1)^2(x-1)^2 - 2(x-1)(x+1)^3}{(x-1)^4} =$

$= \frac{3(x+1)^2(x-1) - 2(x+1)^3}{(x-1)^3} = \frac{(x+1)^2(3x-3-2x-2)}{(x-1)^3} =$

$= \frac{(x+1)^2(x-5)}{(x-1)^3}$



$x = 5$ - точка минимума.

$y(5) = \frac{6^3}{4^2} = \frac{216}{16} = \frac{108}{8} = \frac{54}{4} = \frac{27}{2} = 13{,}5$

$y(x)$ - возрастает при $x \in (-\infty; -1) \cup (-1; 1) \cup (5; +\infty)$

- убывает при $x \in (1; 5)$

IV.

$$y'' = \left(\frac{(x+1)^2(x-5)}{(x-1)^3}\right)' = \frac{\left((x+1)^3(x-5)\right)'(x-1)^3 - \left((x-1)^3\right)'(x+1)^3(x-5)}{(x-1)^6} =$$

$$= \frac{(x+1)^2(4x-14)(x-1)^3 - 3(x-1)^2(x+1)^3(x-5)}{(x-1)^6} =$$

$$= \frac{(x+1)^2(x-1)(4x-14) - 3(x+1)^3(x-5)}{(x-1)^4} = \frac{(x+1)^2\left((x-1)(4x-14) - 3(x+1)(x-5)\right)}{(x-1)^4}$$

$$= \frac{(x+1)^2(x^2-6x+29)}{(x-1)^4}$$

$+ \quad -1 \quad + \quad 1 \quad +$

?

$$= \frac{\left((x+1)^2(x-5)\right)'(x-1)^3 - \left((x-1)^3\right)'(x+1)^2(x-5)}{(x-1)^6} =$$

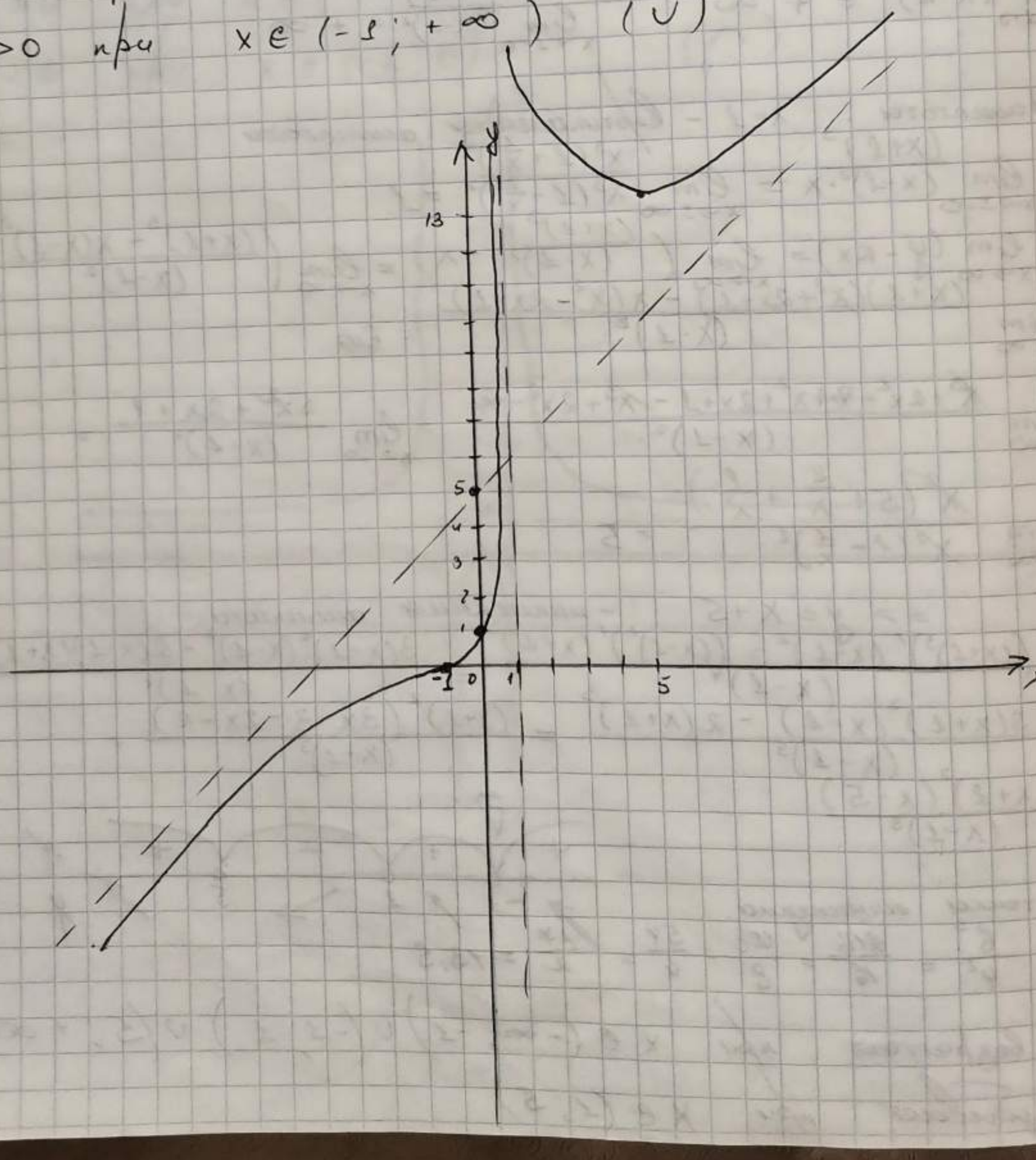
$$= \frac{3(x+1)(x-3)(x-1)^3 - 3(x-1)^2(x+1)^2(x-5)}{(x-1)^6} = \frac{3(x+1)(x-3)(x-1) - 3(x+1)^2(x-5)}{(x-1)^4}$$

$$= \frac{3(x+1)\left((x-3)(x-1) - (x+1)(x-5)\right)}{(x-1)^4} = \frac{24(x+1)}{(x-1)^4}$$

$- \quad -1 \quad + \quad 1 \quad +$

$y'' < 0$ при $x \in (-\infty; -1)$ $(\cap)$

$y'' > 0$ при $x \in (-1; +\infty)$ $(\cup)$



4. $y = \dfrac{x^2(x-1)}{(x+1)^2}$

I. 1) $D(y)$: $x \in (-\infty; -1) \cup (-1; +\infty)$

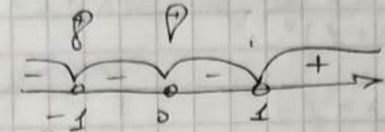
2) $y(-x) = \dfrac{x^2(-x-1)}{(-x+1)^2} \neq -y(x) \Rightarrow$ $y(x)$ - общего вида.

3) $x = 0 \to y = \dfrac{-1 \cdot 0}{1} = -1 \cdot 0 = 0$

$y = 0 \to x = 0$ или $x = 1$

$\Rightarrow (0; 0)$ и $(1; 0)$ - т. пересеч. с Ox и Oy.

4) $y = \dfrac{x^2(x-1)}{(x+1)^2} > 0$



$y > 0$ при $x \in (1; +\infty)$

$y < 0$ при $x \in (-\infty; -1) \cup (-1; 0) \cup (0; 1)$

II. $\lim\limits_{x \to +\infty} \dfrac{x^2(x-1)}{(x+1)^2} = +\infty$ $\quad \lim\limits_{x \to -\infty} \dfrac{x^2(x-1)}{(x+1)^2} = -\infty$

$\lim\limits_{x \to -1+0} \dfrac{x^2(x-1)}{(x+1)^2} = -\infty$ $\quad \lim\limits_{x \to -1-0} \dfrac{x^2(x-1)}{(x+1)^2} = -\infty$

Асимптоты: $x = -1$ - вертикальная асимптота.

$k = \lim\limits_{x \to \pm\infty} \dfrac{x^2(x-1)}{x(x+1)^2} = \lim\limits_{x \to \pm\infty} \dfrac{x^2 - x}{x^2 + 2x + 1} = \lim\limits_{x \to \pm\infty} \dfrac{x^2(1 - \frac{1}{x})}{x^2(1 + \frac{2}{x} + \frac{1}{x^2})} = 1$

$b = \lim\limits_{x \to \infty} (y - kx) = \lim\limits_{x \to \infty} \left(\dfrac{x^2(x-1)}{(x+1)^2} - x \right) = \lim\limits_{x \to \infty} \left(\dfrac{x^2(x-1) - x(x+1)^2}{(x+1)^2} \right) =$

$= \lim\limits_{x \to \infty} \dfrac{\cancel{x}(x(x-1) - x^2 - 2x - 1)}{x^{\cancel{2}}(1 + \frac{1}{x})^2} = \lim\limits_{x \to \infty} \dfrac{\cancel{x^2} - x - \cancel{x^2} - 2x - 1}{x(1 + \frac{1}{x})^2} =$

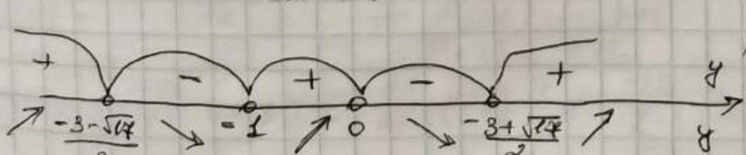
$= \lim\limits_{x \to \infty} \dfrac{-3x - 1}{x(1 + \frac{1}{x})^2} = \underline{-3}$

$y = x - 3$ - наклонная асимптота.

III $y' = \left(\dfrac{x^3 - x^2}{(x+1)^2} \right)' = \dfrac{(3x^2 - 2x)(x+1)^2 - 2(x+1)(x^3 - x^2)}{(x+1)^4} =$ $x_1 =$

$= \dfrac{(3x^2 - 2x)(x+1) - 2(x^3 - x^2)}{(x+1)^3} = \dfrac{x^3 + 3x^2 - 2x}{(x+1)^3} = \dfrac{x(x^2 + 3x - 2)}{(x+1)^3} =$

$= \dfrac{x\left(x - \left(\frac{-3 - \sqrt{17}}{2}\right)\right)\left(x - \left(\frac{-3 + \sqrt{17}}{2}\right)\right)}{(x+1)^3}$



~~$y(-1) \approx$~~ $\quad y\left(\dfrac{-3 + \sqrt{17}}{2}\right) \approx -0{,}057$; $y\left(\dfrac{-3 - \sqrt{17}}{2}\right) \approx -8{,}818$

$y(x)$ возрастает при $x \in \left(-\infty; \frac{-3-\sqrt{17}}{2}\right) \cup (-1; 0) \cup \left(\frac{-3+\sqrt{17}}{2}; +\infty\right)$

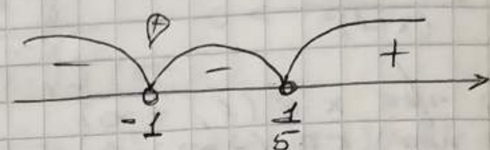
убывает при $x \in \left(\frac{-3-\sqrt{17}}{2}; -1\right) \cup \left(0; \frac{-3+\sqrt{17}}{2}\right)$

IV $y'' = \left(\frac{x^3+3x^2-2x}{(x+1)^3}\right)' = \frac{(x^3+3x^2-2x)'(x+1)^3 - 3(x+1)^2(x^3+3x^2-2x)}{(x+1)^6} =$

$= \frac{(3x^2+6x-2)(x+1) - 3(x^3+3x^2-2x)}{(x+1)^4} =$

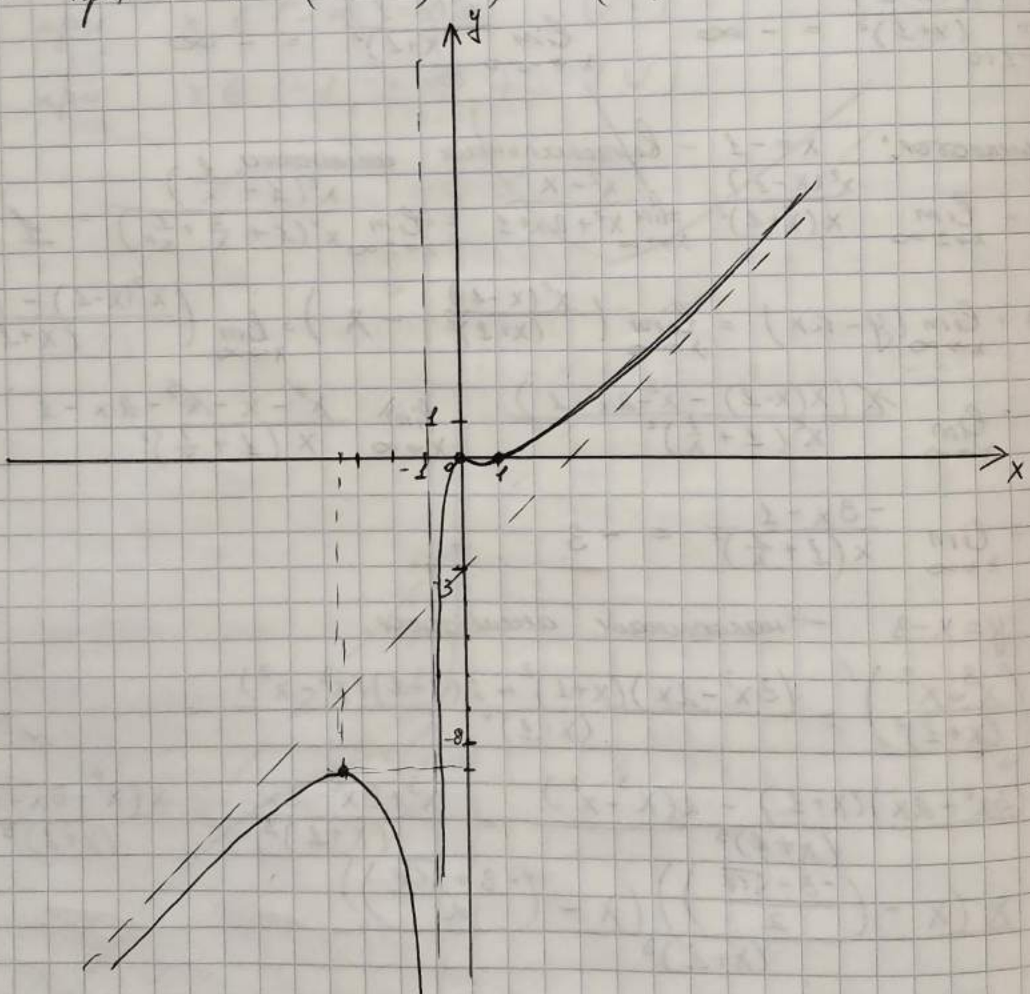
$= \frac{\cancel{3x^3}+\cancel{3x^2}+\cancel{6x^2}+6x-2x-2-\cancel{3x^3}-\cancel{9x^2}+6x}{(x+1)^4} =$

$= \frac{10x-2}{(x+1)^4} = \frac{2(5x-1)}{(x+1)^4}$



$y'' > 0$ при $x \in \left(\frac{1}{5}; +\infty\right)$ $(\cup)$

$y'' < 0$ при $x \in (-\infty; -1) \cup \left(-1; \frac{1}{5}\right)$ $(\cap)$



17.11.2020

Разложение основных функций по формуле Тейлора (по степеням x)

I. $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!} + o(x^n)$

II. $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + o(x^{2n+2})$

III. $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots + (-1)^n \frac{x^{2n}}{(2n)!} + o(x^{2n+1})$

IV. $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots + (-1)^{n-1} \frac{x^n}{n} + o(x^n)$

V. $(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!} x^2 + \ldots + \frac{\alpha(\alpha-1)\ldots(\alpha-n+1)}{n!} x^n + o(x^n)$

VI. $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots + x^n + o(x^n)$

VII. $\frac{1}{1+x} = 1 - x + x^2 - x^3 + \ldots + (-1)^n x^n + o(x^n)$

Разложения VI и VII получаются как частные случаи из V при $\alpha = -1$.

(1) Многочлен $P(x) = 1 + 3x + 5x^2 - 2x^3$ разложить по целым неотриц. степеням двучлена $(x+1)$

Пок-ть, что разложение в ряд Тейлора по степеням x совпадает с исходным видом многочлена. $(x_0 = 0)$

$x_0 = -1$

$P(x) = 1 + 3x + 5x^2 - 2x^3$; $P(x_0) = P(-1) = 1 - 3 + 5 + 2 = 5$

$P(0) = 1$.

$P'(x) = 3 + 10x - 6x^2$ $\quad P(-1) = 3 - 10 - 6 = -13$

$P(0) = 3$

$P''(x) = 10 - 12x$ $\quad P(-1) = 22$

$P(0) = 10$

$P'''(x) = -12$ $\quad \to P(-1) = -12$

$P(0) = -12$

1: $P(x) = 5 + \frac{-13}{1!}(x+1) + \frac{22}{2!}(x+1)^2 + \frac{-12}{\underbrace{2 \cdot 3}_{=3!}}(x+1)^3 = 5 - 13(x+1) + 11(x+1)^2 + -2(x+1)^3$

Метод неопред. коэфф.

$1+3x+5x^2-2x^3 = -2x^3+5x^2+3x+1 = A(x+1)^3+B(x+1)^2+C(x+1)+D$

$= x^3\cdot\underbrace{A}_{=-2} + \underbrace{x^2(\ldots)}_{5} + \underbrace{x(\ldots)}_{3} + \underbrace{(\ldots)}_{1}$

Разложить по формулам и составить систему

2: Разложение по степеням x:

$P(x) = 1 + \frac{3}{1!}\cdot x + \frac{10}{2}\cdot x^2 + \frac{(-12)}{2\cdot 3}\cdot x^3 =$

$= 1+3x+5x^2-2x^3$

(2) Написать разложение по целым неотрицательным степеням переменной x до члена с x^3 включительно для функции $f(x)=\sin(\sin x)$

$x_0 = 0$

$f(x) = \sin(\sin x)$ $\qquad f(0)=0$

$f'(x) = \cos(\sin x)\cdot\cos x$ $\qquad f'(0)=1$

$f''(x) = -\sin(\sin x)\cdot\cos^2 x + \sin x\cdot\cos(\sin x)$ $\qquad f''(0)=0$

$f'''(x) = \cos x\cdot\cos(\sin x) - \sin(\sin x)\cdot\cos x\cdot\sin x + (2\cos x\cdot(-\sin x)\cdot(-\sin(\sin x)) +$
$+ (-1)\cos(\sin x)\cdot\cos x\cdot\cos^2 x) =$

$= -\cos x\cdot\cos(\sin x) - \sin(\sin x)\cdot\cos x\cdot\sin x + 2\cos x\cdot\sin x\cdot\sin(\sin x) -$
$- \cos x\cdot\cos^2(x)\cdot\cos(\sin x)$

$f'''(0) = -1 - 0 + 0 - 1 = -2$

$\sin(\sin x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}\cdot x^3 + o(x^3)$

$\Rightarrow \sin(\sin x) = x - \frac{1}{3}x^3 + o(x^3)$

(3) $f(x) = tg\,x$

Написать разложение по целым неотрицательным степеням переменной x, до члена с x^5 включительно для функции $y = tg\,x$

$f(x) = tg\,x$ $\qquad f(0) = 0$

$$f'(x) = \frac{1}{\cos^2 x} = 1 + tg^2 x = 1 + f^2(x)$$

$$f'(0) = 1$$

$$f''(x) = 2tg x \cdot \frac{1}{\cos^2 x} = 2f(x) \cdot f'(x)$$

$$f''(0) = 0$$

$$f'''(x) = 2(f'(x))^2 + 2f(x) \cdot f''(x)$$

$$f'''(0) = 2$$

$$f^{IV}(x) = 4f'(x) \cdot f''(x) + 2f'(x) \cdot f''(x) + 2f(x) \cdot f'''(x) = 6f'(x) \cdot f''(x) + 2f(x) f'''(x)$$

$$f^{IV}(0) = 0$$

$$f^{V}(x) = \left(6f'(x) \cdot f''(x) + 2f(x) \cdot f'''(x)\right)' =$$

$$= 6\left(f''(x) \cdot f''(x) + f'(x) \cdot f'''(x)\right) + 2\left(f'(x) \cdot f'''(x) + f(x) \cdot f^{IV}(x)\right) =$$

$$= 6 \cdot (f''(x))^2 + 6f'(x) f'''(x) + 2f'(x) \cdot f'''(x) + 2f(x) f^{IV}(x) =$$

$$= 6 \cdot (f''(x))^2 + 8f'(x) f'''(x) + 2f(x) f^{IV}(x)$$

$$f^{V}(0) = 0 + 8 \cdot 2 + 0 = 16$$

$$tg\, x = 0 + \frac{1}{1!} x + \frac{2}{3!} x^3 + \frac{16}{5!} x^5 = x + \frac{1}{3} x^3 + \frac{2}{15} x^5 + o(x^5)$$

④ $$\lim_{x \to 0} \frac{\cos x - e^{-\frac{x^2}{2}}}{x^4} = \lim_{x \to 0} \frac{1}{x^4} \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} + o(x^4) - \left(1 - \frac{x^2}{2} + \frac{x^4}{4 \cdot 2!} + o(x^4)\right)\right)$$

$$= \lim_{x \to 0} \frac{1}{x^4} \left(1 - \frac{x^2}{2} + \frac{x^4}{24} + o(x^4) - 1 + \frac{x^2}{2} - \frac{x^4}{8} - o(x^4)\right) =$$

$$= \lim_{x \to 0} \frac{1}{x^4} \left(\frac{x^4}{24} - \frac{x^4}{8} + o(x^4)\right) = \lim_{x \to 0} \left(\frac{\left(\frac{1}{4!} - \frac{1}{4 \cdot 2!}\right) \cdot x^4}{x^4} + \underbrace{\frac{o(x^4)}{x^4}}_{\to 0}\right) =$$

$$= \frac{1}{24} - \frac{1}{8} = -\frac{2}{24} = -\frac{1}{12}$$

Можно исп. когда ф-я стремится к 0.

⑤ $$\lim_{x \to \infty} \left(x - x^2 \ln\left(1 + \frac{1}{x}\right)\right) =$$

$$= \lim_{x \to \infty} \left(x - x^2\left(\frac{1}{x} - \frac{1}{2x^2} + o\left(\frac{1}{x^2}\right)\right)\right) = \lim_{x \to \infty} \left(x - x + \frac{1}{2} - \underbrace{x^2 \cdot o\left(\frac{1}{x^2}\right)}_{0}\right) =$$

$$= \frac{1}{2}$$

⑥ $\lim\limits_{x\to 0}\left(\frac{1}{x}-\frac{1}{\sin x}\right)=\lim\limits_{x\to 0}\left(\frac{\sin x - x}{x\sin x}\right)=$

$$=\lim_{x\to 0}\frac{\cancel{x}-\frac{x^3}{3!}+\frac{x^5}{5!}+o(x^6)-\cancel{x}}{x\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}+o(x^6)\right)}=$$

$$=\lim_{x\to 0}\frac{-\frac{x^3}{3!}+\frac{x^5}{5!}+o(x^6)}{x^2\left(1-\frac{x^2}{3!}+\frac{x^4}{5!}+\frac{o(x^6)}{x}\right)}=\lim_{x\to 0}\frac{-\frac{x}{3!}+\frac{x^3}{5!}+\frac{o(x^6)}{x^2}}{1-\frac{x}{3!}+\frac{x^4}{5!}+\frac{o(x^6)}{x}}=0$$

Домашняя работа.

10.1 $f(x)=e^{2x-x^2}$, $f(0)=e^0=1$.

$f'(x)=(2x-x^2)'\cdot e^{2x-x^2}=(2-2x)\cdot e^{2x-x^2}=(2-2x)\cdot f(x)$

$\to f'(0)=2$

$f''(x)=(2-2x)'f(x)+f'(x)(2-2x)=-2f(x)+f'(x)(2-2x)$

$\to f''(0)=-2f(0)+f'(0)(2-0)=-2\cdot 1+2\cdot 2=2$

$f'''(x)=-2f'(x)+f''(x)(2-2x)+(2-2x)'f'(x)=$

$=-2f'(x)+f''(x)(2-2x)-2f'(x)=f''(x)(2-2x)-4f'(x)$

$\to f'''(0)=2\cdot 2-4\cdot 2=4-8=-4$

$f^{IV}(x)=f'''(x)(2-2x)+(2-2x)'f''(x)-4f''(x)=$

$=f'''(x)(2-2x)-2f''(x)-4f''(x)=f'''(x)(2-2x)-6f''(x)$

$\to f^{IV}(0)=-4\cdot 2-6\cdot 2=-8-12=-20$

$f^{V}(x)=f^{IV}(x)(2-2x)-2f'''(x)-6f'''(x)=f^{IV}(x)(2-2x)-8f'''(x)$

$\to f^{V}(0)=-20\cdot 2-8\cdot(-4)=-40+32=-8$

$\Rightarrow f(x)=1+2x+x^2-\frac{2}{3}x^3-\frac{\cancel{20}^{5}}{2\cdot 3\cdot \cancel{4}_{2}}x^4-\frac{\cancel{8}^{1}}{\cancel{2\cdot 3\cdot 4}\cdot 5}x^5+o(x^5)=$

$=1+2x+x^2-\frac{2}{3}x^3-\frac{5}{6}x^4-\frac{1}{15}x^5+o(x^5)$

10.2 $f(x)=\ln(\cos x)$

$f(0)=\ln(1)=0$

$f'(x)=\frac{1}{\cos x}\cdot(-\sin x)=-\operatorname{tg}x$

$\to f'(0)=0$

$$f''(x) = -\left(\frac{1}{\cos^2 x}\right) = -(1+tg^2 x) = -1 - tg^2 x = -1 + (f'(x))^2 =$$
$$= (f'(x))^2 - 1.$$

$$\to f''(0) = -1$$

$$f'''(x) = 2f'(x)\cdot f''(x)$$

$$\to f'''(0) = 0$$

$$f^{IV}(x) = 2(f''(x))^2 + 2f'(x)\cdot f'''(x)$$

$$\to f^{IV}(0) = 2 + 0 = 2$$

$$f^{V}(x) = \underbrace{4f''(x)\cdot f'''(x)} + \underbrace{2f''(x)\cdot f'''(x)} + 2f'(x)\, f^{IV}(x) =$$
$$= 6f''(x)\cdot f'''(x) + 2f'(x) f^{IV}(x)$$

$$\to f^{V}(0) = 0 + 0 = 0$$

$$f^{VI}(x) = \underbrace{6(f'''(x))^2}_{=0} + 6f''(x)\cdot f^{IV}(x) + 2f''(x)\, f^{IV}(x) + \underbrace{2f'(x)\, f^{V}(x)}_{=0}$$

$$f^{VI}(0) = 8\cdot(-1)\cdot 2 = -16$$

$$\Rightarrow f(x) = -\frac{1}{2}x^2 + \frac{2}{2\cdot 3\cdot 4}x^4 - \frac{16}{2\cdot 3\cdot 4\cdot 5\cdot 6}x^6 + o(x^6) =$$

$$= -\frac{x^2}{2} + \frac{x^4}{12} - \frac{x^6}{45} + o(x^6)$$

10.3 $f(x) = \sqrt{x}$

$$x_0 = 1.$$

$$f(x_0) = f(1) = 1$$

$$f'(x) = \frac{1}{2\sqrt{x}} \Rightarrow f'(1) = \frac{1}{2}$$

$$f''(x) = -\frac{2(\sqrt{x})'}{4x} = -\frac{1}{2x\cdot 2\sqrt{x}} = -\frac{1}{4x\cdot\sqrt{x}}$$

$$f''(1) = -\frac{1}{4}.$$

$$\Rightarrow f(x) = 1 + \frac{1}{2}(x-1) - \frac{1}{8}(x-1)^2 + o((x-1)^2)$$

10.4. $\lim\limits_{x\to 0} \frac{e^x + e^{-x} - 2}{x^2} = \lim\limits_{x\to 0} \frac{\left(1 + x + \frac{x^2}{2} + o(x^2)\right) + \left(1 - x + \frac{x^2}{2} + o(x^2)\right) - 2}{x^2}$

$= \lim\limits_{x\to 0} \frac{\cancel{1} + \cancel{x} + \frac{x^2}{2} + o(x^2) + o(x^2) + \frac{x^2}{2} - \cancel{x} + \cancel{1} \cancel{- 2}}{x^2} =$

$= \lim\limits_{x\to 0} \frac{x^2}{x^2} = 1.$

10.5. $\lim\limits_{x\to 0} \frac{1}{x}\left(\frac{1}{x} - \operatorname{ctg} x\right) = \lim\limits_{x\to 0} \frac{1}{x^2} - \frac{\cos x}{x \sin x} =$

$= \lim\limits_{x\to 0} \frac{\sin x - x\cos x}{x^2 \cdot \sin x} = \lim\limits_{x\to 0} \frac{\left(x - \frac{x^3}{3!} + o(x^8)\right) - x\left(1 - \frac{x^2}{2!} + o(x^5)\right)}{x^2 \cdot \sin x} =$

$= \lim\limits_{x\to 0} \frac{\cancel{x} - \frac{x^3}{6} + o(x^8) - \cancel{x} + \frac{x^3}{2} + x \cdot o(x^5)}{x^2 \cdot \sin x} =$

$= \lim\limits_{x\to 0} \frac{\frac{x^3}{3} + o(x^8) + x \cdot o(x^5)}{x^2 \cdot \sin x} = \lim\limits_{x\to 0} \frac{x^3\left(\frac{1}{3} + \frac{o(x^8)}{x^3} + \frac{o(x^5)}{x^2}\right)}{x^2 \cdot \sin x} =$

$= \frac{1}{3}$

10.6. $\lim\limits_{x\to 0} \frac{e^x \sin x - x(1+x)}{x^3} = \lim\limits_{x\to 0} \frac{e^x \sin x - x - x^2}{x^3} =$

$= \lim\limits_{x\to 0} \frac{\left(1 + x + \frac{x^2}{2} + o(x^2)\right)\left(x + o(x^4)\right) - x - x^2}{x^3} =$

$= \lim\limits_{x\to 0} \frac{x + o(x^4) + x^2 + x \cdot o(x^4) + \frac{x^3}{2} + \frac{x^2}{2} o(x^4) + x \cdot o(x^2) + o(x^2) \cdot o(x^4) - x - x^2}{x^3} =$

$= \lim\limits_{x\to 0} \frac{\left(1 + x + \frac{x^2}{2} + \frac{x^3}{6}\right) \sin x - x - x^2}{x^3} = \lim\limits_{x\to 0} =$

$= \lim\limits_{x\to 0} \frac{1 \cdot \left(x - \frac{x^3}{6}\right) - x - x^2 + o(x^8)}{x^3} =$

$= \lim\limits_{x\to 0} x - \frac{x^3}{6} - x - x^2$

10.6. $\lim\limits_{x\to 0} \frac{e^x \sin x - x(1+x)}{x^3} = \lim\limits_{x\to 0} \frac{\left(1 + x + \frac{x^2}{2} + o(x^2)\right) \cdot \sin x - x - x^2}{x^3} =$

$= \lim\limits_{x\to 0} \frac{\left(1 + x + \frac{x^2}{2} + o(x^2)\right)\left(x + o(x^4)\right) - x(1+x)}{x^3}$

10.6 $\lim\limits_{x\to 0}\dfrac{e^x\sin x - x(1+x)}{x^3} = \lim\limits_{x\to 0}\dfrac{\left(1+x+\frac{x^2}{2}+o(x^2)\right)\left(x-\frac{x^3}{6}+o(x^8)\right)-x-x^2}{x^3} =$

$$= \lim_{x\to 0}\frac{\cancel{x}-\frac{x^3}{6}+o(x^8)+\cancel{x^2}-\frac{x^4}{6}+x\cdot o(x^8)+\frac{x^3}{2}-\frac{x^5}{12}+\frac{x^2}{2}\cdot o(x^8)+x\cdot o(x^2)-\ldots-\cancel{x+x^2}}{x^3} =$$

$$= \lim_{x\to 0}\frac{-\frac{x^3}{6}+o(x^8)-\frac{x^4}{6}+x\cdot o(x^8)+\frac{x^3}{2}-\frac{x^5}{12}+\frac{x^2}{2}\cdot o(x^8)+x\cdot o(x^2)-o(x^2)\cdot\frac{x^3}{6}+o(x^2)\cdot o(x^8)}{x^3} =$$

$$= \lim_{x\to 0}\frac{\cancel{x^3}\left(-\frac{1}{6}+\overset{\to 0}{\frac{o(x^8)}{x^3}}-\overset{\to 0}{\frac{x}{6}}+\overset{\to 0}{\frac{o(x^8)}{x^2}}+\frac{1}{2}-\overset{\to 0}{\frac{x^2}{12}}+\overset{\to 0}{\frac{o(x^8)}{2x}}+\overset{\to 0}{\frac{o(x^2)}{x^2}}-\overset{\to 0}{\frac{o(x^2)}{6}}+\frac{o(x^2)\cdot o(x^8)}{x^3}\right)}{\cancel{x^3}} =$$

$$= \frac{1}{3}$$

18.11.2020г.

Приложения производных.

① Написать уравнение касательной и нормали к кривой $y=(x+1)\sqrt[3]{3-x}$

а) $A(-1;0)$

б) $B(2;3)$

в) $C(3;0)$

а) $A(-1;0) \to x_0=-1$; $f(x_0)=0$; $f'(x_0)=\sqrt[3]{4}$

$\Rightarrow y = 0+\sqrt[3]{4}(x+1) = \sqrt[3]{4}(x+1)$

$$f'(x)=\sqrt[3]{3-x}+(1+x)\cdot\frac{1\cdot(-1)}{3\sqrt[3]{(3-x)^2}}=\sqrt[3]{3-x}-\frac{x+1}{3\sqrt[3]{(3-x)^2}}=\frac{3\cdot(3-x)-x-1}{3\sqrt[3]{(3-x)^2}}=\frac{8-4x}{3\sqrt[3]{(3-x)^2}}$$

б) $B(2;3) \to x_0=2$; $f(x_0)=3$; $f'(x_0)=0 \to y=3$

в) $C(3;0) \to x_0=3$ $f(x_0)=0$; $f'(x_0)=\infty$

$\Rightarrow x=3$ - уравнение касательной.

Нормаль: $y=f(x_0)-\frac{1}{f'(x_0)}(x-x_0)$

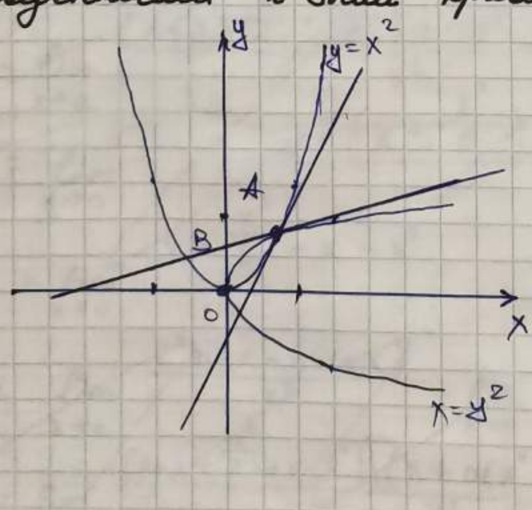
а) $y=-\frac{1}{\sqrt[3]{4}}(x+1)$

б) $x=2$

в) $y=0$

② Под какими углами пересекаются кривые $x=y^2$ и $y=x^2$

Под углом между кривыми $y=f(x)$ и $y=g(x)$ в их общей точке $M_0(x_0;y_0)$ понимается угол α между касательными, проведёнными к этим кривым в точке M_0



$y=k_1x+b_1$

$y=k_2x+b_2$

$tg\alpha=\frac{k_2-k_1}{1+k_1\cdot k_2}$

$f(x)=x^2$ $A(1;1)$

$g(x)=\sqrt{x}$ $B(0;0)$

$$tg\alpha=\frac{k_2-k_1}{1+k_1\cdot k_2}=\frac{f'(x_0)-g'(x_0)}{1+f'(x_0)\cdot g'(x_0)}=\frac{2-\frac{1}{2}}{1+2\cdot\frac{1}{2}}=\frac{3}{4};\ \alpha=arctg\frac{3}{4}$$

$f'(x_0)=2$; $g'(x_0)=\frac{1}{2}$;

$B(0;0) \to \alpha=90°$

③ Доказать неравенство: $e^x > 1+x$; при $x \neq 0$

$f(x) = e^x - x - 1$

$f'(x) = e^x - 1$

а) $x > 0$

$e^x > e^0 = 1$

$f'(x) > 0 \Rightarrow f(x)$ - возрастает при $x > 0$

$x > 0 \Rightarrow f(x) > f(0)$

$e^x - x - 1 > 0 \to \underline{e^x > x+1}$

б) $x < 0$

$f'(x) < 0 \Rightarrow f(x)$ - убывает при $x < 0$

$x < 0 \Rightarrow f(x) > f(0)$

$\underline{e^x > x+1}$

④ Док-ть неравенство: $x - \frac{x^3}{6} < \sin x < x$ при $x > 0$

1) $\sin x < x$

При $x > 1$ очевидно

$0 < x \leq 1$

$f(x) = \sin x - x$

$f'(x) = \cos x - 1 < 0$ $\quad$ - $f(x)$ - убывает.

$x > 0 \Rightarrow f(x) < f(0)$

$\sin x - x < 0$

$\sin x < x$

2) $x - \frac{x^3}{6} < \sin x$ при $x > 0$

$f(x) = x - \frac{x^3}{6} - \sin x$

$f'(x) = 1 - \frac{x^2}{2} - \cos x$

$f''(x) = \sin x - x < 0$ при $x > 0$

Т.к. $f''(x) < 0$, то $f'(x)$ - убывает

Из неравенства $x > 0 \Rightarrow f'(x) < f'(0) = 0$, т.е. $f'(x) < 0$, значит $f(x)$ - убывает.

Если $x > 0$, то $f(x) < f(0)$

$$x - \frac{x^3}{6} - \sin x < 0 \quad \text{и} \quad x - \frac{x^3}{6} < \sin x$$

(5) Найти наименьшее и наибольшее значение функции $f(x) = |x^2 - 3x + 2|$ на отрезке $[-10; 10]$.

$$f(x) = \begin{cases} x^2 - 3x + 2, & \text{если } x^2 - 3x + 2 \geq 0 \\ -x^2 + 3x - 2, & \text{если } x^2 - 3x + 2 < 0 \end{cases}$$

$$f(x) = \begin{cases} x^2 - 3x + 2, & x < 1,\ x > 2 \\ -x^2 + 3x - 2, & 1 < x < 2 \end{cases}$$

$$f'(x) = \begin{cases} 2x - 3, & x < 1,\ x > 2 \\ -2x + 3, & 1 < x < 2 \end{cases}$$

$f'(x) = 0$ при $x = \frac{3}{2}$

$x = 1$ и $x = 2$ - возможно критические точки.

$f(x) = |x-1| \cdot |x-2|$

$$f'_+(x) = \lim_{x \to 1+0} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1+0} \frac{|x-1| \cdot |x-2|}{x-1} = 1.$$

$$f'_-(x) = \lim_{x \to 1-0} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1-0} \frac{(x-1) \cdot |x-2|}{x-1} = -1$$

$x = 1$ - критическая точка $\rightarrow$

$x = 2$ - критическая точка $\rightarrow$ т.к.:

$f(1) = 0 \quad f\left(\frac{3}{2}\right) = \frac{1}{4}$

$f(2) = 0 \quad f(10) = 72$

$f(-10) = 132$

$f_{наим.} = 0$ при $x = 1,\ x = 2$

$f_{наиб.} = 132$ при $x = -10$

(6) Найти нижнюю грань ($\inf f(x)$) и верхнюю грань $\sup f(x)$

$$f(x) = \frac{1 + x^2}{1 + x^4} \quad \text{на } (0; +\infty)$$

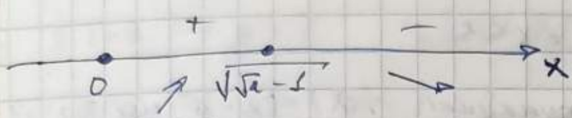
$$f'(x) = \frac{2x(1 + x^4) - 4x^3(1 + x^2)}{(1 + x^4)^2} = \frac{2x}{(1+x^4)^2} \cdot (-x^4 - 2x^2 + 1) = -\frac{2x(x^4 + 2x^2 - 1)}{(1+x^4)^2}$$

$x^4+2x^2-1 = t^2+2t-1 = (t-(\sqrt{2}-1))\cdot(t-(-1-\sqrt{2}))$

$t_{1,2} = -1 \pm \sqrt{2}$

$\ominus -\frac{2x}{(1+x^4)^2}\cdot(x^2+1+\sqrt{2})\cdot(x^2-(\sqrt{2}-1)) = -\frac{2x}{(1+x^4)^2}\cdot(x^2+1+\sqrt{2})\cdot(x-\sqrt{\sqrt{2}-1})\cdot$

$\cdot(x+\sqrt{\sqrt{2}-1})$



$x=\sqrt{\sqrt{2}-1}$ (•) max.

$f(\sqrt{\sqrt{2}-1}) = \frac{1+\sqrt{2}-1}{1+(\sqrt{2}-1)^2} = \frac{\sqrt{2}}{1+2-2\sqrt{2}+1} = \frac{\sqrt{2}+1}{2}$

$\lim_{x\to+0} f(x) = \lim_{x\to+0}\frac{1+x^2}{1+x^4} = 1$; $\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty}\frac{1+x^2}{1+x^4} = 0$

$\inf f(x) = 0$

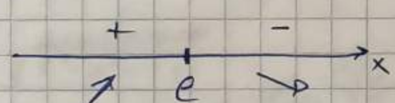
$\sup f(x) = \frac{\sqrt{2}+1}{2}$

④ Определите наибольший член последовательности $x_n = \sqrt[n]{n}$, $n\in\mathbb{N}$

$f(x) = \sqrt[x]{x}$ $\quad 1\le x<+\infty$

$f'(x) = (e^{\frac{1}{x}\ln x})' = \sqrt[x]{x}\cdot\left(-\frac{1}{x^2}\cdot\ln x+\frac{1}{x^2}\right) = -\frac{\sqrt[x]{x}}{x^2}(1-\ln x) = 0$

$f(e) = \sqrt[e]{e}$



$f(1) = 1$

$\lim_{x\to+\infty}\sqrt[x]{x} = 1$

$\inf f(x) = 1$

$\sup f(x) = \sqrt[e]{e}$

$n=2$ $\qquad n=3$

$\sqrt{2} \quad \vee \quad \sqrt[3]{3}$ ↑ в 6 степень

$8 \quad \vee \quad 9$

$8<9$

=> Наибольший член последовательности равен $\sqrt[3]{3}$

8 Доказать неравенство

$|3x - x^3| \le 2$ при $|x| \le 2$

В основе док-ва лежит тот факт, что все значения непрерывной на отрезке функции лежат между её наименьшим и наибольшим значениями:

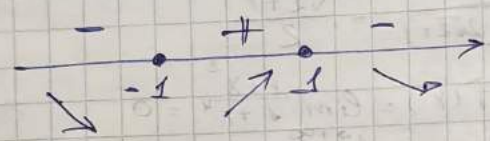
$$f_{наим} \le f(x) \le f_{наиб.}$$

$$|3x - x^3| \le 2 \text{ при } |x| \le 2$$

Надо док-ть: $-2 \le 3x - x^3 \le 2$ при $-2 \le x \le 2$

Найдём наим. и наиб. значение функции $f(x) = 3x - x^3$ на $[-2;2]$

$$f'(x) = 3 - 3x^2 = 3(1 - x^2) = 3(1-x)(1+x)$$



$x = -1$ – точка минимума
$x = 1$ – точка максимума

$f(-1) = -2$ ~ наим.
$f(1) = 2$ ~ наиб.
$f(2) = -2$ ~ наим.
$f(-2) = 2$ ~ наиб.

$$\Rightarrow \quad -2 \le 3x - x^3 \le 2$$

6

20. 11. 2020

Построение графиков иррациональных функций.

① $y=\sqrt[3]{x^3-x^2-x+1}$

I. 1) $D(y): x\in\mathbb{R}$

2) $y(-x)=\sqrt[3]{(-x)^3-(-x)^2-(-x+1)}=\sqrt[3]{-x^3-x^2+x+1}$

$\to y(x)$ - ф-я общего вида.

3) $x=0 \to y=1$. $M_1(0,1)$ - пер. с осью Oy.

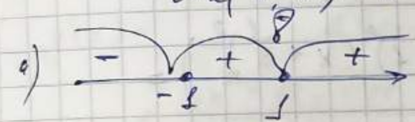
$y=0 \to x^3-x^2-x+1=0$

$x^2(x-1)-(x-1)=0$

$(x-1)(x^2-1)=0 \to (x-1)^2(x+1)=0$

$\Rightarrow x_{1,2}=\pm 1.$

$M_2(1;0)$ и $M_3(-1;0)$ - пер. с осью OX.

4) 

$y(x)<0$ при $x<-1$

$y(x)\geq 0$ при $x\geq -1$.

II $\lim\limits_{x\to+\infty}\sqrt[3]{x^3-x^2-x+1}=+\infty$; $\lim\limits_{x\to-\infty}\sqrt[3]{x^3-x^2-x+1}=-\infty$

Найдём наклонные асимптоты.

$k=\lim\limits_{x\to\infty}\frac{y}{x}=\lim\limits_{x\to\infty}\frac{\sqrt[3]{x^3-x^2-x+1}}{x}=\lim\limits_{x\to\infty}\sqrt[3]{1-\frac{1}{x}-\frac{1}{x^2}+\frac{1}{x^3}}=1.$

$b=\lim\limits_{x\to\infty}(y-kx)=\lim\limits_{x\to\infty}\left(\sqrt[3]{x^3-x^2-x+1}-x\right)=$

$=\lim\limits_{x\to\infty}\frac{\left(\sqrt[3]{x^3-x^2-x+1}-x\right)\left(\sqrt[3]{(x^3-x^2-x+1)^2}+x\cdot\sqrt[3]{x^3-x^2-x+1}+x^2\right)}{\left(\sqrt[3]{(x^3-x^2-x+1)^2}+x\sqrt[3]{x^3-x^2-x+1}+x^2\right)}=$

$=\lim\limits_{x\to\infty}\frac{\cancel{x^3}-x^2-x+1-\cancel{x^3}}{\left(\sqrt[3]{(x^3-x^2-x+1)^2}+x\sqrt[3]{x^3-x^2-x+1}+x^2\right)}=\lim\limits_{x\to\infty}=$

$=\lim\limits_{x\to\infty}\frac{-1-\frac{1}{x}+\frac{1}{x^2}}{\left(\left(\frac{\sqrt[3]{x^3-x^2-x+1}}{x}\right)^2+\frac{\sqrt[3]{x^3-x^2-x+1}}{x}+1\right)}=-\frac{1}{3}$

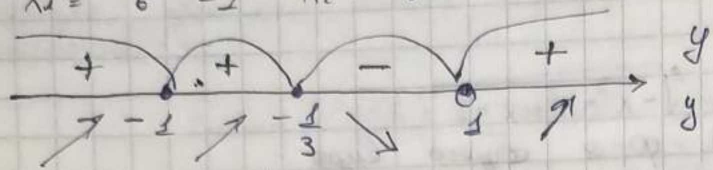
$\to 1 \qquad \to 1 \qquad \to 1$

$\Rightarrow y=x-\frac{1}{3}$ - наклонная асимптота.

III. $y' = \dfrac{1\cdot(3x^2-2x-1)}{3\sqrt[3]{(x^3-x^2-x+1)^2}} = \dfrac{\cancel{3}(x-1)(x+\frac{1}{3})}{\cancel{3}\sqrt[3]{(x^3-x^2-x+1)^2}} = \dfrac{(x-1)(x+\frac{1}{3})}{\sqrt[3]{(x^3-x^2-x+1)^2}}$

$D = b^2-4ac = 4+4\cdot 3 = 16,\ \sqrt{D}=4$

$x_1 = \dfrac{2+4}{6} = 1 \qquad x_2 = \dfrac{2-4}{6} = -\dfrac{1}{3}$



В т. 1 – произв. не существует

$\Rightarrow$ ф-я возрастает на $(-\infty; -\frac{1}{3}),\ (1; +\infty)$,
убывает на $(-\frac{1}{3}; 1)$

$y_{max} = y(-\frac{1}{3}) = \sqrt[3]{-\frac{1}{27}-\frac{1}{9}+\frac{1}{3}+1} = \sqrt[3]{\dfrac{-1-3+9+27}{27}} = \sqrt[3]{\dfrac{32}{27}} = \dfrac{2\sqrt[3]{4}}{3} \approx 1{,}08$

IV. $y'' = \dfrac{1}{3}\left(\dfrac{(6x-2)\sqrt[3]{(x^3-x^2-x+1)^2} - (3x^2-2x-1)\left(\dfrac{2\cdot(3x^2-2x-1)}{3\sqrt[3]{x^3-x^2-x+1}}\right)}{\sqrt[3]{(x^3-x^2-x+1)^4}}\right) =$

$= \dfrac{1}{3}\left(\dfrac{(6x-2)\cdot 3(x^3-x^2-x+1) - 2(3x^2-2x-1)(3x^2-2x-1)}{3\sqrt[3]{(x^3-x^2-x+1)^5}}\right) =$

$= \dfrac{1}{\cancel{3}}\left(\dfrac{\cancel{3}\cdot 6^{2}(x-\frac{1}{3})(x-1)^2(x+1) - 2\cdot\cancel{3}\cdot\cancel{3}(x-1)^2(x+\frac{1}{3})^2}{\cancel{3}\sqrt[3]{(x^3-x^2-x+1)^5}}\right) =$

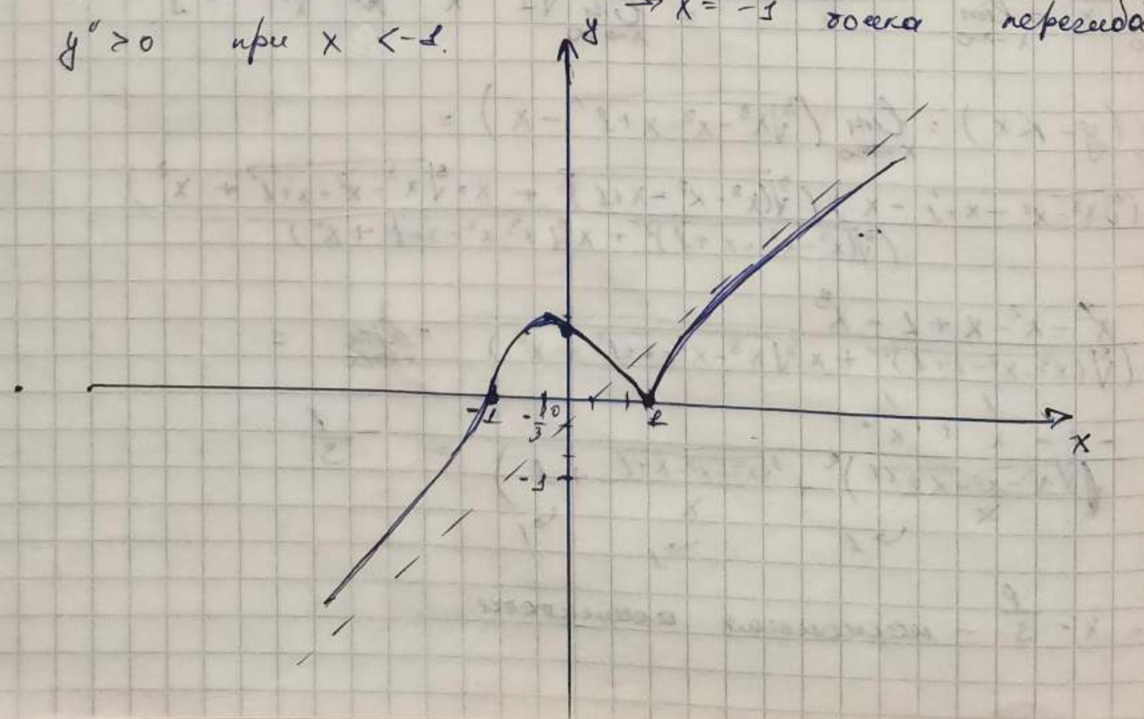
$= \dfrac{2(x-1)^2\left((x-\frac{1}{3})(x+1)-(x+\frac{1}{3})^2\right)}{\sqrt[3]{(x^3-x^2-x+1)^5}} =$

$= \dfrac{2(x-1)^2(-\frac{4}{9})}{\sqrt[3]{(x^3-x^2-x+1)^4}} = -\dfrac{8(x-1)^2}{9\sqrt[3]{(x^3-x^2-x+1)^5}} = -\dfrac{8}{9\underbrace{(\sqrt[3]{x^3-x^2-x+1})^2}_{>0}\cdot(x+1)}$

$y''<0$ при $x>-1$

$y''>0$ при $x<-1$

$\rightarrow x=-1$ точка перегиба



$y = \sqrt{(x-1)(x-2)(x-3)}$

I. 1) $D(y)$: $(x-1)(x-2)(x-3) \geq 0$

$-$ 1 $+$ 2 $-$ 3 $+$

$\Rightarrow x \in [1; 2] \cup [3; +\infty)$

2) $y(x)$ - общего вида.

3) $M_1(1; 0)$; $M_2(2; 0)$; $M_3(3; 0)$ - пер. с OX.

$x = 0 \to y = \sqrt{-1 \cdot (-2) \cdot (-3)}$ $\nexists$

4) $y \geq 0$ при $x \in [1; 2] \cup [3; +\infty)$

II. $\lim\limits_{x \to +\infty} \sqrt{(x-1)(x-2)(x-3)} = +\infty$ из пункта I

$\Rightarrow$ нет асимптот.

III. $$y'(x) = \frac{(x-2)(x-3) + (x-1)(x-3) + (x-1)(x-2)}{2\sqrt{(x-1)(x-2)(x-3)}} =$$

$$= \frac{3x^2 - 12x + 11}{2\sqrt{(x-1)(x-2)(x-3)}} = \frac{3\left(x - \left(\frac{6-\sqrt{3}}{3}\right)\right)\left(x - \left(\frac{6+\sqrt{3}}{3}\right)\right)}{2\sqrt{(x-1)(x-2)(x-3)}}$$

не сущ. 1 $+$ $\frac{6-\sqrt{3}}{2}$ $-$ 2 не сущ. $\frac{6+\sqrt{3}}{3}$ 3 $+$ y'

y: $\nearrow$ на $\left(1; \frac{6-\sqrt{3}}{2}\right)$, $\searrow$ на $\left(\frac{6-\sqrt{3}}{2}; 2\right)$, $\nearrow$ на $(3; +\infty)$

Ф-я возр. на $x \in \left(1; \frac{6-\sqrt{3}}{2}\right) \cup (3; +\infty)$,
убывает: на $x \in \left(\frac{6-\sqrt{3}}{2}; 2\right)$

$y\left(\frac{6-\sqrt{3}}{2}\right) \approx y(1{,}423) = 0{,}62$.

IV. $$y''(x) = \frac{1}{2}\left(\frac{(6x-12)\cdot\sqrt{(x-1)(x-2)(x-3)} - \frac{(3x^2-12x+11)(3x^2-12x+11)}{2\sqrt{(x-1)(x-2)(x-3)}}}{(x-1)(x-2)(x-3)}\right) =$$

$$= \frac{1}{2}\left(\frac{2(6x-12)(x-1)(x-2)(x-3) - (3x^2-12x+11)^2}{2(x-1)(x-2)(x-3)\cdot\sqrt{(x-1)(x-2)(x-3)}}\right)$$

Домашняя работа

① $y=\sqrt{x(x^2-1)}=\sqrt{x(x-1)(x+1)}$

I) 1) $D(y)$: $-$, $+$, $-$, $+$; -1, 0, 1 $\quad x(x-1)(x+1)\geq 0$

$\rightarrow x\in[-1;0]\cup[1;+\infty)$

2) $y(-x)=\sqrt{-x(x^2-1)}$

$\rightarrow y(x)$ - общего вида.

3) $M_1(-1;0)$; $M_2(0;0)$; $M_3(-1;0)$ - т. пер. с Ox, и Oy.

4) $y>0$ при $x\in(-1;0)\cup(1;+\infty)$

II) $\lim\limits_{x\to+\infty}\sqrt{x(x-1)(x+1)}=+\infty$ из пункта I следует, что асимптот нет.

$k=\lim\limits_{x\to\infty}\left(\frac{y}{x}\right)=\lim\limits_{x\to\infty}\sqrt{\frac{x(x-1)(x+1)}{x^2}}=$

III) $y'=\frac{(x-1)(x+1)+x(x+1)+x(x-1)}{2\sqrt{x(x-1)(x+1)}}=\frac{x^2-1+x^2+\cancel{x}+x^2-\cancel{x}}{2\sqrt{x(x-1)(x+1)}}=$

$=\frac{3x^2-1}{2\sqrt{x(x-1)(x+1)}}=\frac{3\left(x^2-\frac{1}{3}\right)}{2\sqrt{x(x-1)(x+1)}}=\frac{3\left(x-\frac{\sqrt3}{3}\right)\left(x+\frac{\sqrt3}{3}\right)}{2\sqrt{x(x-1)(x+1)}}$

$+$, $-$, $+$ $\quad y'$

-1, $-\frac{\sqrt3}{3}$, 0, $\frac{\sqrt3}{3}$, 1 $\quad y$

y возр. $\left(-1;-\frac{\sqrt3}{3}\right)\cup(1;+\infty$

убывает $\left(-\frac{\sqrt3}{3};0\right)$

$y\left(-\frac{\sqrt3}{3}\right)\approx 0{,}62$

IV. $y''=\frac{3}{2}\left(\frac{2x\sqrt{x(x-1)(x+1)}-\frac{(3x^2-1)^2}{2\sqrt{x(x-1)(x+1)}}}{x(x-1)(x+1)}\right)=$

$=\frac{3}{2}\left(\frac{4x\cdot x(x-1)(x+1)-(3x^2-1)^2}{2x(x-1)(x+1)\sqrt{x(x-1)(x+1)}}\right)=$

$=\frac{3}{2}\left(\frac{4x^2(x^2-1)-9x^4+6x^2-1}{2x(x-1)(x+1)\sqrt{x(x-1)(x+1)}}\right)=$

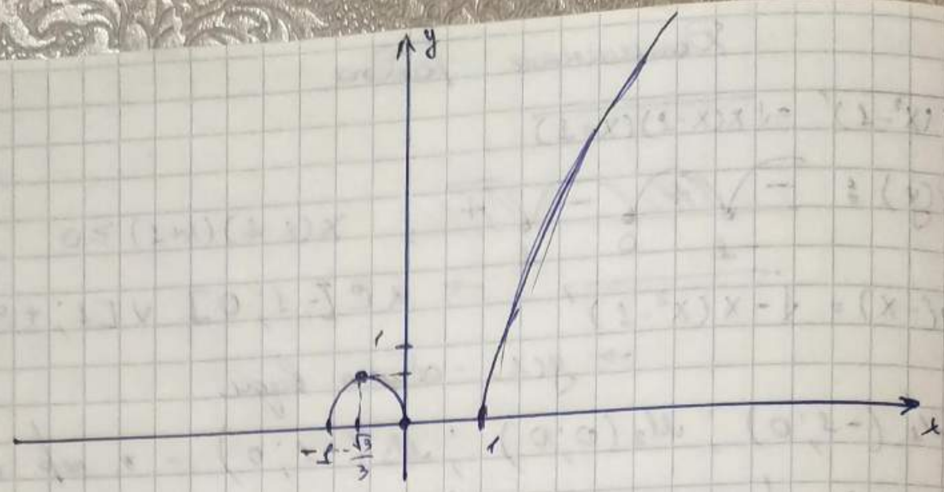
$=\frac{3}{2}\left(\frac{4x^4-4x^2-9x^4+6x^2-1}{2x(x-1)(x+1)\sqrt{x(x-1)(x+1)}}\right)=\frac{3}{2}\left(\frac{-5x^4+2x^2-1}{2x(x-1)(x+1)\sqrt{x(x-1)(x+1)}}\right)=$

$=\frac{-3(5x^4-2x^2+1)}{4x(x-1)(x+1)\sqrt{x(x-1)(x+1)}}$

>0 всегда.

$-$, $-$ $\quad y''$

-1, 0, 1 $\quad \frown$, $\frown$



② $y = \sqrt{x^2(x+3)}$

I. 1) $D(y)$: $x^2(x+3) \geq 0$

$x \in (-3; \infty)$

2) $y(-x) = \sqrt{x^2(-x+3)} \to y(x)$ - общего вида

3) $M_1(0;0)$; $M_2(-3;0)$ - точки пересечения с OX и Oy.

4) $y > 0$ при $x \in (-3; 0) \cup (0; +\infty)$

II. $\lim\limits_{x \to +\infty} \sqrt{x^2(x+3)} = +\infty$ → из пункта I можно сделать вывод, что асимптот нет.

III. $y' = \dfrac{x^2 + 2x(x+3)}{2\sqrt{x^2(x+3)}} = \dfrac{x(x+2x+6)}{2\sqrt{x^2(x+3)}} = \dfrac{x(3x+6)}{2\sqrt{x^2(x+3)}} = \dfrac{3x^2+6x}{2\sqrt{x^2(x+3)}}$

$= \dfrac{3x(x+2)}{2\sqrt{x^2(x+3)}}$

$y(x)$ возр. на $(-3; -2) \cup (0; +\infty)$
убыв на $(-2; 0)$

$f(-2) = 2$; $f(0) = 0$

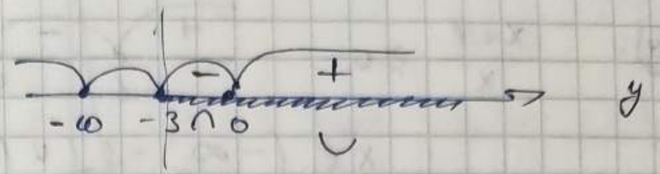
IV. $y'' = \dfrac{3}{2}\left(\dfrac{(x^2+2x)'\sqrt{x^2(x+3)} - \left(\sqrt{x^2(x+3)}\right)'(x^2+2x)}{x^2(x+3)}\right) =$

$= \dfrac{3}{2}\left(\dfrac{(2x+2)\sqrt{x^2(x+3)} - \dfrac{3x(x+2)(x^2+2x)}{2\sqrt{x^2(x+3)}}}{x^2(x+3)}\right) =$

$= \dfrac{3}{2}\left(\dfrac{4(x+1)\cdot x^2(x+3) - 3x^2(x+2)^2}{2x^2(x+3)\sqrt{x^2(x+3)}}\right) =$

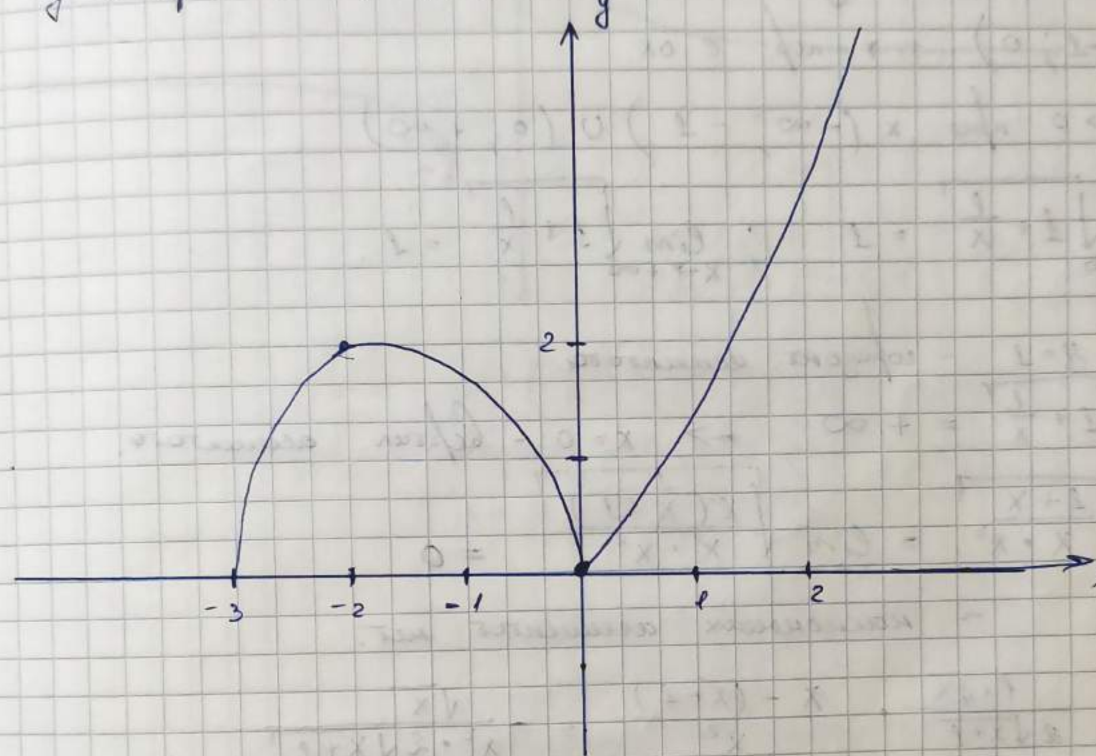
$$= \frac{3}{2}\left(\frac{4(x+8)(x+3) - 3(x^2+2x+4)}{2(x+3)\sqrt{x^2(x+3)}}\right) = \frac{3(x^2+10x)}{2\cdot 2(x+3)\sqrt{x^2(x+3)}} =$$

$$= \frac{3x(x+10)}{4(x+3)\sqrt{x^2(x+8)}}$$



$y'' > 0$ при $x > 0$

$y'' < 0$ при $x \in (-3; 0)$



③ $y=\sqrt{1+\frac{1}{x}}=\sqrt{\frac{x+1}{x}}$

I. 1) $D(y)$: $\frac{x+1}{x}\ge 0$

$+ \quad -1 \quad - \quad 0 \quad +$

$x\in(-\infty;-1]\cup(0;+\infty)$

2) $y(x)$ – общего вида.

3) $(-1;0)$ – т. пер. с Ox.

4) $y>0$ при $x\,(-\infty;-1)\cup(0;+\infty)$

II. $\lim\limits_{x\to-\infty}\sqrt{1+\frac{1}{x}}=1$; $\lim\limits_{x\to+\infty}\sqrt{1+\frac{1}{x}}=1$.

$\Rightarrow$ $y=1$ – горизонт. асимптота.

$\lim\limits_{x\to+0}\sqrt{1+\frac{1}{x}}=+\infty$ $\Rightarrow$ $x=0$ – вертик. асимптота.

$k=\lim\limits_{x\to\infty}\sqrt{\frac{1+x}{x\cdot x^2}}=\lim\limits_{x\to\infty}\sqrt{\frac{x\left(\frac{1}{x}+1\right)}{x\cdot x^2}}=0$

~ наклонных асимптот нет.

III. $y'=\frac{1\cdot\sqrt{x}}{2\sqrt{x+1}}\cdot\frac{x-(x+1)}{x^2}=-\frac{\sqrt{x}}{x^2\cdot 2\sqrt{x+1}}$

$\Rightarrow$ убывает на всей области определ

IV. $y''=\left(-\frac{1}{2}\cdot\frac{1}{\sqrt{x+1}\cdot x^{\frac{3}{2}}}\right)'=-\frac{1}{2}\cdot\left(-\frac{1}{x^3(x+1)}\right)\cdot\left(\frac{1}{2\sqrt{x+1}}\cdot x^{\frac{3}{2}}+\frac{3}{2}\cdot\sqrt{x}\sqrt{x+1}\right.$

$=\frac{1}{2x^3(x+1)}\cdot\left(\frac{x\cdot\sqrt{x}}{2\sqrt{x+1}}+\frac{3\cdot\sqrt{x}\cdot\sqrt{x+1}}{2}\right)=$

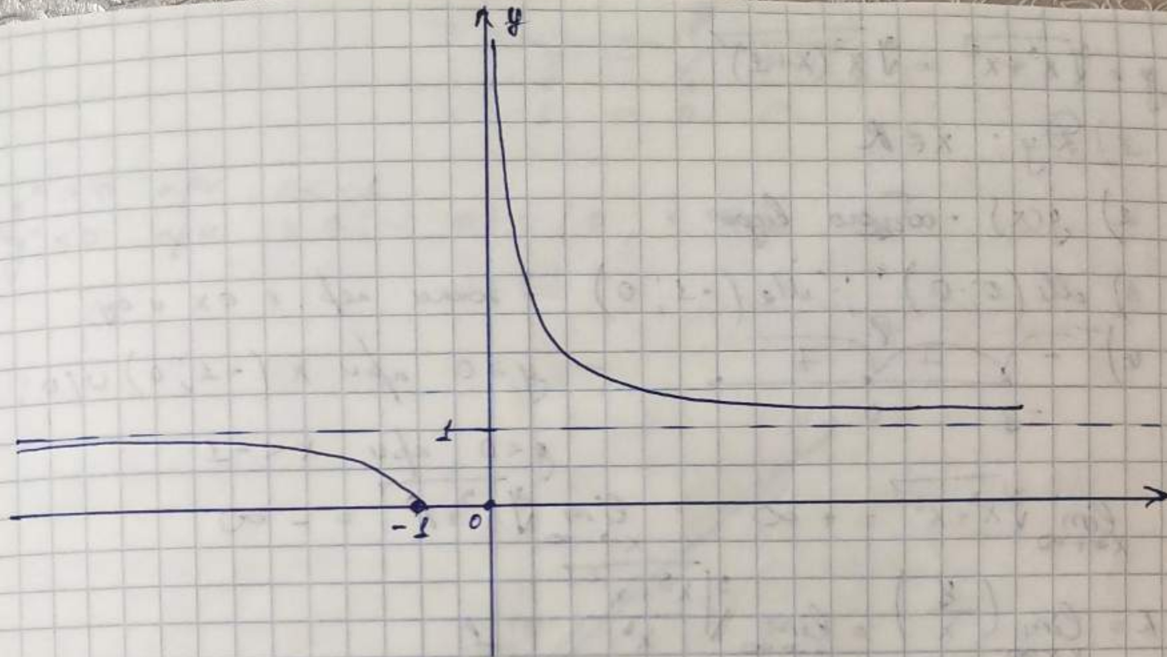
$=\frac{\sqrt{x}}{2x^3(x+1)}\cdot\left(\frac{x+3(x+1)}{2\sqrt{x+1}}\right)=\frac{\sqrt{x}\,(4x+1)}{4x^3(x+1)\sqrt{x+1}}=$

$=\frac{\left(x+\frac{1}{4}\right)}{x^{\frac{5}{2}}(x+1)^{\frac{3}{2}}}$

$- \quad \frown \quad -1 \quad -\frac{1}{4} \quad 0 \quad + \quad \smile \quad y''$

$y''<0$ при $x<-1$

$y''>0$ при $x>0$


y
1
-1
0

④ $y = \sqrt[3]{x^3+x^2} = \sqrt[3]{x^2(x+1)}$

I. 1) D_y: $x \in R$

2) $y(x)$ - общего вида

3) $M_1(0;0)$; $M_2(-1;0)$ - точки пер. с Ox и Oy.

4) $-$ (-1) $+$ (0) $+$ $\Rightarrow$ $y>0$ при $x\,(-1;0)\cup(0;+\infty)$

$y<0$ при $x<-1$

II. $\lim\limits_{x\to+\infty}\sqrt[3]{x^3+x^2} = +\infty$; $\lim\limits_{x\to-\infty}\sqrt[3]{x^3+x^2} = -\infty$

$$k = \lim_{x\to\infty}\left(\frac{y}{x}\right) = \lim_{x\to\infty}\sqrt[3]{\frac{x^3+x^2}{x^3}} = 1.$$

$$b = \lim_{x\to\infty}(y-kx) = \lim_{x\to\infty}\left(\sqrt[3]{x^3+x^2}-x\right) =$$

$$= \lim_{x\to\infty}\frac{\left(\sqrt[3]{x^3+x^2}-x\right)\left(\sqrt[3]{(x^3+x^2)^2}+\sqrt[3]{x^3+x^2}\,x+x^2\right)}{\left(\sqrt[3]{(x^3+x^2)^2}+\sqrt[3]{x^3+x^2}\,x+x^2\right)} =$$

$$= \lim_{x\to\infty}\frac{\cancel{x^3}+x^2-\cancel{x^3}}{\sqrt[3]{(x^3+x^2)^2}+\sqrt[3]{x^3+x^2}\cdot x+x^2)} = \frac{1}{3}$$

$\Rightarrow$ $y = x+\frac{1}{3}$ - наклонная асимптота.

III. $$y' = \left((x^3+x^2)^{\frac{1}{3}}\right)' = \frac{3x^2+2x}{3\sqrt[3]{(x^3+x^2)^2}} = \frac{x\left(x+\frac{2}{3}\right)}{\sqrt[3]{x^2(x+1)}} =$$

$$= \frac{x\left(x+\frac{2}{3}\right)}{x^{\frac{2}{3}}(x+1)^{\frac{2}{3}}} = \frac{\sqrt[3]{x}\,\left(x+\frac{2}{3}\right)}{\sqrt[3]{(x+1)^2}}$$

$+$ (-1) $+$ $\left(-\frac{2}{3}\right)$ $-$ (0) $+$ $\quad y'$

$\nearrow$ $\searrow$ $\nearrow$ $\quad y$

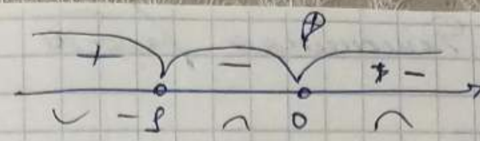
$f\left(-\frac{2}{3}\right) = \frac{\sqrt[3]{4}}{3}$

$y(x)$ возр. при $x\in\left(-\infty;-\frac{2}{3}\right)$
убыв при $x\in(0;+\infty)$

IV. $$y'' = \frac{1}{3}\left(\frac{(6x+2)\cdot\sqrt[3]{(x^3+x^2)^2}-\frac{2\cdot(3x^2+2x)^2}{3\sqrt[3]{(x^3+x^2)}}}{\sqrt[3]{(x^3+x^2)^4}}\right) =$$

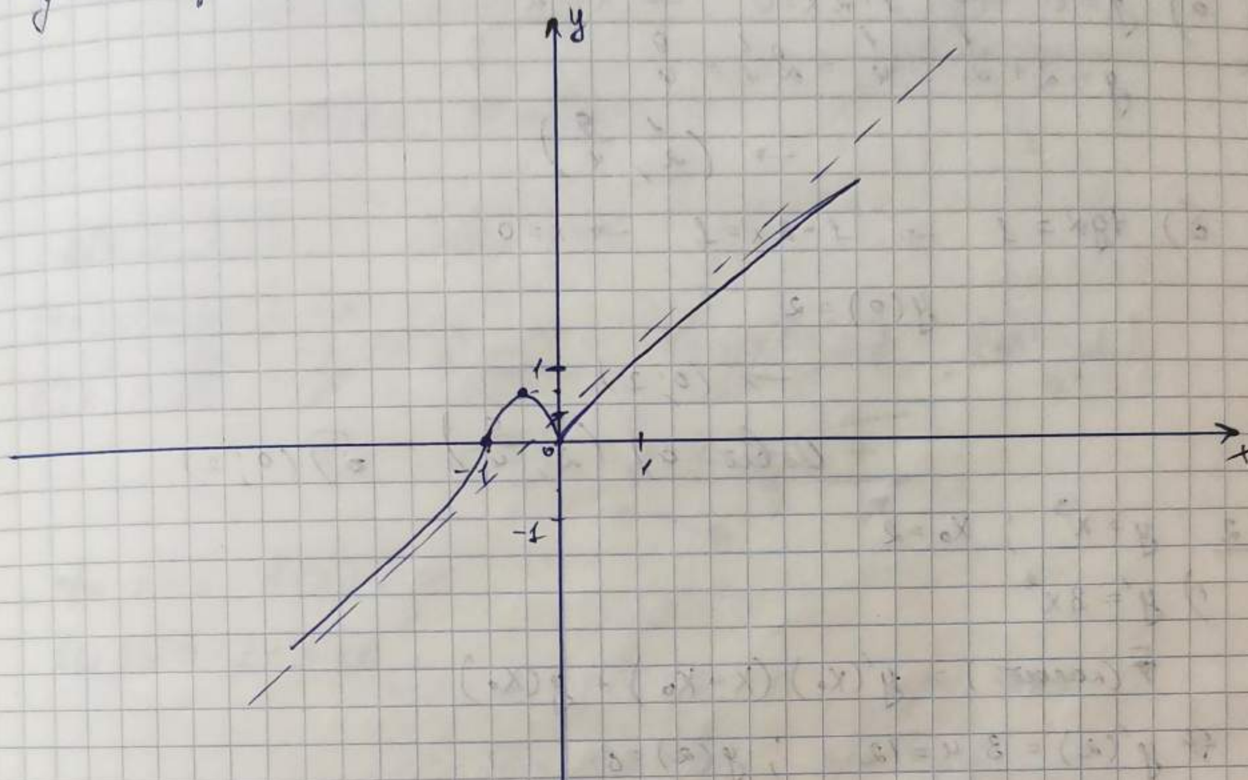
$$= \frac{1}{3}\left(\frac{3(6x+2)(x^3+x^2)-2\cdot(3x^2+2x)^2}{3\sqrt[3]{(x^3+x^2)^5}}\right) = \frac{-2x^2}{9\sqrt[3]{(x^3+x^2)^5}} =$$

$$= -\frac{2}{9}\,\frac{x^2}{\sqrt[3]{(x^2(x+1))^5}} = -\frac{2}{9}\,\frac{x^2}{x^{\frac{10}{3}}(x+1)^{\frac{5}{3}}} = -\frac{2}{9\,x^{\frac{4}{3}}(x+1)^{\frac{5}{3}}}$$



$y'' > 0$ при $x < -1$

$y'' < 0$ при $x \in (-1; 0) \cup (0; +\infty)$



Домашняя работа.

11.1 $y = 2 + x - x^2$

$y' = 1 - 2x$

а) $y' = 0 \to 1 - 2x = 0 \to x = \frac{1}{2}$

$y = 2 + \frac{1}{2} - \frac{1}{4} = 2\frac{1}{4} = \frac{9}{4}$

$\to \left(\frac{1}{2}; \frac{9}{4}\right)$

б) $tg\,\alpha = 1 \to 1 - 2x = 1 \to x = 0$

$y(0) = 2$

$\to (0; 2)$

Ответ: а) $\left(\frac{1}{2}; \frac{9}{4}\right)$ б) $(0; 2)$

11.2 $y = x^3$; $x_0 = 2$

1) $y' = 3x^2$

$f(\text{касат.}) = y'(x_0)(x - x_0) + y(x_0)$

$y'(2) = 3 \cdot 4 = 12$; $y(2) = 8$

$\Rightarrow f = 12(x - 2) + 8 = 12x - 24 + 8 = 12x - 16$

$12x - y - 16 = 0$

2) $y(\text{нормаль}) = -\frac{1}{12}(x - 2) + 8 = -\frac{x}{12} + \frac{1}{6} + 8$

$\frac{x}{12} + y - 8 - \frac{1}{6} \quad | \cdot 12$

$x + 12y - 98 = 0$

Ответ: $12x - y - 16 = 0$ и $x + 12y - 98 = 0$

11.3. $y_1 = \frac{1}{x}$; $y_2 = \sqrt{x}$; $x > 0$

$\frac{1}{x} = \sqrt{x}, x > 0$; $\frac{1}{x^2} = x \quad | \cdot x^2$

$1 = x^3 \Rightarrow x^3 = 1$

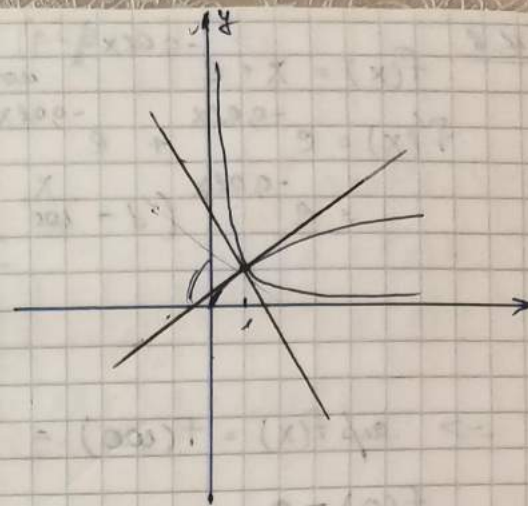
$\Rightarrow x = 1$; $x_0 = 1$.

$y_1' = -\frac{1}{x^2}$; $y_2' = \frac{1}{2\sqrt{x}}$

$k_1 = -1$; $k_2 = \frac{1}{2}$

$$tg\alpha = \frac{\frac{1}{2}+1}{1-\frac{1}{2}} = \frac{3\cdot 2}{2\cdot 1} = 3$$

$\Rightarrow \alpha = arctg(3)$

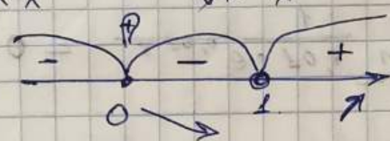


Ответ: $\alpha = arctg(3)$

11.4 $2\sqrt{x} > 3 - \frac{1}{x}$ при $x > 1$

$f(x) = 2\sqrt{x} + \frac{1}{x} - 3$

$$f'(x) = \frac{1}{\sqrt{x}} - \frac{1}{x^2} = \frac{x^2 - \sqrt{x}}{\sqrt{x}\,x^2} = \frac{\sqrt{x}\,(x^{\frac{3}{2}} - 1)}{\sqrt{x}\,x^2} = \frac{(\sqrt{x^3} - 1)}{x^2}$$



при $x > 1$ $f(x) \uparrow$

$f(1) = 2 + 1 - 3 = 0$ $\Rightarrow 2\sqrt{x} + \frac{1}{x} - 3 > 0$ при $x > 1$

$\Rightarrow 2\sqrt{x} > 3 - \frac{1}{x}$ при $x > 1$

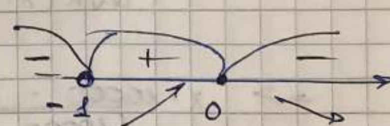
что и треб. док-ть.

11.5 $x - \frac{x^2}{2} < \ln(1+x) < x$ при $x > 0$

1) $\ln(1+x) < x$ при $x > 0$

$f(x) = \ln(1+x) - x$

$$f'(x) = \frac{1}{1+x} - 1 = \frac{1 - 1 - x}{1+x} \Rightarrow -\frac{x}{1+x}$$



при $x > 0$ $f(x) \searrow$

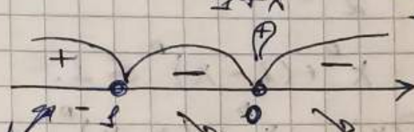
$f(0) = \ln 1 - 0 = 0$ $\Rightarrow \ln(1+x) < x$ при $x > 0$

2) $x - \frac{x^2}{2} < \ln(1+x)$ при $x > 0$

$f(x) = x - \frac{x^2}{2} - \ln(1+x)$

$$f'(x) = 1 - x - \frac{1}{1+x} = \frac{1 + x - x(1+x) - 1}{1+x} = \frac{x - x - x^2}{1+x} = -\frac{x^2}{1+x}$$

при $x > 0$ $f(x) \searrow$



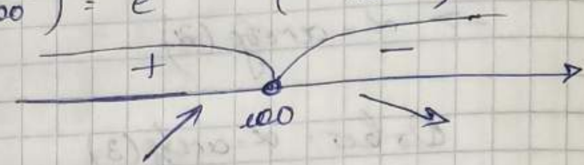
$f(0) = 0$ $\Rightarrow x - \frac{x^2}{2} - \ln(1+x) < 0$ при $x > 0$

$\Rightarrow$ учитывая пункт 1. $\Rightarrow x - \frac{x^2}{2} < \ln(1+x) < x$ при $x > 0$

что и т.д.

11.6 $f(x) = x \cdot e^{-0{,}01x}$ на инт. $(0; +\infty)$

$$f'(x) = e^{-0{,}01x} + e^{-0{,}01x} \cdot (-0{,}01)x =$$
$$= e^{-0{,}01x}\left(1 - \frac{x}{100}\right) = e^{-0{,}01x}\left(\frac{100-x}{100}\right)$$



$$\Rightarrow \sup f(x) = f(100) = \frac{100}{e}$$

$$f(0) = 0$$

$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} x \cdot e^{-0{,}01x} = \lim_{x\to+\infty} \frac{x}{e^{0{,}01x}} = \left(\frac{\infty}{\infty}\right) =$$
$$= \lim_{x\to+\infty} \frac{1}{0{,}01 \cdot e^{0{,}01x}} = 0$$

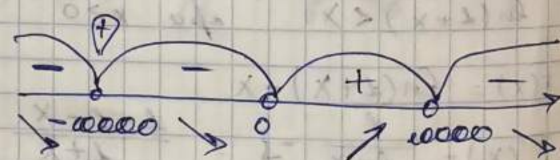
$$\Rightarrow \inf f(x) = 0$$

Ответ: $0; \ \frac{100}{e}$

11.7. $x_n = \frac{\sqrt{n}}{n+10000}$ $(n = 1, 2, 3 \dots)$

$$f(x) = \frac{\sqrt{x}}{x+10000}$$

$$f'(x) = \frac{\frac{1}{2\sqrt{x}}(x+10000) - \sqrt{x}}{(x+10000)^2} = \frac{(x+10000) - 2x}{2\sqrt{x}(x+10000)^2} =$$
$$= \frac{-x+10000}{2\sqrt{x}(x+100000)^2}$$



$\Rightarrow x = 10000$ — т. макс.

$$f(x) = \frac{\sqrt{10000}}{10000+10000} = \frac{100}{20000} = \frac{1}{200}$$

$$\Rightarrow x_{n\,max} = \frac{1}{200}$$

Ответ: $\frac{1}{200}$

11.8. $\frac{1}{2^{p-1}} \le x^p + (1-x)^p \le 1$, если $0 \le x \le 1$ и $p > 1$.

1) $x^p + (1-x)^p \le 1$.

$$f(x) = x^p + (1-x)^p - 1.$$

$$f'(x) = px^{p-1} - p(1-x)^{p-1}$$

$\dots$ $f^{(n)}(x) = p(p-1)\dots(p-p+1) + (-1)^p\dots$

$\cdot (p-2) \dots (p-p+1)$

$f^{(n)}(x) = p! + (-1)^p p!$

н.9. $\frac{2}{3} \le \frac{x^2+1}{x^2+x+1} \le 2$ при $-\infty < x < +\infty$

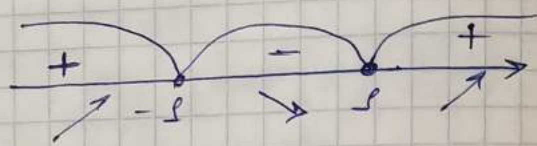
$x \in (-\infty; +\infty)$

1) $\frac{x^2+1}{x^2+x+1} \le 2$

$$f(x) = \frac{x^2+1}{x^2+x+1} - 2 = \frac{x^2+1-2x^2-2x-2}{x^2+x+1} = \frac{-x^2-2x-1}{x^2+x+1}$$

$$f'(x) = \frac{2x(x^2+x+1)-(2x+1)(x^2+1)}{(x^2+x+1)^2} =$$

$$= \frac{\cancel{2x^3}+2x^2+\cancel{2x}-\cancel{2x^3}-\cancel{2x}-x^2-1}{(x^2+x+1)^2} = \frac{x^2-1}{(x^2+x+1)^2} = \frac{(x-1)(x+1)}{(x^2+x+1)^2}$$



$$f(-1) = \frac{-1+2-1}{1-1+1} = 0 \le 0$$

$$f(1) = \frac{-1-2-1}{3} = -\frac{4}{3}$$

$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} \frac{\cancel{x^2}\left(-1-\frac{2}{x}-\frac{1}{x^2}\right)}{\cancel{x^2}\left(1+\frac{1}{x}+\frac{1}{x^2}\right)} = -1. < 0$$

$$\lim_{x\to-\infty} f(x) = -1 < 0 \Rightarrow \frac{x^2+1}{x^2+x+1} - 2 \le 0$$

$$\Rightarrow \frac{x^2+1}{x^2+x+1} \le 2$$

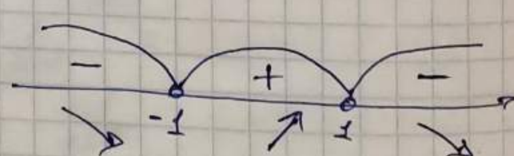
2) $\frac{2}{3} \le \frac{x^2+1}{x^2+x+1}$

$$f(x) = \frac{2}{3} - \frac{x^2+1}{x^2+x+1} = \frac{2(x^2+x+1)-3(x^2+1)}{3(x^2+x+1)} = \frac{2x^2+2x+2-3x^2-3}{3(x^2+x+1)} =$$

$$= \frac{-x^2+2x-1}{3(x^2+x+1)}$$

$$f'(x) = -\frac{2x(x^2+x+1)-(2x+1)(x^2+1)}{(x^2+x+1)^2} = -\frac{x^2-1}{(x^2+x+1)^2} =$$

$$= \frac{-(x-1)(x+1)}{(x^2+x+1)^2}$$



$$f(-1) = \frac{-1-2-1}{3(1-1+1)} = -\frac{4}{3} < 0$$

$$f(1) = \frac{-1+2-1}{3(\cdot 3)} = 0 \le 0$$

$$\lim_{x\to+\infty} \frac{\cancel{x^2}\left(-1+\frac{2}{x}-\frac{1}{x^2}\right)}{\cancel{x^2}\cdot 3\left(1+\frac{1}{x}+\frac{1}{x^2}\right)} = -\frac{1}{3} < 0$$

$$\lim_{x \to -\infty} f(x) = -\frac{1}{3} < 0$$

$$\Rightarrow \quad \frac{2}{3} - \frac{x^2+1}{x^2+x+1} \le 0$$

$$\frac{2}{3} \le \frac{x^2+1}{x^2+x+1}$$

$$\Rightarrow \quad \frac{2}{3} \le \frac{x^2+1}{x^2+x+1} \le 2$$, что и т.д.

24.11.2020

Интегрирование с помощью таблицы.

① $\int x^{\alpha} dx = \frac{x^{\alpha+1}}{\alpha+1} + C$

$\alpha \neq -1.$

② $\int \frac{dx}{x} = \ln|x| + C$

③ $\int dx = x + C$

④ $\int a^x dx = \frac{a^x}{\ln a} + C$

$\int e^x dx = e^x + C$

⑤ $\int \sin x \cdot dx = -\cos x + C$

⑥ $\int \cos x \cdot dx = \sin x + C$

⑦ $\int \frac{dx}{\cos^2 x} = \operatorname{tg} x + C$

⑧ $\int \frac{dx}{\sin^2 x} = -\operatorname{ctg} x + C$

⑨ $\int \operatorname{sh} x \cdot dx = \operatorname{ch} x + C$

⑩ $\int \operatorname{ch} x \cdot dx = \operatorname{sh} x + C$

⑪ $\int \frac{dx}{\operatorname{ch}^2 x} = \operatorname{th} x + C$

⑫ $\int \frac{dx}{\operatorname{sh}^2 x} = -\operatorname{cth} x + C$

⑬ $\int \frac{dx}{\sqrt{1-x^2}} = \arcsin x + C$

⑭ $\int \frac{dx}{1+x^2} = \operatorname{arctg} x + C$

①. $\int (3-x^2)^3 dx = \int (27 - 27x^2 + 9x^4 - x^6)\, dx = \int 27 dx - \int 27x^2 \cdot dx + \int 9x^4 dx - \int x^6 dx =$

$\{(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3\}$

$= 27\int dx - 27\int x^2 dx + 9\int x^4 dx - \int x^6 dx =$

$= 27 \cdot x - 27 \cdot \frac{x^3}{3} + 9 \cdot \frac{x^5}{5} - \frac{x^7}{7} + C =$

$= 27x - 9x^3 + \frac{9}{5}x^5 - \frac{x^7}{7} + C.$

② $\int \left(\frac{a}{x} + \frac{a^2}{x^2} + \frac{a^3}{x^3}\right) dx = a\int \frac{1}{x} \cdot dx + a^2 \int (x^{-2})\, dx + a^3 \int (x^{-3})\, dx =$

$= a \ln|x| - \frac{a^2}{x} - \frac{a^3}{2x^2} + C$

③ $\int \frac{\sqrt{x} - 2\sqrt[3]{x^2} + 1}{\sqrt[4]{x}} dx = \int \frac{\sqrt{x}}{\sqrt[4]{x}} dx - 2\int \frac{\sqrt[3]{x^2}}{\sqrt[4]{x}} \cdot dx + \int \frac{dx}{\sqrt[4]{x}} =$

$= \int x^{\frac{1}{2}-\frac{1}{4}} dx - 2\int x^{\frac{2}{3}-\frac{1}{4}} dx + \int x^{-\frac{1}{4}} dx =$

$= \int x^{\frac{1}{4}} dx - 2\int x^{\frac{5}{12}} dx + \int x^{-\frac{1}{4}} dx =$

$= \frac{x^{\frac{1}{4}+1}}{\frac{5}{4}} - \frac{2\cdot x^{\frac{5}{12}+1}}{\frac{17}{12}} + \frac{x^{\frac{3}{4}}}{\frac{3}{4}} + C = \frac{4}{5}x^{\frac{5}{4}} - \frac{24}{17}x^{\frac{17}{12}} + \frac{4}{3}x^{\frac{3}{4}} + C =$

$= \frac{4}{5}x\sqrt[4]{x} - \frac{24}{17}x\sqrt[12]{x^5} + \frac{4}{3}\sqrt[4]{x^3} + C$

(1) $\int \frac{x^2 dx}{1+x^2} = \int \frac{x^2+1-1}{1+x^2} \cdot dx = \int\left(1 - \frac{1}{1+x^2}\right) dx = \int dx - \int \frac{dx}{1+x^2} =$

$= x - \operatorname{arctg} x + C$

(5) $\int \frac{2^{x+1} - 5^{x-1}}{10^x} dx = \int \frac{2^{x+1}}{10^x} dx - \int \frac{5^{x-1}}{10^x} dx = 2\int\left(\frac{2}{10}\right)^x dx - \frac{1}{5}\int\left(\frac{5}{10}\right)^x dx =$

$= 2\left(\frac{1}{5}\right)^x \cdot \frac{1}{\ln\frac{1}{5}} - \frac{1}{5}\left(\frac{1}{2}\right)^x \cdot \frac{1}{\ln\frac{1}{2}} + C =$

$= -\frac{2}{5^x \ln 5} + \frac{1}{5\cdot 2^x \ln 2} + C$

(6) $\int \frac{e^{3x}+1}{e^x+1} \cdot dx = \int \frac{(e^x+1)(e^{2x} - e^x + 1)}{(e^x+1)} \cdot dx = \int(e^{2x} - e^x + 1) \cdot dx =$

$= \int e^{2x} dx - \int e^x dx + \int dx = \int (\underset{a}{\underbrace{e^2}})^x dx - e^x + x =$

$= \frac{e^{2x}}{\ln e^2} - e^x + x + C = \frac{e^{2x}}{2} - e^x + x + C$

(7) $\int tg^2 x\, dx = \int\left(\frac{1}{\cos^2 x} - 1\right) \cdot dx = tg\, x - x + C$

(8) $\int th^2 x \cdot dx = \int\left(1 - \frac{1}{ch^2 x}\right) dx = \int dx - \int \frac{1\, dx}{ch^2 x} =$

$= x - th\, x + C$

Если $\int f(x)\, dx = F(x) + C$, то

$$\int f(ax+b)\, dx = \frac{F(ax+b)}{a} + C.$$

(9) $\int \frac{dx}{\sqrt{2-5x}} = \frac{(2-5x)^{-\frac{1}{2}+1}}{\left(-\frac{1}{2}+1\right)\cdot(-5)} + C = -\frac{\sqrt{2-5x} \cdot 5}{2} + C$

(10) $\int \frac{\sqrt[5]{1-2x+x^2}}{1-x}\,dx = \int \frac{\sqrt[5]{(1-x)^2}}{(1-x)}\cdot dx = \int (1-x)^{-\frac{3}{5}}\,dx =$

$= \frac{(1-x)^{\frac{2}{5}}}{(-1)\cdot(-\frac{3}{5}+1)} + C = -\frac{5}{2}(1-x)^{\frac{2}{5}} + C$

(11) $\int \frac{dx}{(2+3x^2)} = \int \frac{dx}{2(1+\frac{3}{2}x^2)} = \frac{1}{2}\int \frac{dx}{1+(\sqrt{\frac{3}{2}}\cdot x)^2} = \frac{1}{2}\cdot\frac{\operatorname{arctg}(x\cdot\sqrt{\frac{3}{2}})}{\sqrt{\frac{3}{2}}}$

$= \frac{1}{\sqrt{6}}\operatorname{arctg}\left(x\sqrt{\frac{3}{2}}\right) + C$

(12) $\int \frac{dx}{\sqrt{2-3x^2}} = \int \frac{dx}{\sqrt{2(1-\frac{3}{2}x^2)}} = \frac{1}{\sqrt{2}}\int \frac{dx}{\sqrt{1-(\frac{\sqrt{3}}{\sqrt{2}}x)^2}} =$

$= \frac{1}{\sqrt{2}}\cdot\frac{\arcsin(x\cdot\sqrt{\frac{3}{2}})}{\sqrt{\frac{3}{2}}} + C = \frac{1}{\sqrt{3}}\cdot\arcsin\left(x\sqrt{\frac{3}{2}}\right) + C$

(13) $\int \frac{dx}{1-\cos x} = \int \frac{dx}{2\sin^2\frac{x}{2}} = \frac{1}{2}\cdot\frac{-\operatorname{ctg}\frac{x}{2}}{\frac{1}{2}} + C = -\operatorname{ctg}\frac{x}{2} + C$

(14) $\int \frac{dx}{1+\sin x} = \cancel{\int \frac{1-\sin x}{1-\sin^2 x}\,dx} = \cancel{\int \frac{dx}{1+\frac{2\operatorname{tg}\frac{x}{2}}{1+\operatorname{tg}^2(\frac{x}{2})}} =}$

$= \int \frac{dx}{1+\cos(\frac{\pi}{2}-x)}$ $\cdot\cos(x-\frac{\pi}{2})$ $= \int \frac{dx}{2\cos^2(\frac{x}{2}-\frac{\pi}{4})} = \frac{1}{2}\cdot\frac{\operatorname{tg}(\frac{x}{2}-\frac{\pi}{4})}{\frac{1}{2}} + C =$

$= \operatorname{tg}\left(\frac{x}{2}-\frac{\pi}{4}\right) + C$

$1+\operatorname{ctg}^2 x = \frac{1}{\sin^2 x}$

$\operatorname{ctg}^2 x = \frac{1}{\sin^2 x} - 1$

Домашняя работа:

1) $\int\left(\frac{1-x}{x}\right)^2 dx = \int\left(\frac{1-2x+x^2}{x^2}\right)dx = \int\left(\frac{1}{x^2}-\frac{2}{x}+1\right)dx =$

$= \int (x^{-2})\,dx - 2\int\frac{dx}{x} + \int dx = -\frac{1}{x} + x - 2\ln|x| + C$

2) $\int\frac{(1-x)^3}{x\sqrt[3]{x}}dx = \int\frac{1-3x+3x^2-x^3}{x^{\frac{4}{3}}}dx = \int\left(\frac{1}{x^{\frac{4}{3}}} - \frac{3}{x^{\frac{1}{3}}} + 3x^{\frac{2}{3}} - x^{\frac{5}{3}}\right)dx =$

$= -\frac{3}{x^{\frac{1}{3}}} - \frac{9x^{\frac{2}{3}}}{2} + \frac{9x^{\frac{5}{3}}}{5} - \frac{3x^{\frac{8}{3}}}{8} + C = -\frac{3}{\sqrt[3]{x}}\left(1+\frac{3}{2}x - \frac{3}{5}x^2 + \frac{1}{8}x^3\right) +$

$+ C$

3) $\int\frac{x^2}{1-x^2}dx = \int\frac{1-x^2+1}{1-x^2}dx = \int\left(1-\frac{1}{1-x^2}\right)dx =$

$= \int\frac{x^2-1+1}{-(x^2-1)}dx = \int\left(-1-\frac{1}{x^2-1}\right)dx = \int\left(-1-\frac{1}{(x-1)(x+1)}\right)dx =$

$= -x + \cancel{\int\frac{1}{x^2+1}}\ \int -\frac{1}{x^2-1}dx = \cancel{-x - \frac{1}{2}\ln|x^2-1|} =$

$= -x - \int\frac{1}{2(1-x)} + \frac{1}{2(1+x)}dx = -x - \frac{1}{2}\ln|1-x| + \frac{1}{2}\ln|1+x| =$

$= -x + \frac{1}{2}\ln\left|\frac{1+x}{1-x}\right| + C$

4) $\int(2^x+3^x)^2dx = \int(2^{2x}+2\cdot2^x\cdot3^x+3^{2x})dx = \int(2^{2x}+2\cdot6^x+3^{2x})dx =$

$= \int(4)^x dx + 2\int 6^x dx + \int 9^x dx =$

$= \frac{4^x}{\ln 4} + 2\frac{6^x}{\ln 6} + \frac{9^x}{\ln 9} + C.$

5) $\int ctg^2x\,dx = \int\left(\frac{1}{\sin^2x}-1\right)dx = -ctg\,x - x + C$

6) $\int cth^2x\,dx = \int\left(\frac{1}{sh^2x}+1\right)dx = -cth\,x + x + C$

7) $\int(2x-3)^{10}dx = \frac{(2x-3)^{11}}{2\cdot11}+C$

8) $\int(1-3x)^{\frac{1}{3}}dx = -\frac{(1-3x)^{\frac{4}{3}}}{4}+C$

9) $\int\frac{dx}{(5x-2)^{\frac{5}{2}}} = \int(5x-2)^{-\frac{5}{2}}dx = \frac{-1\cdot2}{5\cdot3(5x-2)^{\frac{3}{2}}}+C = -\frac{2}{15(5x-2)^{\frac{3}{2}}}+C$

10) $\int \frac{dx}{\sin^2(2x+\frac{\pi}{4})} = -\frac{1}{2}ctg\left(2x+\frac{\pi}{4}\right)+C$

11) $\int \frac{dx}{1+\cos x} = \int \frac{dx}{2\cos^2\frac{x}{2}} = \frac{1}{2}\int \frac{dx}{\cos^2\frac{x}{2}} = \frac{1}{2}\, tg\frac{x}{2}\cdot 2 + C =$

$= tg\frac{x}{2} + C$

12) $\int \frac{dx}{sh^2\frac{x}{2}} = -cth\frac{x}{2}\cdot 2 + C = -2cth\frac{x}{2} + C$

Интегрирование внесением под знак дифференциала.

$d(x^2) = 2\boxed{x\cdot dx} \rightarrow x\,dx = \frac{1}{2}dx^2$

$d(\cos x) = -\sin x\,dx \rightarrow \sin x\,dx = -d(\cos x)$

$d(x+c) = dx$

$t^{-\frac{1}{2}}\cdot dt$

① $\int \frac{x\,dx}{\sqrt{1-x^2}} = -\frac{1}{2}\int \frac{d(x^2)}{\sqrt{1-x^2}} = -\frac{1}{2}\int \frac{d(1-x^2)}{\sqrt{1-x^2}} =$

$= -\frac{1}{2}\cdot\sqrt{(1-x^2)}\cdot 2 + C = -\frac{1}{2}\cdot 2\sqrt{1-x^2} + C = -\sqrt{1-x^2} + C$

$(-\sqrt{1-x^2})' = -\frac{1}{2\sqrt{1-x^2}}\cdot(-2x) = \frac{x}{\sqrt{1-x^2}}$

② $\int \frac{x\,dx}{3-2x^2} = \frac{1}{2}\int \frac{d(x^2)}{3-2x^2} = -\frac{1}{4}\int \frac{d(-2x^2)}{3-2x^2} = -\frac{1}{4}\int \frac{d(3-2x^2)}{(3-2x^2)}$

$= -\frac{1}{4}\cdot \ln|3-2x^2| + C$

③ $\int \frac{x\,dx}{\cancel{4+x^4}} = \cancel{\frac{1}{2}\int \frac{d(x^2)}{\ }} =$

$\cancel{d(x^4) = 4x^3dx} \rightarrow \cancel{dx = \frac{d(x^4)}{4x^3}}$

$= \cancel{\int \frac{d(x^4)}{4x^3(4+x^4)}}$

$\int \frac{x\,dx}{4+x^4} = \frac{1}{2}\int \frac{d(x^2)}{4+x^4} = \frac{1}{2}\int \frac{d(x^2)}{4+(x^2)^2} = \frac{1}{8}\int \frac{d(x^2)}{1+(\frac{x^2}{2})^2} =$

$= \frac{1}{4}\int \frac{\frac{1}{2}d(x^2)}{1+(\frac{x^2}{2})^2} = \frac{1}{4}\int \frac{(\frac{dx^2}{2})}{1+(\frac{x^2}{2})^2} = \frac{1}{4}\, arctg\left(\frac{x^2}{2}\right) + C$

④ $\int \frac{dx}{\sqrt{x(1-x)}} = \int \frac{2\, dx}{2\sqrt{x} \cdot \sqrt{1-x}} = 2\int \frac{d(\sqrt{x})}{\sqrt{1-(\sqrt{x})^2}} = 2\arcsin\sqrt{x} + C$

$x(1-x) > 0$
$0 < x < 1$

$d(\sqrt{x}) = \frac{dx}{2\sqrt{x}}$

⑤ $\int \frac{dx \cdot 2}{x\sqrt{x-1}\, 2} = 2\int \frac{d(\sqrt{x-1})}{x} = 2\int \frac{d(\sqrt{x-1})}{x-1+1} = 2\int \frac{d\sqrt{x-1}}{(\sqrt{(x-1)})^2+1} =$

$d(\sqrt{x-1}) = \frac{1 \cdot dx}{2\sqrt{x-1}}$ $\quad = 2\,\mathrm{arctg}(\sqrt{x-1}) + C$

⑥ $\int \frac{e^x dx}{2+e^x} = \int \frac{d(e^x)}{2+e^x} = \int \frac{d(e^x+2)}{2+e^x} = \ln|2+e^x| + C =$

$\quad = \ln(2+e^x) + C$

$d(e^x) = e^x dx$

⑦ $\int \frac{dx}{x\ln x\, \ln(\ln x)} = \int \frac{d(\ln x)}{\ln x \cdot \ln(\ln x)} = \int \frac{d(\ln(\ln x))}{\ln(\ln x)} =$

$d(\ln x) = \frac{1\, dx}{x}$

$= \ln|\ln(\ln x)|$

$\cancel{d(\ln x)}$

⑧ $\int \frac{\sin x}{\sqrt{\cos^3 x}} \cdot dx = \int - \frac{\cdot(-\sin x)\cdot dx}{\cos^{\frac{3}{2}} x} = -\int \frac{d(\cos x)}{\cos^{\frac{3}{2}} x} =$

$\left.\begin{array}{l} d(\sin x) = \cos x \cdot dx \\ d(\cos x) = -\sin x \cdot dx \end{array}\right\}$ $\quad = -\int \cos^{-\frac{3}{2}} x \cdot d(\cos x) =$

$\quad = + \frac{2}{\sqrt{\cos x}} + C$

⑨ $\int \mathrm{ctg}\, x\, dx = \int \frac{\cos x}{\sin x} \cdot dx = \int \frac{d(\sin x)}{\sin x} = \ln|\sin x| + C$

⑩ $\int \frac{dx}{\sin^2 x + 2\cos^2 x} = \int \frac{dx}{1+\cos^2 x} = \int \frac{dx}{\cos^2 x\left(\frac{\sin^2 x}{\cos^2 x} + 2\right)} =$

$= \int \frac{d(\mathrm{tg}\, x)}{\mathrm{tg}^2 x + 2} = \frac{1}{2}\int \frac{\sqrt{2}\, d(\mathrm{tg}\, x)}{1 + \left(\frac{\mathrm{tg}\, x}{\sqrt{2}}\right)^2} = \frac{\sqrt{2}}{2}\,\mathrm{arctg}\left(\frac{\mathrm{tg}\, x}{\sqrt{2}}\right) + C$

⑪ $\int \frac{dx}{\arcsin^2 x \sqrt{1-x^2}} = \int \frac{d(\arcsin x)}{\arcsin^2 x} = -\frac{1}{\arcsin x} + C$

⑫ $\int \frac{dx}{\cos x} = \int \frac{dx}{\sin(x+\frac{\pi}{2})} = \int \frac{dx}{2\sin(\frac{x}{2}+\frac{\pi}{4}) \cdot \cos(\frac{x}{2}+\frac{\pi}{4})} =$

$= \int \frac{2\, d(\frac{x}{2}+\frac{\pi}{4})}{2\,\mathrm{tg}(\frac{x}{2}+\frac{\pi}{4}) \cdot \cos^2(\frac{x}{2}+\frac{\pi}{4})} = \int \frac{d(\mathrm{tg}(\frac{x}{2}+\frac{\pi}{4}))}{\mathrm{tg}(\frac{x}{2}+\frac{\pi}{4})} = \ln|\mathrm{tg}(\frac{x}{2}+\frac{\pi}{4})| + C$

$$tg\,x \cdot \cos^2 x = \frac{\sin x}{\cos x} \cdot \cos^2 x = \sin x \cdot \cos x$$

(13) $$\int \frac{x^2+1}{x^4+1}\,dx = \int \frac{1+\frac{1}{x^2}}{x^2+\frac{1}{x^2}}\,dx \ominus$$

$$d\left(x-\frac{1}{x}\right) = \left(1+\frac{1}{x^2}\right)\cdot dx$$

$$\ominus \int \frac{d\left(x-\frac{1}{x}\right)}{x^2-2\cdot x\cdot\frac{1}{x}+\frac{1}{x^2}+2} = \int \frac{d\left(x-\frac{1}{x}\right)}{\left(x-\frac{1}{x}\right)^2+2} =$$

$$= \frac{1}{\sqrt{2}}\,\mathrm{arctg}\left(\frac{x-\frac{1}{x}}{\sqrt{2}}\right)+C$$

(14) $$\int \frac{x^4\,dx}{(x^5+1)^4} = \frac{1}{5}\int \frac{d(x^5+1)}{(x^5+1)^4} = -\frac{1}{15\,(x^5+1)^3}+C$$

$$d(x^5+1) = 5\,x^4\,dx$$

Домашняя работа.

1.) $$\int \frac{x\,dx}{(1+x^2)^2} = \frac{1}{2}\int \frac{d(1+x^2)}{(1+x^2)^2} = -\frac{1}{2(1+x^2)}+C$$

$$d(x^2) = 2x\,dx \Rightarrow x\,dx = \frac{1}{2}\,d(x^2)$$

2) $$\int \sin\frac{1}{x}\cdot\frac{dx}{x^2} = -\int \sin\left(\frac{1}{x}\right)\cdot d\left(\frac{1}{x}\right) = \cos\left(\frac{1}{x}\right)+C$$

$$d\left(\frac{1}{x}\right) = -\frac{dx}{x^2}$$

3) $$\int \frac{x\,dx}{(x^2-1)^{\frac{3}{2}}} = \frac{1}{2}\int \frac{d(x^2-1)}{(x^2-1)^{\frac{3}{2}}} = -\frac{1}{\sqrt{x^2-1}}+C$$

4) $$\int x\cdot e^{-x^2}\cdot dx \ominus$$

$$d(e^{-x^2}) = -2x\cdot e^{-x^2}\,dx \rightarrow dx = -\frac{d(e^{-x^2})}{2x\cdot e^{-x^2}}$$

$$\ominus -\int \cancel{x\cdot e^{-x^2}}\cdot\frac{d(e^{-x^2})}{\cancel{2x\cdot e^{-x^2}}} = -\int \frac{1}{2}\cdot d(e^{-x^2}) = -\frac{1}{2}e^{-x^2}+C$$

5) $$\int \frac{\ln^2 x}{x}\,dx = \int \ln^2 x\; d(\ln x) = \frac{\ln^3 x}{3}+C$$

$$d(\ln x) = \frac{dx}{x}$$

6) $\int \sin^5 x \cdot \cos x \cdot dx = \int \sin^5 x \cdot d(\sin x) = \frac{\sin^6 x}{6} + C$

$d(\sin x) = \cos x \cdot dx$

7) $\int ctg\, x\, dx = \int \frac{\cos x}{\sin x}\, dx = \cancel{\int \frac{\cos^2 x}{\sin x \cos x} \cdot dx} =$

$= \cancel{\int \frac{\cos x \cdot \sin x}{\sin^2 x}\, dx} = \int \frac{d(\sin x)}{\sin x} = \ln|\sin x| + C$

8) $\int \frac{\sin x + \cos x}{\sqrt[3]{\sin x - \cos x}}\, dx = \int \frac{d(\sin x - \cos x)}{\sqrt[3]{\sin x - \cos x}} =$

$dx \cdot (\sin x + \cos x) = d(\sin x - \cos x)$

$= \frac{3}{2}\sqrt[3]{(\sin x - \cos x)^2} + C = \frac{3}{2}\sqrt[3]{1 - \sin 2x} + C$

9) $\int \frac{dx}{\sin^2 x \sqrt[4]{ctg\, x}} = \cancel{\int \frac{dx}{\sqrt[4]{\frac{\cos x}{\sin x}} \cdot \sin^2 x}} = \cancel{\int \frac{dx}{\sqrt[4]{\cos x \cdot \sin^7 x}}}$

$d(ctg\, x) = -\frac{dx}{\sin^2 x}$

$= -\int \frac{d(ctg\, x)}{\sqrt[4]{ctg\, x}} = -\frac{4}{3} ctg^{\frac{3}{4}} x + C = -\frac{4}{3}\sqrt[4]{ctg^3 x} + C$

10) $\int \frac{dx}{\sin x} = \int \frac{\cos x\, dx}{\sin x \cos x} = \int \frac{d(\sin x)}{\frac{1}{2}\sin 2x} = 2\int \frac{d(\sin x)}{\sin 2x} =$

$=$

$\cancel{d(\sin x) = 2\cos 2x\, dx}$

11) $\int \frac{\arctg x}{1+x^2}\,dx = \int \arctg x \cdot d(\arctg x) = \frac{\arctg^2 x}{2} + C$

12) $\int \frac{2^x \cdot 3^x}{9^x - 4^x}\,dx = \int \frac{2^x \cdot 3^x}{(3^x - 2^x)(3^x + 2^x)} \cdot dx =$

$d(2^x \cdot 3^x) = (2^x \ln 2 \cdot 3^x + 3^x \ln 3 \cdot 2^x) = 2^x 3^x (\ln 2 + \ln 3)\,dx$

?

$d(3^x - 2^x) = (3^x \ln 3 - 2^x \ln 2)\,dx$

$= \int \frac{\left(\frac{3}{2}\right)^x}{\left(\left(\frac{3}{2}\right)^x - 1\right)\left(\left(\frac{3}{2}\right)^x + 1\right)} \cdot dx = \int \frac{1 + \left(\frac{3}{2}\right)^x - 1}{\left(\left(\frac{3}{2}\right)^x - 1\right)\left(\left(\frac{3}{2}\right)^x + 1\right)} \cdot dx =$

$= \int \frac{\left(\frac{3}{2}\right)^x + 1 - 1}{\left(\frac{3}{2}\right)^{2x} - 1}\,dx = \cancel{\int \frac{\left(\frac{3}{2}\right)^x \cdot dx}{\left(\frac{3}{2}\right)^{2x} -}}$

$= \int \frac{\frac{2^x \cdot 3^x}{4^x}}{\frac{9^x - 4^x}{4^x}}\,dx = \int \frac{\left(\frac{2 \cdot 3}{4}\right)^x}{\left(\frac{3}{2}\right)^{2x} - 1}\,dx = \int \frac{\left(\frac{3}{2}\right)^x \cdot dx}{\left(\left(\frac{3}{2}\right)^x - 1\right)\left(\left(\frac{3}{2}\right)^x + 1\right)} =$

$= \left\{ \left(\frac{3}{2}\right)^x = t\ ;\quad dt = \left(\frac{3}{2}\right)^x \ln\frac{3}{2} \cdot dx \right\} =$

01.12.2020

Метод разложения

① $\int \frac{x^2}{x+3}dx = \int \frac{x^2-9+9}{x+3}dx = \int \frac{(x-3)(x+3)+9}{x+3}dx =$

$= \int \left(x-3+\frac{9}{x+3}\right)dx = \int (x-3)dx + 9\int \frac{1}{x+3}dx = \frac{(x-3)^2}{2} +$

$+ 9\ln|x+3| + C$

② $\int \frac{x^2}{(x-1)^{100}}dx = \cancel{\int \frac{x^2}{x^{100}(1-\frac{1}{x})^{100}}} = \int \frac{(x-1+1)^2}{(x-1)^{100}}dx =$

$= \int \frac{(x-1)^2+2(x-1)+1}{(x-1)^{100}}dx = \int \frac{d(x-1)}{(x-1)^{98}} + 2\int \frac{d(x-1)}{(x-1)^{99}} + \int \frac{d(x-1)}{(x-1)^{100}} =$

$= -\frac{1}{97(x-1)^{97}} - \frac{1}{49(x-1)^{98}} - \frac{1}{99(x-1)^{99}} + C$

③ $\int \frac{x^5+1-1}{x+1}dx = \int \frac{(x+1)(x^4-x^3+x^2-x+1)-1}{(x+1)}dx = \int (x^4-x^3+x^2-x+1)dx -$

$- \int \frac{1}{x+1}dx = \frac{x^5}{5} - \frac{x^4}{4} + \frac{x^3}{3} - \frac{x^2}{2} + x - \ln|x+1| + C$

$a^{2n+1} \pm b^{2n+1} = (a \pm b)(a^{2n} \mp a^{2n-1}b + a^{2n-2}b^2 \mp \ldots + b^{2n})$

④ $\int \frac{dx}{x^2+x-2} = \int \frac{1 \cdot dx}{(x+2)(x-1)} = \frac{1}{3}\int \frac{(x+2)-(x-1)}{(x-1)(x+2)}dx =$

$= \frac{1}{3}\left(\int \frac{d(x-1)}{x-1} - \int \frac{d(x+2)}{x+2}\right) = \frac{1}{3}\left(\ln|x-1| - \ln|x+2|\right) + C$

⑤ $\int \frac{x\,dx}{(x+2)(x+3)} = \int \frac{(x+2-2)\,dx}{(x+2)(x+3)} = \int \frac{dx}{x+3} - 2\int \frac{1 \cdot dx}{(x+2)(x+3)} =$

$= \ln|x+3| - 2\int \frac{(x+3)-(x+2)}{(x+2)(x+3)}dx = \ln|x+3| - 2\left(\int \frac{dx}{x+2} - \int \frac{dx}{x+3}\right) =$

$= \ln|x+3| - 2\ln|x+2| + 2\ln|x+3| + C = 3\ln|x+3| - 2\ln|x+2| + C =$

$= \ln\frac{|x+3|^3}{(x+2)^2} + C$

⑥ $\int \frac{x\,dx}{x^4+3x^2+2} = \frac{1}{2}\int \frac{d(x^2)}{x^4+3x^2+2} = \frac{1}{2}\int \frac{dt}{t^2+3t+2} = \frac{1}{2}\int \frac{dt}{(t+2)(t+1)} =$

$= \frac{1}{2}\int \frac{(t+2)-(t+1)}{(t+2)(t+1)}dt = \frac{1}{2}\left(\int \frac{dt}{t+1} - \int \frac{dt}{t+2}\right) =$

$= \frac{1}{2}\ln|t+1| - \frac{1}{2}\ln|t+2| + C = \frac{1}{2}\ln(x^2+1) - \frac{1}{2}\ln(x^2+2) + C =$

$= \frac{1}{2}\ln\frac{x^2+1}{x^2+2} + C$

(7) $\int \cos^2 x\, dx = \frac{1}{2}\int (1+\cos 2x)\, dx = \frac{1}{2}\left(x + \frac{\sin 2x}{2}\right) + C$

(8) $\int \cos\frac{x}{2}\cdot\cos\frac{x}{3}\, dx = \frac{1}{2}\int\left(\cos\frac{5x}{6} + \cos\frac{x}{6}\right) dx =$

$\cos\alpha\cdot\cos\beta = \frac{1}{2}\left(\cos(\alpha+\beta) + \cos(\alpha-\beta)\right)$

$= \frac{3}{5}\sin\frac{5x}{6} + \frac{1}{2}\cdot\frac{6}{1}\sin\frac{x}{6} + C = \frac{3}{5}\sin\frac{5x}{6} + 3\sin\frac{x}{6} + C$

(9) $\int \sin^3 x\, dx = -\int \sin^2 x\, d(\cos x) = -\int (1-\cos^2 x)\, d(\cos x) =$

$= \int (\cos^2 x - 1)\, d(\cos x) = \frac{1}{3}\cos^3 x - \cos x + C$

(10) $\int \sin^4 x\, dx =$

$\sin^4 x = (\sin^2 x)^2 = \left(\frac{1-\cos 2x}{2}\right)^2 = \frac{1}{4}(1 - 2\cos 2x + \cos^2 2x) =$

$= \frac{1}{4} - \frac{1}{2}\cos 2x + \frac{1}{4}\left(\frac{1+\cos 4x}{2}\right) = \frac{1}{4} - \frac{1}{2}\cos 2x + \frac{1}{8} + \frac{1}{8}\cos 4x =$

$= \frac{3}{8} - \frac{1}{2}\cos 2x + \frac{1}{8}\cos 4x.$

$= \int\left(\frac{3}{8} - \frac{1}{2}\cos 2x + \frac{1}{8}\cos 4x\right) dx =$

$= \frac{3}{8}x - \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + C$

(11) $\int tg^3 x\, dx = \int tg^2 x \cdot tg\, x\, dx = \int\left(-1 + \frac{1}{\cos^2 x}\right) tg\, x \cdot dx =$

$= \int\left(\frac{1}{\cos^2 x} - 1\right) tg\, x\, dx = \int tg\, x \cdot d(tg\, x) - \int tg\, x\, dx =$

$= \frac{1}{2} tg^2 x - \int \frac{\sin x \cdot dx}{\cos x} = \frac{1}{2} tg^2 x + \int \frac{d(\cos x)}{\cos x} = \underline{\frac{1}{2} tg^2 x + \ln|\cos x| + C}$

2 способ.

$\int tg^3 x\, dx = \int \frac{\sin^3 x}{\cos^3 x}\, dx = -\int \frac{(1-\cos^2 x)\, d\cos x}{\cos^3 x} =$

$= \int \frac{d(\cos x)}{\cos x} - \int \frac{d(\cos x)}{\cos^3 x} = \underline{\ln|\cos x| + \frac{1}{2\cos^2 x} + C}$

Отлич. с точн. до констант

(12) $\int \frac{1\, dx}{\sin x\cdot\cos^3 x} = \int \frac{(\sin^2 x + \cos^2 x)\, dx}{\sin x\cdot\cos^3 x} = \int \frac{\sin x\, dx}{\cos^3 x} + \int \frac{dx}{\sin x\cdot\cos x} =$

$= -\int \frac{d(\cos x)}{\cos^3 x} + \int \frac{dx}{\cos^2 x\cdot tg\, x} = \frac{1}{2\cos^2 x} + \ln|tg\, x| + C$

~~2-й способ~~.

2-й способ для 1 из интегралов:

$$\int \frac{\sin x\, dx}{\cos^3 x} = \int \frac{\sin x}{\cos x} \frac{dx}{\cos^2 x} = \int \operatorname{tg} x\, d(\operatorname{tg} x) = \frac{\operatorname{tg}^2 x}{2} + C$$

(13) $$\int \frac{dx}{\cos^4 x} = \int \frac{dx}{\cos^2 x \cos^2 x} = \int \frac{d \operatorname{tg} x}{\cos^2 x} = \int (1+\operatorname{tg}^2 x)\, d(\operatorname{tg} x) = \operatorname{tg} x + \frac{\operatorname{tg}^3 x}{3} + C$$

(14) $$\int \frac{1 \cdot dx}{1+e^x} = \int \frac{(1+e^x) - e^x}{1+e^x} dx = \int dx - \int \frac{e^x dx}{1+e^x} = x - \int \frac{d(e^x+1)}{e^x+1} = x - \ln(e^x+1) + C$$

(15) $$\int \frac{dx \cdot 1}{\operatorname{sh}^2 x \cdot \operatorname{ch}^2 x} = \int \frac{\operatorname{ch}^2 x - \operatorname{sh}^2 x}{\operatorname{sh}^2 x\, \operatorname{ch}^2 x} dx = \int \frac{dx}{\operatorname{sh}^2 x} - \int \frac{dx}{\operatorname{ch}^2 x} = -\operatorname{cth} x - \operatorname{th} x + C.$$

Домашняя работа.

1) $$\int x^2(2-3x^2)^2 dx = \int x^2(4-12x^2+9x^4)dx = \int (4x^2 - 12x^4 + 9x^6)dx = \frac{4x^3}{3} - \frac{12x^5}{5} + \frac{9}{7}x^7 + C$$

2) $$\int x(1-x)^{10} dx = \int (1+x-1)(1-x)^{10} dx = -\int (1-x+1)(1-x)^{10} dx = \left\{ \begin{matrix} t = x-1 \\ x = t+1 \\ dx = dt \end{matrix} \right\} = \int t^{10}(t+1)dt = \int (t^{11} + t^{10}) dt = \frac{(x-1)^{12}}{12} + \frac{(x-1)^{11}}{11} + C$$

3) $$\int \frac{x^2}{1+x} dx = \int \frac{((x+1)-1)^2}{(x+1)} dx = \int \frac{(x+1)^2 - 2(x+1) + 1}{(x+1)} dx =$$
$$= \int (x+1)dx - 2\int dx + \int \frac{dx}{x+1} = \frac{(x+1)^2}{2} - 2x + \ln|x+1| + C =$$
$$= \frac{x^2+2x+1}{2} - 2x + \ln|x+1| + C = \frac{1}{2}(x-1)^2 + \ln|x+1| + C$$

4) $$\int \frac{(1+x)^2}{1+x^2} dx = \int \frac{1+2x+x^2}{1+x^2} dx = \int \frac{1+x^2+2x}{1+x^2} dx = \int dx + 2\int \frac{x}{1+x^2}dx = x + \int \frac{d(x^2+1)}{x^2+1} = x + \ln(x^2+1) + C$$

5) $$\int \frac{dx}{(x-1)(x+3)} dx = \frac{1}{4}\int \frac{(x+3)-(x-1)}{(x-1)(x+3)} dx = \frac{1}{4}\int \frac{dx}{x-1} - \frac{1}{4}\int \frac{dx}{x+3} =$$
$$= \frac{1}{4}\ln|x-1| - \frac{1}{4}\ln|x+3| + C = \frac{1}{4}\ln\left|\frac{x-1}{x+3}\right| + C$$

6) $\int \frac{dx}{(x^2+a^2)(x^2+b^2)} = \frac{1}{b^2-a^2}\int \frac{(x^2+b^2)-(x^2+a^2)}{(x^2+a^2)(x^2+b^2)}\,dx =$

$= \frac{1}{b^2-a^2}\int \frac{dx}{a^2+x^2} - \frac{1}{b^2-a^2}\int \frac{dx}{x^2+b^2} = \frac{1}{b^2-a^2}\left(\frac{1}{a}\,\mathrm{arctg}\,\frac{x}{a} - \frac{1}{b}\,\mathrm{arctg}\right.$

7) $\int \sin^2 x\,dx = \int \frac{1-\cos 2x}{2}\,dx = \frac{1}{2}\int (1-\cos 2x)\,dx =$

$= \frac{x}{2} - \frac{\sin 2x}{4} + C$

8) $\int \sin 3x \cdot \sin 5x\,dx = \frac{1}{2}\int\left(\cos(-2x) - \cos(8x)\right)dx =$

$= \frac{1}{2}\int \cos(-2x)\,dx - \frac{1}{2}\int \cos 8x\,dx =$

$= -\frac{1}{4}\sin(-2x) - \frac{1}{16}\sin 8x + C = \frac{1}{4}\sin 2x - \frac{1}{16}\sin 8x + C$

9) $\int \cos^3 x\,dx = \int \cos^2 x \cdot \cos x\,dx = \int (1-\sin^2 x)\,d(\sin x) =$

$= \sin x - \frac{\sin^3 x}{3} + C$

10) $\int \cos^4 x\,dx \ =$

$\cos^4 x = (\cos^2 x)^2 = \left(\frac{1+\cos 2x}{2}\right)^2 = \frac{1+2\cos 2x+\cos^2 2x}{4} =$

$= \frac{1}{4} + \frac{1}{2}\cos 2x + \frac{\cos^2 2x}{4} = \frac{1}{4} + \frac{1}{2}\cos 2x + \frac{1+\cos 4x}{8}$

$= \int\left(\frac{1}{4}\right)dx + \frac{1}{2}\int \cos 2x\,dx + \int\left(\frac{1}{8}\right)dx + \frac{1}{8}\int \cos 4x\,dx =$

$= \frac{x}{4} + \frac{\sin 2x}{4} + \frac{x}{8} + \frac{\sin 4x}{32} + C = \frac{3}{8}x + \frac{\sin 2x}{4} + \frac{\sin 4x}{32} + C$

11) $\int \mathrm{ctg}^2 x\,dx = \int\left(\frac{1}{\sin^2 x} - 1\right)dx = \int \frac{dx}{\sin^2 x} - \int dx = -\mathrm{ctg}\,x - x + C$

12) $\int \frac{dx}{\sin^2 x \cos^2 x} = \int \frac{(\sin^2 x + \cos^2 x)\,dx}{\sin^2 x \cos^2 x} = \int \frac{dx}{\cos^2 x} + \int \frac{dx}{\sin^2 x} =$

$= \mathrm{tg}\,x - \mathrm{ctg}\,x + C$

13) $\int \frac{dx}{\sin^2 x \cos x} = \int \frac{\cos x\,dx}{\cancel{\sin^2 x \cos^2 x}} \cancel{= \int \frac{d(\sin x)}{\sin^2 x(1-\sin^2 x)}} =$

$\cancel{= \int \frac{d(\sin x)}{\sin^2 x - \sin^4 x}} \cancel{= \int \frac{dt}{t^2 - t^4}} = \int \frac{dt}{\cancel{t^2(1-t^2)}}$

$= \int \frac{(\sin^2 x + \cos^2 x)\,dx}{\sin^2 x \cos x} = \int \frac{dx}{\cos x} + \int \frac{\cos x\,dx}{\sin^2 x} = \int \frac{\cos\,dx}{\cos^2 x} + \int \frac{d(\sin x)}{\sin^2 x}$

$= \int \frac{d(\sin x)}{1-\sin^2 x} + \int \frac{d(\sin x)}{\sin^2 x} =$

$$= \frac{1}{2}\ln\left|\frac{1+\sin x}{1-\sin x}\right| - \frac{1}{\sin x} + C$$

11) $$\int \frac{(1+e^x)^2}{1+e^{2x}}\,dx = \int \frac{1+2e^x+e^{2x}}{1+e^{2x}}\,dx = \int dx + 2\int \frac{e^x}{1+e^{2x}}\,dx =$$

$$= \int dx + 2\int \frac{d(e^x)}{1+e^{2x}} = x + 2\operatorname{arctg}(e^x) + C$$

02.12.2020г

Замена переменной в неопределённом интеграле.

1) $$\int \frac{e^{5\sqrt{x+3}}}{\sqrt{x+3}}\,dx = \left\{\begin{array}{l} \sqrt{x+3} = t \\ x+3 = t^2 \Rightarrow dx = 2t\,dt \\ x = t^2-3 \end{array}\right\} = \int \frac{e^{5t}\cdot 2t\,dt}{t} =$$

$$= 2\int e^{5t}\,dt = \frac{2}{5}e^{5t} + C = \frac{2}{5}e^{5\sqrt{x+3}} + C$$

2) $$\int \frac{\sin 2\sqrt{5x+4}}{\sqrt{5x+4}}\,dx = \left\{\begin{array}{l} \sqrt{5x+4} = t \\ 5x+4 = t^2 \rightarrow dx = \frac{2}{5}t\,dt \\ x = \frac{t^2}{5} - \frac{4}{5} \end{array}\right\} =$$

$$= \int \frac{\sin 2t}{\cancel{t}}\cdot\frac{2}{5}\cancel{t}\,dt = \frac{2}{5}\int \sin 2t\,dt = -\frac{1}{5}\cos 2t + C = -\frac{1}{5}\cos 2\sqrt{5x+4} + C$$

3) $$\int \frac{dx}{\sqrt{2x}\cdot\cos^2\sqrt{2x}} = \left\{\begin{array}{l} \sqrt{2x} = t \\ 2x = t^2 \Rightarrow dx = t\,dt \\ x = \frac{t^2}{2} \end{array}\right\} = \int \frac{\cancel{t}\,dt}{\cancel{t}\cdot\cos^2 t} = \int \frac{dt}{\cos^2 t} =$$

$$= \operatorname{tg} t + C = \operatorname{tg}\sqrt{2x} + C$$

4) $$\int \frac{x}{\sqrt[3]{x+5}}\,dx = \left\{\begin{array}{l} \sqrt[3]{x+5} = t \\ x+5 = t^3 \Rightarrow dx = 3t^2\,dt \\ x = t^3-5 \end{array}\right\} = \int \frac{(t^3-5)\cdot 3t^2\,dt}{t} =$$

$$= 3\int (t^4 - 5t)\,dt = 3\int t^4\,dt - 5\int t\,dt = \frac{3t^5}{5} - \frac{5t^2}{2} + C =$$

$$= \frac{3(\sqrt[3]{x+5})^5}{5} - \frac{15(\sqrt[3]{x+5})^2}{2} + C = \cancel{\frac{6(\sqrt[3]{x+5})^5 -}{10}} =$$

$$= \frac{3}{10}\sqrt[3]{(x+5)^2}\,\big(2(x+5) - 25\big) + C = 0{,}3\sqrt[3]{(x+5)^2}\,(2x-15) + C$$

5) $$\int \frac{dx}{\sqrt{x}+\sqrt[4]{x}} = \left\{\begin{array}{l} \sqrt[4]{x} = t \quad dx = 4t^3\,dt \\ x = t^4 \end{array}\right\} = \int \frac{4t^3\,dt}{t^2+t} = 4\int \frac{t^2\,dt}{t+1} =$$

$$= 4\int \frac{((t+1)-1)^2\,dt}{t+1} = 4\int \frac{(t+1)^2 - 2(t+1) + 1}{t+1}\,dt = 4\int \left((t+1) - 2 + \frac{1}{t+1}\right)dt =$$

$$= 4\left(\frac{(t+1)^2}{2} - 2t + \ln|t+1|\right) + C = 4\left(\frac{t^2+2t+1-4t}{2} + \ln|t+1|\right) + C$$

$$= 2(t-1)^2 + 4\ln|t+1| + C = 2(\sqrt[4]{x}-1)^2 + 4\ln(\sqrt[4]{x}+1) + C$$

(6) $$\int x^5 (x^3-1)^{\frac{2}{3}}\,dx = \left\{ \begin{array}{l} \cancel{\sqrt[3]{(x^3-1)^2} = t} \\ \cancel{(x^3-1)^2 = t^3} \\ \cancel{x^6 - 2x^3 + 1} \end{array} \quad \begin{array}{l} \cancel{x^3-1 = \sqrt{t^3}} \\ x^2 = \sqrt{t^3}+1 \end{array} \quad \begin{array}{l} \sqrt[3]{x^3-1} = t \\ x^3 - 1 = t^3 \\ x^3 = t^3 + 1 \end{array} \quad \begin{array}{l} 3x^2dx = 3t^2dt \\ x^2dx = t^2dt \end{array} \right.$$

$$= \int t^2dt \cdot (t^3+1)(t^3)^{\frac{2}{3}} = \int (t^3+1)\cdot t^2 \cdot t^2\,dt =$$

$$= \int (t^7+t^4)\,dt = \frac{t^8}{8} + \frac{t^5}{5} + C = \frac{(\sqrt[3]{x^3-1})^8}{8} + \frac{(\sqrt[3]{x^3-1})^5}{5} + C =$$

$$= \frac{(\sqrt[3]{x^3-1})^5}{40}(5(x^3-1)+8) + C = \frac{1}{40}\sqrt[3]{(x^3-1)^2}\,(5x^6-2x^3-3) + C$$

(7) $$\int \frac{\cos^3 x}{\sqrt{1+\sin x}}\,dx = \left\{ \begin{array}{l} \cancel{\sqrt{1+\sin x} = t} \\ \cancel{1+\sin x = t^2} \end{array} \right. = \int \frac{\cos^2 x \cdot d(\sin x)}{\sqrt{1+\sin x}} =$$

$$= \int \frac{(1-\sin x)(1+\sin x)\;d(\sin x)}{\sqrt{1+\sin x}} = \left\{ \begin{array}{l} \sqrt{1+\sin x} = t \\ 1+\sin x = t^2 \to d(\sin x) = 2t\,dt \\ \sin x = t^2 - 1 \end{array} \right\} =$$

$$= \int \frac{(1-t^2+1)(1+t^2-1)\cdot 2\cancel{t}\,dt}{\cancel{t}} = 2\int (2-t^2)\,t^2\,dt =$$

$$= 2\int (2t^2 - t^4)\,dt = \frac{2t^3}{3} - \frac{2t^5}{5} + C = \frac{t^3}{15}(10 - 6t^2) + C =$$

$$= \frac{2(\sqrt{1+\sin x})^3}{15}(5 - 3 - 3\sin x) + C = \frac{2}{15}\sqrt{1+\sin x}\,(1+\sin x)\,\cdot$$

$$\cdot\,(2 - 3\sin x) + C$$

? (забыл двойку!)

(8) $$\int \frac{dx}{\sqrt{e^x+4}} = \left\{ \begin{array}{l} \sqrt{e^x+4} = t \\ e^x + 4 = t^2 \end{array} \quad \begin{array}{l} e^x = t^2 - 4 \\ x = \ln|t^2-4| \\ dx = \frac{1}{t^2-4}\cdot 2t\,dt \end{array} \right\} =$$

$$= \int \frac{2\cancel{t}\,dt}{(t^2-4)\cancel{t}} = 2\int \frac{dt}{t^2-4} = -2\int \frac{dt}{4-t^2} = 2\int \frac{dt}{(t-2)(t+2)} =$$

$$= \frac{1}{2}\int \frac{t+2-(t-2)}{(t+2)(t-2)}\,dt = \frac{1}{2}\left(\int \frac{dt}{t-2} - \int \frac{dt}{t+2}\right) = \frac{1}{2}\left(\ln|t-2| - \ln|t+2|\right) =$$

$$= \frac{1}{2}\ln\left|\frac{t-2}{t+2}\right| + C = \frac{1}{2}\ln\left|\frac{\sqrt{e^x+4}-2}{\sqrt{e^x+4}+2}\right| + C =$$

$$= \frac{1}{2}\ln\frac{(\sqrt{e^x+4}-2)^4}{e^x+4-4} + C = \frac{1}{2}\ln(\sqrt{e^x+4}-2)^2 - \frac{1}{2}\ln e^x + C =$$

$$= \frac{2}{2}\ln(\sqrt{e^x+4}-2) - \frac{1}{2}x + C$$

(9) $\int \frac{dx}{x^2\sqrt{4-x^2}} = \left\{ \begin{array}{l} \cancel{\sqrt{4-x^2}=t} \\ \cancel{4-x^2=t^2} \\ \cancel{x^2=4-t^2} \\ \cancel{2x\,dx=-2t\,dt} \end{array} \quad \begin{array}{l} \cancel{x^2=t} \\ \cancel{2x\,dx=dt} \end{array} \right.$

$\left\{ \begin{array}{l} 4-x^2>0 \\ (2-x)(2+x)>0 \\ -2<x<2 \\ x=2\sin t \\ dx=2\cos t\,dt \\ x^2=4\sin^2 t \end{array} \right. \quad t=\arcsin\frac{x}{2}, \quad t\in\left(-\frac{\pi}{2};\frac{\pi}{2}\right) \Bigg\}$

$$= \int \frac{2\cos t\,dt}{4\sin^2 t\,\underbrace{\sqrt{4-4\sin^2 t}}_{=2|\cos t|}} = \int \frac{2d(\sin t)}{8\sin^2 t\sqrt{\cos^2 t}} =$$

$=> |\cos t|>0$

$$= \frac{1}{4}\int \frac{\cos t\,dt}{\sin^2 t\cdot\cos t} = -\frac{1}{4}\,\mathrm{ctg}\,t + C = -\frac{1}{4}\,\mathrm{ctg}\left(\arcsin\frac{x}{2}\right) + C =$$

$$= -\frac{1}{4}\cdot\frac{\sqrt{1-\sin^2 t}}{\sin t} + C = -\frac{2}{4}\cdot\frac{\sqrt{1-\frac{x^2}{4}}}{x} + C =$$

$$= -\frac{1}{4}\,\frac{\sqrt{4-x^2}}{x} + C = -\frac{\sqrt{4-x^2}}{4x} + C$$

(10) $\int \sqrt{\frac{a-x}{a+x}}\,dx = \left\{ \begin{array}{l} \cancel{\sqrt{\frac{a-x}{a+x}}=t} \\ \cancel{\frac{a-x}{a+x}=t^2} \end{array} \quad \begin{array}{l} \cancel{a-x=at^2+xt^2} \\ \cancel{-dx=2at\,dt} \end{array} \right.$

$$= \left\{ \begin{array}{ll} \frac{a-x}{a+x}\geq 0 & -a<x\leq a \\ x=a\cos t, & t\in[0;\pi) \end{array} \quad dx=-a\sin t \right\}$$

$$= -a\int\sqrt{\frac{a-a\cos t}{a+a\cos t}}\,dt\cdot\sin t = -a\int\sqrt{\frac{1-\cos t}{1+\cos t}}\cdot\sin t\,dt =$$

$$= -a\int\sqrt{\frac{2\sin^2\frac{t}{2}}{2\cos^2\frac{t}{2}}}\,\sin t\cdot dt = -a\int\left|\mathrm{tg}\,\frac{t}{2}\right|\sin t\,dt =$$

$\mathrm{tg}\,\frac{t}{2}>0$, т.к. наход. в 1 четверти.

$$= -a\int \mathrm{tg}\,\frac{t}{2}\,\sin t\,dt = -a\int\frac{\sin\frac{t}{2}}{\cos\frac{t}{2}}\cdot 2\sin\frac{t}{2}\cdot\cos\frac{t}{2}\,dt =$$

$$= -2a\int\sin^2\frac{t}{2}\,dt = -2a\int\frac{1-\cos t}{2}\,dt =$$

$$= -a\int(1-\cos t)\,dt = -at + a\sin t + C = a\sin t - at + C =$$

$$= a\left(\sqrt{1-\frac{x^2}{a^2}} - \arccos\frac{x}{a}\right) + C = \sqrt{a^2-x^2} - a\cdot\arccos\frac{x}{a} + C$$

Домашняя работа

1) $\int x^2\sqrt[3]{1-x}\,dx = \left\{\begin{array}{l}\sqrt[3]{1-x}=t \\ 1-x=t^3,\quad x=1-t^3\end{array}\quad dx=-3t^2dt\right\} =$

$= \int(1-t^3)^2(-3)t^2\,dt = -3\int(1-2t^3+t^6)t^2\,dt = -3\int(t^2-2t^5+t^8)\,dt$

$= -3\left(\frac{t^3}{3}-\frac{2t^6}{6}+\frac{t^9}{9}\right)+C = -\frac{3}{140}(9+12x+14x^2)\sqrt[3]{(1-x)^4}+C$

2) $\int\frac{x^2}{\sqrt{2-x}}\,dx = \left\{\begin{array}{l}\sqrt{2-x}=t \\ 2-x=t^2 \Rightarrow x=2-t^2\end{array}\quad dx=-2t\,dt\right\} =$

$= -2\int\frac{(2-t^2)^2t\,dt}{t} = -2\int(4-2t^2+t^4)\,dt = -2\left(4t-\frac{2t^3}{3}+\frac{t^5}{5}\right)+C$

$= -2t\left(4-\frac{2t^2}{3}+\frac{t^4}{5}\right)+C = -2\sqrt{2-x}\left(4-\frac{2(2-x)}{3}+\frac{(2-x)^2}{5}\right)+C=$

$= -\frac{2}{15}(32+8x+3x^2)\sqrt{2-x}+C$

3) $\int\frac{x^5}{\sqrt{1-x^2}}\,dx = \left\{\begin{array}{l}\sqrt{1-x^2}=t \\ 1-x^2=t^2 \Rightarrow x^2=1+t^2\end{array}\quad \cancel{2}x\,dx=\cancel{2}t\,dt\right\}=$

$= \int\frac{x^4\cdot x\,dx}{\sqrt{1-x^2}} = \int\frac{(1+t^2)^2t\,dt}{t} = \int(1+2t^2+t^4)\,dt=$

$= t+\frac{2t^3}{3}+\frac{t^5}{5}+C = t\left(1+\frac{2t^2}{3}+\frac{t^4}{5}\right)+C=$

$= \sqrt{1-x^2}\left(1+\frac{2(1-x^2)}{3}+\frac{(1-x^2)^2}{5}\right)+C = -\frac{1}{15}(8+4x^2+3x^4)\sqrt{1-x^2}$

4) $\int\cos^5x\cdot\sqrt{\sin x}\,dx = \left\{\begin{array}{l}\sin^2x=t^4\quad\sqrt{\sin x}=t \\ \sin x=t^2,\quad d\sin x=2t\,dt\end{array}\right\}=$

$= \int\cos^4x\sqrt{\sin x}\,d(\sin x) = \int(1-\sin^2x)^2\,t\cdot2t\,dt =$

$= \int(1-t^4)^2t\cdot2t\,dt = \int(1-2t^4+t^8)2t^2\,dt=$

$= 2\int(t^2-2t^6+t^{10})\,dt = 2\left(\frac{t^3}{3}-\frac{2t^7}{7}+\frac{t^4}{11}\right)+C=$

$= 2t^3\left(\frac{1}{3}-\frac{2t^4}{7}+\frac{t^8}{11}\right)+C = \sqrt{\sin^3x}\cdot\left(\frac{2}{3}-\frac{4}{7}\sin^2x+\frac{2}{11}\sin^4x\right)$

5) $\int\frac{\sin x\cos^3x}{1+\cos^2x}\,dx = \left\{\begin{array}{l}\cos^2x=t \\ 2\cos x\sin x\,dx=dt \Rightarrow\end{array}\quad dx=\frac{dt}{2\cos x\sin x}\right\}=$

$= \frac{1}{2}\int\frac{\cancel{\sin x\cos x}\ t\,dt}{(1+t)\cancel{\cos x\sin x}} = \frac{1}{2}\int\frac{(t+1)-1}{(t+1)}\,dt =$

$= \frac{1}{2}\int\left(1-\frac{1}{t+1}\right)dt = \frac{1}{2}t-\frac{1}{2}\ln|t+1|+C=$

$= \frac{1}{2}\cos^2x-\frac{1}{2}\ln|1+\cos^2x|+C$

6) $\int \frac{\ln x \, dx}{x\sqrt{1+\ln x}} = \left\{ \begin{array}{l} \sqrt{1+\ln x} = t \\ 1+\ln x = t^2 \end{array} \right| \begin{array}{l} \ln x = t^2-1 \\ x = e^{t^2-1} \end{array} \quad dx = 2t\,dt \cdot e^{t^2-1} \Big\} =$

$= \int \frac{(t^2-1)\, 2\cancel{t}\, \cancel{e^{t^2-1}}\, dt}{\cancel{e^{t^2-1}} \cdot \cancel{t}} = 2\int (t^2-1)\, dt = \frac{2t^3}{3} - 2t + C =$

$= t\left(\frac{2t^2}{3} - 2\right) + C = \sqrt{1+\ln x}\left(\frac{2}{3}(1+\ln x) - 2\right) + C =$

$= \frac{2}{3}(\ln x - 2)\sqrt{1+\ln x} + C$

7) $\int \frac{dx}{\sqrt{1+e^x}} = \left\{ \begin{array}{l} \sqrt{1+e^x} = t \\ 1+e^x = t^2 \end{array} \right. \Rightarrow x = \ln|t^2-1| \Rightarrow dx = \frac{2t\,dt}{t^2-1} \Big\} =$

$= \int \frac{2\cancel{t}\,dt}{(t^2-1)\cancel{t}} = 2\int \frac{dt}{t^2-1} = -2\int \frac{dt}{1-t^2} = -2\left(\frac{1}{2}\ln\left|\frac{1+t}{1-t}\right|\right) + C =$

$= -\ln\left|\frac{1+\sqrt{1+e^x}}{1-\sqrt{1+e^x}}\right| + C$

8) $\cancel{\int \frac{\operatorname{arctg}\sqrt{x}}{\sqrt{x}} \cdot \frac{dx}{1+x} = \left\{ \begin{array}{l} \operatorname{arctg}\sqrt{x} = t \\ \sqrt{x} = tg\, t \end{array} \right. \Rightarrow x = tg^2 t \Rightarrow dx = 2tg\,t \cdot \frac{dt}{\cos^2 t}}$

$\cancel{\frac{2\sin t\, dt}{\cos t\, (}}$

$\cancel{= \int \frac{t \cdot 2tg\,t \cdot dt}{tg\,t \cdot (1+tg^2 t)\cos^2 t} \; 2\int \frac{t\,dt}{(1+tg^2 t)\cos^2 t} = 2\int \frac{t\, d(tg\,t)}{(1+tg^2 t)}}$

8) $\int \frac{\operatorname{arctg}\sqrt{x}}{\sqrt{x}} \cdot \frac{dx}{1+x} = \left\{ \begin{array}{l} \operatorname{arctg}\sqrt{x} = t \\ \sqrt{x} = tg\,t \Rightarrow x = tg^2 t \end{array} \right. \quad dx = 2tg\,t \cdot \frac{dt}{\cos^2 t} \Big\} =$

$= \int \frac{t}{\cancel{tg\,t}} \cdot \frac{\cancel{2tg\,t} \cdot dt}{(1+tg^2 t)\cos^2 t} = 2\int \frac{t\,dt}{(1+tg^2 t)\cdot \cos^2 t}$

9) $\int\sqrt{1-x^2}\,dx = \begin{cases}1-x^2\ge 0\\ (1-x)(1+x)\ge 0\end{cases}$ $\quad - \;\; + \;\; -$; -1, 1 $\qquad t=\arcsin x$ $\qquad -1\le x\le 1$

$\Rightarrow \sin t = x \qquad dx=\cos t\,dt \qquad t\in(-\frac{\pi}{2};$

$\int\sqrt{1-\sin^2 t}\cdot\cos t\,dt = \int\sqrt{\cos^2 t}\cdot\cos t\,dt = \int\cos^2 t\,dt =$

$=\int\frac{1+\cos 2t}{2}\,dt = \int\left(\frac{1}{2}+\frac{\cos 2t}{2}\right)dt = \frac{1}{2}t+\frac{\sin 2t}{4}+C=$

$=\frac{1}{2}\arcsin x+\frac{\sin(2\arcsin x)}{4}+C$

10) $\int\sqrt{\frac{a+x}{a-x}}\,dx = \left\{\frac{a+x}{a-x}\ge 0\right.$ $\quad - \;\; + \;\; -$; $-a$, a $\qquad t\in[0;\pi]$

$-a\le x<a \qquad x=a\cos t;\; dx=-a\sin t$

$=\int\sqrt{\frac{a+a\cos t}{a-a\cos t}}\,(-a)\sin t\,dt = -a\int\operatorname{ctg}\frac{t}{2}\cdot\sin t\,dt =$

$=-a\int\frac{\cos\frac{t}{2}\cdot 2\sin\frac{t}{2}\cdot\cos\frac{t}{2}}{\sin\frac{t}{2}}\,dt = -2a\int\cos^2\frac{t}{2}\,dt =$

$=-2a\int\frac{1+\cos t}{2}\,dt = -a\int(1+\cos t)\,dt = -a(t+\sin t)+C$

$=-a\left(\arccos\frac{x}{a}+\sin\left(\arccos\frac{x}{a}\right)\right)+C=$

$=-a\cdot\arccos\frac{x}{a}-a\sin\left(\arccos\frac{x}{a}\right)+C.$

04.11.2020

Интегрирование по частям.

$$\int u\,dv = uv - \int v\,du$$

① $$\int \ln x\,dx = \left\{ \begin{matrix} u=\ln x \\ dv=dx \end{matrix} \quad du=\frac{dx}{x};\quad v=\int dx = x \right\} =$$

$$= x\ln x - \int \cancel{x}\cdot\frac{dx}{\cancel{x}} = x\ln x - x + C = x(\ln x - 1)+C$$

② $$\int\left(\frac{\ln x}{x}\right)^2 dx = \left\{ \begin{matrix} u=\ln^2 x \\ \frac{dx}{x^2}=dv \end{matrix} \quad v=\int\frac{dx}{x^2}=-\frac{1}{x} \quad du=2\ln x\,\frac{dx}{x} \right\} =$$

$$= -\frac{\ln^2 x}{x} + 2\int\frac{1}{x}\cdot\frac{\ln x\cdot dx}{x} = -\frac{\ln^2 x}{x} + 2\int\frac{\ln x}{x^2}\,dx =$$

$$\int\frac{\ln x}{x^2}\,dx = \left\{ \begin{matrix} u=\ln x \\ \frac{dx}{x^2}=dv \end{matrix} \quad \begin{matrix} du=\frac{dx}{x} \\ v=\int\frac{dx}{x^2}=-\frac{1}{x} \end{matrix} \right\} =$$

$$= -\frac{\ln x}{x} + \int\frac{1\cdot dx}{x\cdot x} = -\frac{\ln x}{x} - \frac{1}{x} + C$$

$$= -\frac{\ln^2 x}{x} + 2\left(-\frac{\ln x}{x} - \frac{1}{x}\right) + C = -\frac{\ln^2 x}{x} - \frac{2\ln x}{x} - \frac{2}{x} + C =$$

$$= -\frac{1}{x}\left(\ln^2 x + 2\ln x + 2\right) + C$$

③ $$\int x^3 e^{-x^2}dx = \frac{1}{2}\int x^2 e^{-x^2}d(x^2) = \{t=x^2\} = \frac{1}{2}\int t\cdot e^{-t}dt =$$

$$= \left\{ \begin{matrix} u=t \\ e^{-t}dt=dv \end{matrix} \quad du=dt \Rightarrow v=\int e^{-t}dt=-e^{-t} \right\} =$$

$$= \frac{1}{2}t\cdot(-e^{-t}) + \frac{1}{2}\int e^{-t}\cdot dt = \frac{1}{2}\left(-t\cdot e^{-t} - e^{-t}\right) + C =$$

$$= -\frac{1}{2}e^{-t}(t+1) + C = -\frac{1}{2}e^{-x^2}(x^2+1)+C$$

④ $$\int x^2\sin 3x\,dx = \left\{ \begin{matrix} u=x^2 \quad du=2x\,dx \\ \sin 3x\,dx=dv \\ v=\int\sin 3x\,dx = -\frac{1}{3}\cos 3x \end{matrix} \right\} =$$

$$= -\frac{1}{3}x^2\cos 3x + \frac{2}{3}\int x\cos 3x\,dx =$$

$$\int x\cos 3x\,dx =$$

⑤ $\int \operatorname{arctg}\sqrt{x}\,dx = \left\{ \begin{matrix} \sqrt{x} = t \\ x = t^2 \end{matrix} \quad dx = 2t\,dt \right\} =$

$= \int 2t \cdot \operatorname{arctg} t\,dt = \left\{ \begin{matrix} u = \operatorname{arctg} t \\ 2t\,dt = dv \end{matrix} \quad v = \int 2t\,dt = t^2 \quad du = \frac{dt}{1+t^2} \right\}$

$= t^2 \cdot \operatorname{arctg} t - \int \frac{t^2}{t^2+1}\,dt = t^2 \cdot \operatorname{arctg} t - \int \left(1 - \frac{1}{t^2+1}\right) dt =$

$= t^2 \operatorname{arctg} t - t + \operatorname{arctg} t + C = \operatorname{arctg} t\,(t^2+1) - t + C =$

$= \operatorname{arctg}\sqrt{x}\,(x+1) - \sqrt{x} + C$

⑥ $\int \frac{x \ln(x + \sqrt{1+x^2})}{\sqrt{1+x^2}}\,dx = \left\{ \begin{matrix} u = \ln(x+\sqrt{1+x^2}) \\ \frac{x}{\sqrt{1+x^2}}\,dx = dv \end{matrix} \right.$

$du = \frac{1}{x+\sqrt{1+x^2}} \cdot \left(1 + \frac{2x}{2\sqrt{1+x^2}}\right) dx = \frac{2\sqrt{1+x^2} + 2x}{2\sqrt{1+x^2}\,(x+\sqrt{1+x^2})} = \frac{dx}{\sqrt{1+x^2}}$

$\left. v = \sqrt{x^2+1} \right\} =$

$= \sqrt{1+x^2} \cdot \ln(x + \sqrt{1+x^2}) - \int \sqrt{x^2+1} \cdot \frac{dx}{\sqrt{1+x^2}} =$

$= \sqrt{1+x^2} \ln(x + \sqrt{1+x^2}) - x + C$

⑦ $\int x \sin\sqrt{x}\,dx = \left\{ \begin{matrix} \sqrt{x} = t \\ x = t^2 \\ dx = 2t\,dt \end{matrix} \right\} = 2\int t \cdot t^2 \sin t\,dt =$

$= \left\{ \begin{matrix} t^3 = u \quad du = 3t^2\,dt \\ \sin t\,dt = dv \\ v = -\cos t \end{matrix} \right\} = 2\left(-t^3 \cos t + 3\int t^2 \cos t\,dt\right) =$

$= -2t^3 \cos t + 6\int t^2 \cos t\,dt = \left\{ \begin{matrix} u = t^2 \quad du = 2t\,dt \\ \cos t\,dt = dv \\ v = \sin t \end{matrix} \right\} =$

$= -2t^3 \cos t + 6\left(t^2 \sin t - 2\int \sin t \cdot t\,dt\right) =$

$= -2t^3 \cos t + 6t^2 \sin t - 12\int t \sin t\,dt = -2t^3 \cos t + 6t^2 \sin t - 12\left(-t \cdot \cos t + \int \cos t\,dt\right) = -2t^3 \cos t + 6t^2 \sin t + 12t \cos t - 12 \sin t + C =$

$= 2t \cos t\,(-t^2 + 6) + 6 \sin t\,(t^2 - 2) + C =$

$= 2\sqrt{x} \cos\sqrt{x}\,(6 - x) + 6 \sin\sqrt{x}\,(x - 2) + C$

8) $I = \int \sin(\ln x)\, dx = \begin{cases} \sin(\ln x) = u \\ dx = dv \\ v = x \end{cases} \quad du = \frac{\cos(\ln x)\, dx}{x} \Bigg\}$

$= x \cdot \sin(\ln x) - \int \frac{\cancel{x} \cdot \cos(\ln x)\, dx}{\cancel{x}} = x \sin(\ln x) - \int \cos(\ln x)\, dx =$

$= \begin{cases} \cos(\ln x) = u \\ dx = dv \to v = x \end{cases} \quad du = \frac{-\sin(\ln x)}{x} \cdot dx \Bigg\} =$

$= x \sin(\ln x) - x \cos(\ln x) - \underbrace{\int \frac{x \cdot \sin(\ln x)}{x}\, dx}_{= I}$

$I = x \sin(\ln x) - x \cos(\ln x) - I + C$

$2I = x(\sin(\ln x) - \cos(\ln x)) + C \Rightarrow I = \frac{x}{2}(\sin(\ln x) - \cos(\ln x)) + C$

9) $\int \frac{dx}{(x^2+a^2)^2}$

$\frac{1}{a} \operatorname{arctg}\frac{x}{a} = \int \frac{dx}{x^2+a^2} = \begin{cases} u = \frac{1}{x^2+a^2} \\ dx = dv \to v = x \end{cases} \quad du = -\frac{2x}{(x^2+a^2)^2}\, dx \Bigg\} =$

$= \frac{x}{x^2+a^2} + 2\int \frac{x^2+a^2-a^2}{(x^2+a^2)^2}\, dx = \frac{x}{x^2+a^2} + 2\int \frac{dx}{(x^2+a^2)} - 2a^2 \int \frac{dx}{(x^2+a^2)^2}$

$\frac{1}{a} \operatorname{arctg}\frac{x}{a} = \frac{x}{x^2+a^2} + \frac{2}{a} \operatorname{arctg}\frac{x}{a} - 2a^2 \int \frac{dx}{(x^2+a^2)^2}$

$2a^2 \int \frac{dx}{(x^2+a^2)^2} = \frac{x}{x^2+a^2} + \underbrace{\frac{2}{a} \operatorname{arctg}\frac{x}{a} - \frac{1}{a} \operatorname{arctg}\frac{x}{a}}_{\frac{1}{a} \operatorname{arctg}\frac{x}{a}}$

$\int \frac{dx}{(x^2+a^2)^2} = \frac{x}{2a^2(x^2+a^2)} + \frac{1}{2a^3} \operatorname{arctg}\frac{x}{a} + C$

10) $\int \sqrt{x^2+a^2}\, dx =$

$\left(\ln(x + \sqrt{x^2+a^2})\right)' = \frac{1}{x + \sqrt{x^2+a^2}}\left(1 + \frac{x}{\sqrt{x^2+a^2}}\right) = \frac{1}{\sqrt{x^2+a^2}}$

const

$\int \frac{dx}{\sqrt{x^2+a^2}} = \ln(x + \sqrt{x^2+a^2}) + C$

$= \begin{cases} u = \sqrt{x^2+a^2} \\ dv = dx \end{cases} \quad du = \frac{x\, dx}{\sqrt{x^2+a^2}}, \; v = x \Bigg\} =$

$= x\sqrt{x^2+a^2} - \int \frac{x^2\, dx}{\sqrt{x^2+a^2}} = x\sqrt{x^2+a^2} - \int \frac{x^2+a^2-a^2}{\sqrt{x^2+a^2}}\, dx =$

$$= x\sqrt{x^2+a^2} - \int\left(\sqrt{x^2+a^2} - \frac{a^2}{\sqrt{x^2+a^2}}\right)dx =$$

$$= x\sqrt{x^2+a^2} - \int\sqrt{x^2+a^2}\,dx + a^2\int\frac{dx}{\sqrt{a^2+x^2}} =$$

$$= x\sqrt{x^2+a^2} - \int\sqrt{x^2+a^2}\,dx + a^2 \cdot \ln\left(x+\sqrt{x^2+a^2}\right)$$

$$2\int\sqrt{x^2+a^2}\,dx = x\sqrt{x^2+a^2} + a^2\ln\left(x+\sqrt{x^2+a^2}\right)$$

$$\int\sqrt{x^2+a^2}\,dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\ln\left(x+\sqrt{x^2+a^2}\right) + C$$

(17) $\int e^{ax}\cos bx\,dx = \left\{\begin{matrix} u = e^{ax} & du = a e^{ax}dx \\ dv = \cos bx\,dx & v = \frac{\sin bx}{b} \end{matrix}\right\} =$

дорешать, решить!

$=$

$$e^{i\varphi} = \cos\varphi + i\sin\varphi$$

$$e^{ax}\left(e^{ibx}\right) = (\cos bx + i\sin bx)\cdot e^{ax}$$

$$Re\left(e^{ax}e^{ibx}\right) = \cos bx \cdot e^{ax}$$

$$Re\left(e^{x(a+ib)}\right) = e^{ax}\cdot\cos bx$$

$$Re\int e^{x(a+ib)}dx = \int e^{ax}\cdot\cos bx$$

$$Re\int e^{x(a+ib)}dx = Re\frac{e^{x(a+ib)}}{a+ib} + C =$$

$$= Re\frac{e^{ax}(\cos bx + i\sin bx)}{(a+ib)}\cdot\frac{(a-ib)}{(a-ib)} =$$

$$= Re\frac{(a-ib)\,e^{ax}(\cos bx + i\sin bx)}{a^2+b^2} = \frac{e^{ax}(a\cos bx + b\sin bx)}{a^2+b^2} + C$$

Домашняя работа.

$$du = -e^{-x}\quad u = xe^{-x}$$
$$du = e^{-x}dx - x\cdot e^{-x}dx$$

1) $\int x e^{-x} dx = \left\{ \begin{array}{l} \cancel{u = e^{-x}} \\ \cancel{dv = x\,dx} \end{array} \right.$

$= \left\{ \begin{array}{ll} u = x & \Rightarrow du = dx \\ dv = e^{-x} dx & v = -e^{-x} \end{array} \right\} =$

$= -xe^{-x} + \int e^{-x} dx = -xe^{-x} - e^{-x} + C = -e^{-x}(x+1) + C$

2) $\int x^2 e^{-2x} dx = \left\{ \begin{array}{ll} u = x^2 & du = 2x\,dx \\ dv = e^{-2x} dx & \Rightarrow v = -\frac{1}{2} e^{-2x} \end{array} \right\} =$

$= -x^2 \cdot \left(\frac{1}{2} e^{-2x}\right) + \frac{1}{2} \int e^{-2x} 2x\,dx = -\frac{x^2 e^{-2x}}{2} + \int x \cdot e^{-2x} dx =$

$= \left\{ \begin{array}{ll} u = x & \Rightarrow du = dx \\ dv = e^{-2x} dx & \Rightarrow v = -\frac{e^{-2x}}{2} \end{array} \right\} =$

$= -\frac{x^2 e^{-2x}}{2} + \left(-\frac{x \cdot e^{-2x}}{2} + \frac{1}{2} \int e^{-2x} dx\right) =$

$= -\frac{x^2 e^{-2x}}{2} - \frac{x \cdot e^{-2x}}{2} - \frac{1}{4} e^{-2x} + C = -\frac{1}{4} e^{-2x} (2x^2 + 2x + 1) + C$

3) $\int x \cos x \cdot dx = \left\{ \begin{array}{ll} u = x & \Rightarrow du = dx \\ dv = \cos x \cdot dx & \Rightarrow v = \sin x \end{array} \right\} =$

$= x \sin x - \int \sin x\, dx = x \sin x + \cos x + C.$

4) $\int (x+2) \sin 5x\, d(x) = \left\{ \begin{array}{ll} u = (x+2) & \Rightarrow du = dx \\ \sin 5x\, dx = dv & \Rightarrow v = -\frac{\cos 5x}{5} \end{array} \right\} =$

$= -\frac{(x+2)\cos 5x}{5} + \int \frac{\cos 5x}{5} dx = -\frac{(x+2)\cos 5x}{5} + \frac{\sin 5x}{25} + C$

5) $\int (x^2+x+1) \cos(2x+1) dx = \left\{ \begin{array}{ll} u = x^2+x+1 & \Rightarrow du = (2x+1) dx \\ dv = \cos(2x+1) dx & \Rightarrow v = \frac{\sin(2x+1)}{2} \end{array} \right\} =$

$= (x^2+x+1) \frac{\sin(2x+1)}{2} - \frac{1}{2} \int \sin(2x+1)(2x+1) dx$

$= \frac{(x^2+x+1)\sin(2x+1)}{2} - \frac{1}{2}\left(\int \sin(2x+1) \cdot 2x\, dx + \int \sin(2x+1) dx\right) =$

$= \frac{(x^2+x+1)\sin(2x+1)}{2} - \int x \sin(2x+1) dx + \frac{\cos(2x+1)}{4} =$

$= \frac{(x^2+x+1)\sin(2x+1)}{2} + \frac{x\cos(2x+1)}{2} - \frac{\sin(2x+1)}{4} + \frac{\cos(2x+1)}{4} =$

$= \frac{1}{4} \sin(2x+1)(2x^2+2x+1) + \frac{1}{4} \cos(2x+1)(2x+1) + C$

6) $\int \operatorname{arctg} x \, dx = \left\{ \begin{matrix} u = \operatorname{arctg} x & du = \dfrac{dx}{1+x^2} \\ dv = dx & \to v = x \end{matrix} \right\} =$

$= x \operatorname{arctg} x - \int \frac{x\,dx}{1+x^2} =$ $\qquad d(x^2) = 2x\,dx$

$= x \operatorname{arctg} x - \frac{1}{2} \int \frac{d(x^2+1)}{(x^2+1)} = x \operatorname{arctg} x - \frac{1}{2} \ln(1+x^2) + C$

7) $\int \ln(x+\sqrt{1+x^2})\, dx = \left\{ \begin{matrix} u = \ln(x+\sqrt{1+x^2}) & du = \dfrac{\left(1+\frac{2x}{2\sqrt{1+x^2}}\right)}{x+\sqrt{1+x^2}} \cdot dx = \dfrac{dx}{\sqrt{1+x^2}} \\ dv = dx & \to v = x \end{matrix} \right\} =$

$= x \ln(x+\sqrt{1+x^2}) - \int \frac{x\,dx}{\sqrt{1+x^2}} = x \ln(x+\sqrt{1+x^2}) - \frac{1}{2} \int (1+x^2)^{-\frac{1}{2}} d(1+x^2) =$

$= x \ln(x+\sqrt{1+x^2}) - \sqrt{1+x^2} + C$

8) $\int \sin x \cdot \ln(\operatorname{tg} x)\, dx = \left\{ \begin{matrix} u = \ln(\operatorname{tg} x) & du = \dfrac{1\,dx}{\operatorname{tg} x \cos^2 x} = \dfrac{dx}{\sin x \cos x} = \dfrac{2dx}{\sin 2x} \\ dv = \sin x\, dx & \to v = -\cos x \end{matrix} \right\}$

$= -\cos x \ln(\operatorname{tg} x) + \int \cos x \frac{2dx}{\sin 2x} = -\cos x \ln(\operatorname{tg} x) + \int \frac{\cancel{\cos x}\, dx}{\sin x \cancel{\cos x}} =$

$= -\cos x \ln(\operatorname{tg} x) + \underline{\int \frac{dx}{\sin x}} = \ln\left|\operatorname{tg} \frac{x}{2}\right| - \cos x \cdot \ln(\operatorname{tg} x) + C$

9) $\int x^5 e^{x^3} dx = \int x^3 \cdot x^2 dx \cdot e^{x^3} = \frac{1}{3} \int x^3 \cdot e^{x^3} d(x^3) = \{t = x^3\} =$

$= \frac{1}{3} \int t \cdot e^t\, dt = \left\{ \begin{matrix} u = t & du = dt \\ dv = e^t dt & \to v = e^t \end{matrix} \right\} =$

$= \frac{1}{3}\left(te^t - \int e^t \cdot dt\right) = \frac{1}{3}(t \cdot e^t - e^t) + C = \frac{1}{3} e^t (t-1) + C =$

$= \frac{1}{3} \cdot e^{x^3} (x^3 - 1) + C.$

10) $\int (\arcsin x)^2 dx = \left\{ \begin{matrix} u = (\arcsin x)^2 & du = 2\arcsin x \cdot \dfrac{dx}{\sqrt{1-x^2}} \\ dv = dx & \to v = x \end{matrix} \right\} =$

$= x(\arcsin x)^2 - 2\int \frac{x \arcsin x \cdot dx}{\sqrt{1-x^2}} = x(\arcsin x)^2 - \int \frac{\arcsin x\, dx^2}{\sqrt{1-x^2}} =$

$= \left\{ \begin{matrix} u = \arcsin x & du = \dfrac{dx}{\sqrt{1-x^2}} \\ dv = \dfrac{dx^2}{\sqrt{1-x^2}} & \Rightarrow v = -2\sqrt{1-x^2} \end{matrix} \right\} =$

$= x(\arcsin x)^2 + 2\sqrt{1-x^2} \arcsin x - 2x + C$

11) $\int \cos(\ln x)\, dx = \left\{ \begin{matrix} u = \cos(\ln x) & du = -\dfrac{\sin(\ln x)}{x} dx \\ dv = dx & \to v = x \end{matrix} \right\} =$

$= x\cos(\ln x) + \underline{\int \sin(\ln x)\, dx} = \left\{ \begin{matrix} u = \sin(\ln x) & \to du = \dfrac{\cos(\ln x)}{x} \\ dv = dx & \to v = x \end{matrix} \right\} =$

$$= x\cos(\ln x) + x\sin(\ln x) - \int \cos(\ln x)\,dx$$

$$2\int \cos(\ln x)\,dx = x\cos(\ln x) + x\sin(\ln x)$$

$$\Rightarrow \int \cos(\ln x)\,dx = \frac{x}{2}\Big(\cos(\ln x) + \sin(\ln x)\Big) + C.$$

12) $$\int e^{ax}\sin bx\cdot dx = \left\{\begin{array}{ll} u = e^{ax} & \to du = a e^{ax}\cdot dx \\ dv = \sin bx\cdot dx & v = -\dfrac{\cos bx}{b} \end{array}\right\} =$$

$$= -\frac{e^{ax}\cos bx}{b} + \underline{\frac{a}{b}\int e^{ax}\cos bx\cdot dx} = \left\{\begin{array}{ll} u = e^{ax} & du = a e^{ax}\,dx \\ dv = \cos bx\cdot dx & \to v = \dfrac{\sin bx}{b} \end{array}\right.$$

$$= -\frac{e^{ax}\cos bx}{b} + \frac{a}{b}\left(\frac{e^{ax}\sin bx}{b} - \frac{a}{b}\int e^{ax}\sin bx\cdot dx\right) =$$

$$= -\frac{e^{ax}\cos bx}{b} + \frac{a e^{ax}\sin bx}{b^2} - \frac{a^2}{b^2}\int e^{ax}\sin bx\cdot dx$$

$$\left(1 + \frac{a^2}{b^2}\right)\int e^{ax}\sin bx\cdot dx = \frac{a e^{ax}\sin bx}{b^2} - \frac{e^{ax}\cos bx}{b}$$

$$\int e^{ax}\sin bx\cdot dx = \frac{(a e^{ax}\sin bx - b e^{ax}\cos bx)\cancel{b^2}}{\cancel{b^2}(a^2+b^2)} + C$$

$$\int e^{ax}\sin bx\cdot dx = \frac{e^{ax}(a\sin bx - b\cos bx)}{a^2+b^2} + C.$$

8.12.2020

Интегрирование рациональных дробей.

① $\int \frac{2x^2+41x-91}{(x-1)(x+3)(x-4)}dx =$

$$\frac{2x^2+41x-91}{(x-1)(x+3)(x-4)} = \frac{A}{x-1}^{(x+3)(x-4)} + \frac{B}{x+3}^{(x-1)(x-4)} + \frac{C}{x-4}^{(x-1)(x+3)} = \frac{A(x+3)(x-4)+B(x-1)(x-4)+C(x-1)(x+3)}{(x-1)(x+3)(x-4)} =$$

$\Rightarrow 2x^2+41x-91 = A(x+3)(x-4)+B(x-1)(x-4)+C(x-1)(x+3)$

при $x=-3$ $\quad 2\cdot(-3)^2+41\cdot(-3)-91 = B(-3-1)(-3-4)$

$28B = -196 \Rightarrow B=-7.$

$x=4$ $\quad 2\cdot 16+41\cdot 4-91 = C(3\cdot 7)$

$21C = 105 \Rightarrow C=5$

$x=1$ $\quad 2+41-91 = (-3)\cdot 4A$

$-12A = -48 \Rightarrow A=4.$

$$\frac{2x^2+41x-91}{(x-1)(x+3)(x-4)} = \frac{4}{x-1} + \frac{(-7)}{x+3} + \frac{5}{x-4}$$

$= 4\int\frac{dx}{x-1} - 7\int\frac{dx}{x+3} + 5\int\frac{dx}{x-4} = 4\ln|x-1| - 7\ln|x+3| + 5\ln|x-4| + C =$

$= \ln\left|\frac{(x-1)^4(x-4)^5}{(x+3)^7}\right| + C$

2 способ: $2x^2+41x-91 = A(x^2-x-12)+B(x^2-5x+4)+C(x^2+2x-3)$

$2x^2+41x-91 = Ax^2-Ax-12A+Bx^2-5Bx+4B+Cx^2+2Cx-3C$

$2x^2+41x-91 = x^2(A+B+C)+x(2C-A-5B)-(3C-4B+12A)$

Методом Крамера решаем систему.

$$\begin{cases} A+B+C=2 \\ 2C-A-5B=41 \\ 3C-4B+12A=91 \end{cases} \Leftrightarrow \quad \Delta = \begin{vmatrix} 1 & 1 & 1 \\ -1 & -5 & 2 \\ 12 & -4 & 3 \end{vmatrix} = (-15+24+4+60+3+8) = 84$$

$$\begin{cases} A+B+C=2 \\ -A-5B+2C=41 \\ 12A-4B-3C=91 \end{cases} \quad \Delta A = \begin{vmatrix} 2 & 1 & 1 \\ 41 & -5 & 2 \\ 91 & -4 & -3 \end{vmatrix} = (30+182-164+455+123+16) = 642 = 336$$

Не сошлось ?

$$\Delta B = \begin{vmatrix} 1 & 2 & 1 \\ -1 & 41 & 2 \\ 12 & 91 & -3 \end{vmatrix} = (-123-91+48-492-6-182) = \cancel{-846} = -396$$

$$\Delta C = \begin{vmatrix} 1 & 1 & 2 \\ -1 & -5 & 41 \\ 12 & -4 & 91 \end{vmatrix} = (-455+8+492+120+91+164) = 420 =$$

$A = \frac{\Delta A}{\Delta} = 4 \qquad B = \frac{\Delta B}{\Delta} = -7 \qquad C = \frac{\Delta C}{\Delta} = 5$

② $\int \frac{dx}{x(x+1)(x^2+x+1)} =$

$$\frac{1}{x(x+1)(x^2+x+1)} = \frac{A}{x} + \frac{B}{x+1} + \frac{Cx+D}{x^2+x+1}$$

$$A(x+1)(x^2+x+1) + Bx(x^2+x+1) + (Cx+D)(x^2+x) = 1$$

$x=-1$ $\quad B(-1)(\cancel{1}-\cancel{1}+1)=1 \quad \Rightarrow B-1$

$x=0$ $\quad A=1$

$x=i$ $\quad (1+i)(\cancel{i^2}+i+\cancel{1}) - i(\cancel{i^2}+i+\cancel{1}) + (Ci+D)\cdot i(i+1)=1$

$$i(1+i) - i^2 + (i^2+i)\cdot(Ci+D)=1$$

$$i+\cancel{i^2}-\cancel{i^2}+(-1+i)(Ci+D)=1$$

$$i+(i-1)(Ci+D)=1$$

$$i+(i-1)Ci+(i-1)D=1$$

$$i+Ci^2-Ci+Di-D=1$$

$$\underbrace{i(1-C+D)-C-D}_{a+bi}\ \underbrace{=1}_{=c+di}$$

$$\begin{cases} 1-C+D=0 \\ -C-D=1 \end{cases} \quad \begin{matrix} 1+D+1+D=0 \\ \Rightarrow C=-D-1 \end{matrix} \quad \Rightarrow \cancel{D=-1},\ \cancel{C=0} \qquad \begin{matrix} D=-1 \\ C=0 \end{matrix}$$

$$= \int\frac{dx}{x} - \int\frac{dx}{x+1} - \int\frac{dx}{x^2+x+1} = \ln|x| - \ln|x+1| - \int\frac{dx}{x^2+x+\frac{1}{4}+\frac{3}{4}} =$$

$$= \ln|x| - \ln|x+1| - \underbrace{\int\frac{d(x+\frac{1}{2})}{(x+\frac{1}{2})^2+\frac{3}{4}}}_{\text{по формуле}} = \ln|x| - \ln|x+1| - \frac{2}{\sqrt{3}}\operatorname{arctg}\frac{2}{\sqrt{3}}\left(\frac{2x+1}{2}\right)$$

$$= \ln\left|\frac{x}{x+1}\right| - \frac{2}{\sqrt{3}}\operatorname{arctg}\frac{2x+1}{\sqrt{3}} + C$$

③ $\int \frac{dx}{x^4+x^2+1} = \{$

$$x^4+x^2+1 = (x^2+ax+b)(x^2+cx+d) = (x^4 + x^3\underbrace{(c+a)}_{=0} + x^2\underbrace{(d+b+ac)}_{=1} +$$

$$+ x^1\underbrace{(ad+bc)}_{=0} + \underbrace{bd}_{=1})$$

$$\begin{cases} c+a=0 \\ d+b+ac=1 \\ ad+bc=0 \\ bd=1 \end{cases}$$

Подберем: $b=1$, $d=1$

$a+b=0 \to a=-1$

$a+c=0 \quad a+c=0$

$2+ac=1 \to ac=-1 \quad c=1$

Если

$$\begin{cases} b=-1 \\ d=-1 \\ \text{не получилось бы.} \end{cases}$$

$b=1$
$d=1$
$a=-1$
$c=1.$

$$x^4+x^2+1=(x^2-x+1)(x^2+x+1)$$

2 способ: $x^4+x^2+1 = x^4+2x^2+1-x^2=(x^2+1)^2-x^2=$

$=(x^2+1-x)(x^2+1+x)=(x^2-x+1)(x^2+x+1)$

$$\frac{1}{x^4+x^2+1}=\frac{Ax+B}{x^2-x+1}+\frac{Cx+D}{x^2+x+1}$$

$$(Ax+B)(x^2+x+1)+(Cx+D)(x^2-x+1)=1.$$

$x=0 \to B+D=1$

$$\begin{cases} x^3: & A+C=0 & \to A=-C \\ x^2: & A+B-C+D=0 & -C+1-\cancel{D}-C+\cancel{D}=0 \\ x^1: & A+B+C-D=0 & \\ x^0: & B+D=1 & B=1-D \end{cases}$$

$-2C+1=0$
$C=\frac{1}{2}.$
$A=-\frac{1}{2}$
$B=$

$-2C+2D=0$
$C=D=\frac{1}{2}$

$\Rightarrow A=-\frac{1}{2};\ B=\frac{1}{2},\ C=\frac{1}{2},\ D=\frac{1}{2}$

$$=-\frac{1}{2}\int\frac{(x-1)}{x^2-x+1}dx+\frac{1}{2}\int\frac{(x+1)}{x^2+x+1}dx=$$

Посчитаем по отдельности:

$$\int\frac{(x-1)}{x^2-x+1}dx=\cancel{\int\frac{(x-\frac{1}{2})-\frac{1}{2}}{(x-\frac{1}{2})^2+\frac{3}{4}}dx}=$$

$(x^2-x+1)' = 2x-1$ $\Rightarrow$ $= \frac{1}{2}\int \frac{2(x-1)}{x^2-x+1}dx = \frac{1}{2}\int\frac{2x-2}{x^2-x+1}dx =$

чтобы внести в дифференциал.

$$= \frac{1}{2}\int\frac{2x-1-1}{x^2-x+1}dx = \frac{1}{2}\int\frac{(2x-1)\,dx}{x^2-x+1} - \frac{1}{2}\int\frac{dx}{x^2-x+1} =$$

$$= \frac{1}{2}\int\frac{d(x^2-x+1)}{(x^2-x+1)} - \frac{1}{2}\int\frac{dx}{(x-\frac{1}{2})^2+\frac{3}{4}} =$$

$$= \frac{1}{2}\ln(x^2-x+1) - \frac{1}{\cancel{2}}\cdot\frac{\cancel{2}}{\sqrt{3}}\operatorname{arctg}\frac{2x-1}{\sqrt{3}}$$

$$\int\frac{(x+1)dx}{x^2+x+1} \ominus$$

$(x^2+x+1)' = 2x+1 \Rightarrow \ominus \frac{1}{2}\int\frac{2x+2}{x^2+x+1}dx = \frac{1}{2}\int\frac{(2x+1)+1}{x^2+x+1}dx =$

$$= \frac{1}{2}\int\frac{2x+1}{x^2+x+1}dx + \frac{1}{2}\int\frac{dx}{x^2+x+1} =$$

$$= \frac{1}{2}\int\frac{d(x^2+x+1)}{x^2+x+1} + \frac{1}{2}\int\frac{dx}{(x+\frac{1}{2})^2+\frac{3}{4}} =$$

$$= \frac{1}{2}\ln(x^2+x+1) + \frac{1}{\cancel{2}}\cdot\frac{\cancel{2}}{\sqrt{3}}\operatorname{arctg}\frac{2x+1}{\sqrt{3}}$$

Исх. интеграл: $\ominus \frac{1}{4}\ln\frac{x^2+x+1}{x^2-x+1} + \frac{1}{2\sqrt{3}}\left(\operatorname{arctg}\frac{2x+1}{\sqrt{3}} + \operatorname{arctg}\frac{2x-1}{\sqrt{3}}\right)$

Домашняя работа.

① $\int\frac{2x+3}{(x-2)(x+5)}dx \ominus$

$$\frac{2x+3}{(x-2)(x+5)} = \frac{A}{x-2} + \frac{B}{x+5} = \frac{Ax+5A+Bx-2B}{(x-2)(x+5)} = \frac{x(A+B)+(5A-2B)}{(x-2)(x+5)}$$

$\begin{cases} A+B=2 \\ 5A-2B=3 \end{cases}$ $-\begin{cases} 5A+5B=10 \\ 5A-2B=3 \end{cases}$ $7B=7 \to B=1.$ $A=1.$

$$\ominus \int\frac{dx}{x-2} + \int\frac{dx}{x+5} = \ln|x-2| + \ln|x+5| + C$$

② $\int\frac{x\,dx}{(x+1)(x+2)(x+3)} \ominus$

$$\frac{x}{(x+1)(x+2)(x+3)} = \frac{A}{x+1} + \frac{B}{x+2} + \frac{C}{x+3}$$

$$A(x^2+5x+6) + B(x^2+4x+3) + C(x^2+3x+2) = x$$

$$Ax^2+Bx^2+Cx^2+5Ax+4Bx+3Cx+6A+3B+2C=x$$

$$x^2(A+B+C)+x(5A+4B+3C)+(6A+3B+2C)=x$$

$$\begin{cases} A+B+C=0 \\ 5A+4B+3C=1 \\ 6A+3B+2C=0 \end{cases}$$

$$A=-B-C=1+2C-C=1+C$$

$$5(-B-C)+4B+3C=1$$

$$-5B-5C+4B+3C=1$$

$$-B-2C=1 \Rightarrow B=-1-2C$$

$$6(1+C)+3(-1-2C)+2C=0$$

$$6+\cancel{6C}-3-\cancel{6C}+2C=0 \qquad 2C+3=0$$

$$C=-\frac{3}{2}$$

$$B=2$$

$$A=-\frac{1}{2}$$

$$= -\frac{1}{2}\int\frac{dx}{(x+1)}+2\int\frac{dx}{x+2}-\frac{3}{2}\int\frac{dx}{x+3}=$$

$$=-\frac{1}{2}\ln|x+1|+2\ln|x+2|-\frac{3}{2}\ln|x+3|+C=$$

$$=\frac{1}{2}\left(\ln(x+2)^4-\ln|x+1|-3\ln|x+3|\right)+C=$$

$$=\frac{1}{2}\ln\left|\frac{(x+2)^4}{(x+1)(x+3)^3}\right|+C$$

③ $$\int\frac{x^3+1}{x^3-5x^2+6x}dx=\int\frac{x^3-5x^2+6x+1+5x^2-6x}{x^3-5x^2+6x}dx=$$

$$=\int dx+\int\frac{5x^2-6x+1}{x(x-3)(x-2)}dx$$

$$\frac{5x^2-6x+1}{x(x-3)(x-2)}=\frac{A}{x}+\frac{B}{x-3}+\frac{C}{x-2}=\cancel{A(x^2-5x+6)+B(x^2-2x}$$

$$A(x-3)(x-2)+B(x-2)x+C(x-3)x=5x^2-6x+1.$$

$x=3$: $3B=28 \rightarrow B=\frac{28}{3}$

$x=2$: $-2C=9 \rightarrow C=-\frac{9}{2}$

$x=0$: $6A=1 \rightarrow A=\frac{1}{6}$

$$=\int dx+\frac{1}{6}\int\frac{dx}{x}+\frac{28}{3}\int\frac{dx}{x-3}-\frac{9}{2}\int\frac{dx}{x-2}=$$

$$=x+\frac{1}{6}\ln|x|+\frac{28}{3}\ln|x-3|-\frac{9}{2}\ln|x-2|+C$$

$$④\ \int \frac{x^4}{x^4+5x^2+4}\,dx = \int \frac{x^4+5x^2+4-5x^2-4}{x^4+5x^2+4}\,dx =$$

$$= \int dx - \int \frac{5x^2+4}{x^4+5x^2+4}\,dx$$

$$x^4+5x^2+4 = x^4+4x^2+4+x^2 = (x^2+2)^2+x^2$$

$$x^4+5x^2+4 = x^4+6x^2+9-(x^2+5)$$

$$x^4+5x^2+4 = (x^2+ax+b)(x^2+cx+d) = (x^4+x^3(c+a)+x^2(d+b+ac)+$$
$$+x^1(ad+bc)+bd)$$

$$\begin{cases} c+a=0 \\ d+b+ac=5 \\ ad+bc=0 \\ bd=4 \end{cases} \Longleftrightarrow \quad b=1 \quad d=4 \quad a=0 \quad c=0$$

$$x^4+5x^2+4 = (x^2+1)(x^2+4)$$

$$\frac{5x^2+4}{(x^2+1)(x^2+4)} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{x^2+4}$$

$$(Ax+B)(x^2+4) + (Cx+D)(x^2+1) = 5x^2+4$$

$$Ax^3 + 4Ax + Bx^2 + 4B + Cx^3 + Cx + Dx^2 + D = 5x^2+4$$

$$x^3(A+C) + x^2(B+D) + x(4A+C) + (4B+D) = 5x^2+4$$

$$\begin{cases} A+C=0 \\ B+D=5 \\ 4A+C=0 \\ 4B+D=4 \end{cases} \qquad \begin{aligned} &A=-C \\ &-4C+C=0 \Rightarrow C=0 \ ; \ A=0 \\ &B=5-D \\ &4(5-D)+D=4 \\ &20-4D+D=4 \quad \Rightarrow 3D=16 \qquad D=\frac{16}{3} \end{aligned}$$

$$B = 5-\frac{16}{3} = \frac{15-16}{3} = -\frac{1}{3}$$

$$\textcircled{=}\ \int dx - \frac{16}{3}\int\frac{dx}{4+x^2} + \frac{1}{3}\int\frac{dx}{1+x^2} =$$

$$= x - \frac{16}{3}\cdot\frac{1}{2}\,\mathrm{arctg}\,\frac{x}{2} + \frac{1}{3}\,\mathrm{arctg}\,x + C = x - \frac{8}{3}\,\mathrm{arctg}\,\frac{x}{2} + \frac{1}{3}\,\mathrm{arctg}\,x$$

$$⑤\ \int \frac{x^2+5x+4}{x^4+5x^2+4}\,dx \ \textcircled{=}$$

$$\frac{x^2+5x+4}{(x^2+1)(x^2+4)} = \frac{Ax+B}{(x^2+1)} + \frac{Cx+D}{(x^2+4)}$$

$$x^3(A+C) + x^2(B+D) + x(4A+C) + (4B+D) = x^2+5x+4$$

$$\begin{cases} A+C=0 \\ B+D=1 \\ 4A+C=5 \\ 4B+D=4 \end{cases} \qquad A=-C$$

$$-4C+C=5 \qquad C=-\frac{5}{3} \quad \Rightarrow A=\frac{5}{3}$$

$$B=1-D$$

$$\cancel{4}-4D+D=\cancel{4} \rightarrow D=0 \quad \Rightarrow B=1.$$

$$= \int \frac{\frac{5}{3}x+1}{x^2+1}dx + -\frac{5}{3}\int \frac{x\,dx}{x^2+4}dx =$$

$$= \cancel{\frac{5}{3}\int\left(x+\frac{3}{5}\right.} \ \frac{1}{3}\int\frac{5x+3}{x^2+1}dx - \frac{5}{6}\int\frac{d(x^2+4)}{x^2+4}dx =$$

$$= \frac{1}{3}\int\frac{5x+3}{x^2+1}dx - \frac{5}{6}\ln(x^2+4) =$$

$$\frac{1}{3}\int\frac{5x+3}{x^2+1}dx = \frac{1}{3}\int\frac{2x+1+3x+2}{x^2+1}dx = \frac{1}{3}\left(\int\frac{2x+1}{x^2+1} + \int\frac{3x+2}{x^2+1}dx\right) =$$

$$\boxed{(x^2+1)' = 2x+1} \quad \frac{1}{3}\left(\int\frac{d(x^2+1)}{x^2+1} + \int\frac{2x+1+x+1}{x^2+1}dx\right) =$$

$$= \frac{1}{3}\left(\ln(x^2+1) + \int\frac{d(x^2+1)}{x^2+1} + \frac{1}{2}\int\frac{2x+1+1}{x^2+1}dx\right) = \cancel{\frac{1}{6}(3\ln(x^2+1)+}$$

$$= \frac{1}{3}\left(2\ln(x^2+1) + \frac{1}{2}\ln(x^2+1) + \frac{1}{2}\int\frac{dx}{1+x^2}\right) =$$

$$= \frac{5}{6}\ln(x^2+1) + \frac{1}{6}\operatorname{arctg}x + C$$

$$= \frac{1}{6}\operatorname{arctg}x + \frac{5}{6}\ln\frac{x^2+1}{x^2+4} + C$$

⑥ $\int \frac{x\,dx}{x^3-1} = \int \frac{x\,dx}{(x-1)(x^2+x+1)}$

$$\frac{x}{(x-1)(x^2+x+1)} = \frac{A}{(x-1)} + \frac{Bx+C}{(x^2+x+1)}$$

$$A(x^2+x+1) + (Bx+C)(x-1) = x$$

$$\underline{Ax^2} + Ax + A + \underline{Bx^2} - Bx + Cx - C = x$$

$$x^2(A+B) + x(A-B+C) + (A-C) = x$$

$$\begin{cases} A+B=0 \quad \to A=-B \\ A-B+C=1 \\ A-C=0 \\ A=C \end{cases} \to \quad C+C-B=1 \quad 2C=1+B$$

$$C=-B \quad B=-C \quad C+C+C=1 \quad \to C=\frac{1}{3}$$

$$\to B=-\frac{1}{3}$$

$$A=\frac{1}{3}$$

$$\frac{1}{3}\int\frac{dx}{x-1} + \int\frac{-\frac{1}{3}x+\frac{1}{3}}{x^2+x+1}dx = \frac{1}{3}\ln|x-1| - \frac{1}{3}\int\frac{(x-1)\,dx}{x^2+x+1} \ominus$$

$$\int\frac{(x-1)}{x^2+x+1}dx = \frac{1}{2}\int\frac{2x+1-3}{x^2+x+1}dx =$$

$$\boxed{(x^2+x+1)' = 2x+1} \quad = \frac{1}{2}\left(\int\frac{d(x^2+x+1)}{x^2+x+1} - 3\int\frac{dx}{x^2+x+1}\right) =$$

$$= \frac{1}{2}\left(\ln|x^2+x+1| - 3\left(\frac{2}{\sqrt{3}}\operatorname{arctg}\frac{2x+1}{\sqrt{3}}\right)\right) =$$

$$= \frac{1}{2}\ln|x^2+x+1| - \frac{3}{\sqrt{3}}\operatorname{arctg}\frac{2x+1}{\sqrt{3}} + C$$

$$\ominus \frac{1}{3}\ln|x-1| - \frac{1}{6}\ln|x^2+x+1| + \frac{1}{\sqrt{3}}\operatorname{arctg}\frac{2x+1}{\sqrt{3}} + C =$$

$$= \frac{1}{6}\ln\frac{(x-1)^2}{x^2+x+1} + \frac{1}{\sqrt{3}}\operatorname{arctg}\frac{2x+1}{\sqrt{3}} + C.$$

Интегрирование рациональных дробей

① $\int \frac{x^2+1}{(x+1)^2(x-1)} dx =$

$$\frac{x^2+1}{(x+1)^2(x-1)} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$$

$$A(x+1)^2 + B(x+1)(x-1) + C(x-1) = x^2+1.$$

$x=-1 \rightarrow -2C = 2 \rightarrow C = -1.$

$x=1 \rightarrow A = \frac{1}{2}$

$\rightarrow B = \frac{1}{2}$

$$= \frac{1}{2}\int \frac{dx}{x-1} + \frac{1}{2}\int \frac{dx}{x+1} - \int \frac{dx}{(x+1)^2} = \frac{1}{2}\ln|x-1| + \frac{1}{2}\ln|x+1| + \frac{1}{x+1} + C =$$

$$= \frac{1}{2}\ln|x^2-1| + \frac{1}{x+1} + C$$

② $\int \frac{x\,dx}{(x-1)^2(x^2+2x+2)} =$

$$\frac{x}{(x-1)^2(x^2+2x+2)} = \frac{A}{(x-1)} + \frac{B}{(x-1)^2} + \frac{Cx+D}{x^2+2x+2}$$

$$x = A(x-1)(x^2+2x+2) + B(x^2+2x+2) + (Cx+D)(x-1)^2$$

$$x = A(x^3+2x^2+\cancel{2x}-x^2-\cancel{2x}-2) + Bx^2+2Bx+2B + (Cx+D)(x^2-2x+1)$$

$$x = A(x^3+x^2-2) + Bx^2+2Bx+2B + Cx^3-2Cx^2+Cx+Dx^2-2Dx+D$$

$$x = Ax^3+Cx^3 + Ax^2+Bx^2-2Cx^2+Dx^2+2Bx+Cx-2Dx+D+2B-2A$$

$$x = x^3(A+C) + x^2(A+B-2C+D) + x(2B+C-2D) + (D+2B-2A)$$

$$\begin{cases} A+C=0 \\ A+B-2C+D=0 \\ 2B+C-2D=1 \\ D+2B-2A=0 \end{cases}$$

$A=-C$

$D = 2A-2B = -2C+10C = 8C$

$-C+B-2C+2A-2B=0$

$-C+B-2C-2C-2B=0$

$-5C-B=0 \Rightarrow B=-5C$

~~$-C-5C-2C+8C=0$~~

$-10C+C-16C=1$; $-25C=1 \rightarrow C = -\frac{1}{25}$

$-26C+C=1$;

$\rightarrow A = \frac{1}{25}$

$\rightarrow B = \frac{1}{5}$

$\rightarrow D = -\frac{8}{25}$

$$= \frac{1}{25}\int \frac{dx}{x-1} + \frac{1}{5}\int \frac{dx}{(x-1)^2} + \int \frac{\left(-\frac{1}{25}x - \frac{8}{25}\right)dx}{x^2+2x+2} =$$

$$= \frac{1}{25}\ln|x-1| - \frac{1}{5(x-1)} - \frac{1}{25}\int\frac{(x+8)\,dx}{x^2+2x+2}$$

$$(x^2+2x+2)' = 2x+2$$

$$\frac{1}{25}\int\frac{(x+8)\,dx}{x^2+2x+2} = \frac{1}{50}\int\frac{2x+16}{x^2+2x+2}\,dx =$$

$$= \frac{1}{50}\int\frac{(2x+2)+14}{x^2+2x+2}\,dx = \frac{1}{50}\left(\int\frac{d(x^2+2x+2)}{x^2+2x+2} + 14\int\frac{dx}{x^2+2x+2}\right) =$$

$$= \frac{1}{50}\left(\ln|x^2+2x+2| + 14\,\mathrm{arctg}(x+1)\right)$$

$$= \frac{1}{25}\ln|x-1| - \frac{\cancel{1}}{5(x-1)} - \frac{1}{50}\ln(x^2+2x+2) - \frac{14}{50}\,\mathrm{arctg}(x+1) + C =$$

$$= \frac{1}{50}\ln\left|\frac{(x-1)^2}{x^2+2x+2}\right| - \frac{1}{5(x-1)} - \frac{14}{50}\,\mathrm{arctg}(x+1) + C$$

Домашняя работа

1) $\int\frac{x\,dx}{x^3-3x+2} = \int\frac{x\,dx}{(x+2)(x-1)^2} =$

$$\frac{x}{(x+2)(x-1)^2} = \frac{A}{x+2} + \frac{B}{x-1} + \cancel{\frac{Cx+D}{(x-1)^2}} + \frac{C}{(x-1)^2}$$

$$\cancel{A(x-1)^2 + B(x-1)(x+2) + (Cx+D)(x+2) = x}$$
$$\cancel{A(x^2-2x+1) + B(x^2+x-2) + Cx^2 + 2Cx + Dx + 2D = x}$$
$$\cancel{Ax^2 - 2Ax + A + Bx^2 + Bx - 2B + Cx^2 + 2Cx + Dx + 2D = x}$$
$$\cancel{x^2(A+B+C) + x(B-2A+2C+D) + (A-2B+2D) = x}$$
$$\cancel{A+B+C=0}$$
$$\cancel{B-2A+2C+D=1}$$

$$A(x-1)^2 + B(x-1)(x+2) + C(x+2) = x$$

$x=1 \Rightarrow 3C = 1 \Rightarrow C = \frac{1}{3}$

$x=-2 \Rightarrow 9A = -2 \Rightarrow A = -\frac{2}{9}$

$$Ax^2 - 2Ax + A + Bx^2 + Bx - 2B + Cx + 2C = x$$
$$x^2(A+B) + x(B-2A+C) + (A-2B+2C) = x$$

$$\begin{cases} A+B=0 \\ B-2A+C=1. \\ A-2B+2C=0 \end{cases} \Rightarrow A = -\frac{2}{9};\ C = \frac{1}{3};\ B = \frac{2}{9}$$

$$= -\int\frac{2\,dx}{9(x+2)} + \frac{2}{9}\int\frac{dx}{x-1} + \frac{1}{3}\int\frac{dx}{(x-1)^2} = -\frac{2}{9}\ln|x+2| + \frac{2}{9}\ln|x-1| - \frac{1}{3(x-1)}$$
$$+ C = -\frac{1}{3(x-1)} + \frac{2}{9}\ln\left|\frac{x-1}{x-2}\right| + C$$

2) $\int \left(\frac{x}{x^2-3x+2}\right)^2 dx = \int \frac{x^2\,dx}{(x^2-3x+2)^2} = \int \frac{x^2\,dx}{(x-1)^2\cdot(x-2)^2} =$

$$\frac{x^2}{(x-1)^2\cdot(x-2)^2} = \frac{A}{(x-1)} + \frac{B}{(x-2)} + \frac{C}{(x-1)^2} + \frac{D}{(x-2)^2}$$

$$A(x-1)(x-2)^2 + B(x-1)^2(x-2) + C(x^2-4x+4) + D(x^2-2x+1) = x^2$$

$$A(x-1)(x^2-4x+4) + B(x^2-2x+1)(x-2) + Cx^2 - 4Cx + 4C + Dx^2 - 2Dx + D = x^2$$

$$Ax^3 - 5Ax^2 + 8Ax - 4A + Bx^3 - 4Bx^2 + 5Bx - 2B + Cx^2 - 4Cx + 4C + Dx^2 - 2Dx + D = x^2$$

$$x^3(A+B) + x^2(-5A-4B+C+D) + x(8A+5B-4C-2D) + (-4A-2B+4C+D) = x^2$$

$$\begin{cases} A+B=0 \\ -5A-4B+C+D=1 \\ 8A+5B-4C-2D=0 \\ -4A-2B+4C+D=0 \end{cases} \Longrightarrow A=4;\ B=-4;\ C=1;\ D=4$$

$$= 4\int\frac{dx}{x-1} - 4\int\frac{dx}{x-2} + \int\frac{dx}{(x-1)^2} + 4\int\frac{dx}{(x-2)^2} =$$

$$= 4\ln|x-1| - 4\ln|x-2| - \frac{1}{(x-1)} - \frac{4}{(x-2)} + C = 4\ln\left|\frac{x-1}{x-2}\right| - \frac{5x-6}{x^2-3x+2} + C$$

3) $\int \frac{dx}{(x^2-4x+4)(x^2-4x+5)} = \int \frac{dx}{(x-2)^2(x^2-4x+5)} =$

$$\frac{1}{(x-2)^2(x^2-4x+5)} = \frac{A}{(x-2)} + \frac{B}{(x-2)^2} + \frac{Cx+D}{x^2-4x+5}$$

$$A(x-2)(x^2-4x+5) + B(x^2-4x+5) + (Cx+D)(x^2-4x+4) = 1$$

$$Ax^3 - 6Ax^2 + 13Ax - 10A + Bx^2 - 4Bx + 5B + Cx^3 - 4Cx^2 + 4Cx + Dx^2 - 4Dx + 4D = 1$$

$$x^3(A+C) + x^2(-6A+B-4C+D) + x(13A-4B+4C-4D) + (-10A+5B+4D) = 1$$

$$\begin{cases} A+C=0 \\ -6A+B-4C+D=0 \\ 13A-4B+4C-4D=0 \\ -10A+5B+4D=1 \end{cases} \Longleftrightarrow A=0;\ B=1;\ C=0;\ D=-1$$

$$= \int\frac{dx}{(x-2)^2} - \int\frac{dx}{x^2-4x+5} = \int\frac{dx}{(x-2)^2} - \int\frac{dx}{(x-2)^2+1} =$$

$$(x^2-4x+5)' = 2x-4 = 2(x-2)$$

$$= -\frac{1}{(x-2)} - \operatorname{arctg}(x-2) + C$$

4) $\int \frac{x^2 dx}{(x^2+2x+2)^2} = \int \frac{x^2 dx}{(x^2+2x+2)(x^2+2x+2)} \ominus$

$$\frac{x^2}{(x^2+2x+2)^2} = \frac{Ax+B}{(x^2+2x+2)} + \frac{Cx+D}{(x^2+2x+2)^2}$$

$$(Ax+B)(x^4+4x^3+8x^2+8x+4) + (Cx+D)(x^2+2x+2) = x^2$$

$$Ax^5 + 4Ax^4 + 8Ax^3 + 8Ax^2 + 4Ax + Bx^4 + 4Bx^3 + 8Bx^2 + 8Bx + 4B + Cx^3 + 2Cx^2 + 2Cx + Dx^2 + 2Dx + 2D = x^2$$

$$x^5 A + x^4(4A+B) + x^3(8A+4B+C) + x^2(8A+8B+2C+D) + x(4A+8B+2C+2D) + (4B+2D) = x^2$$

$$\begin{cases} A=0 \\ 4A+B=0 \\ 8A+4B+C=0 \\ 8A+8B+2C+D=1 \\ 4A+8B+2C+D=0 \\ 4B+2D=0 \end{cases} \quad \Leftrightarrow$$

$$(Ax+B)(x^2+2x+2) + Cx + D = x^2$$

$$Ax^3 + 2Ax^2 + 2Ax + Bx^2 + 2Bx + 2B + Cx + D = x^2$$

$$Ax^3 + x^2(2A+B) + x(2A+2B+C) + 2B + D = x^2$$

$A=0$

$2A+B=1 \Rightarrow B=1$

$2A+2B+C=0 \Rightarrow 2+C=0 \Rightarrow C=-2$

$2B+D=0 \Rightarrow 2+D=0 \Rightarrow D=-2$

$$\ominus \int \frac{dx}{(x^2+2x+2)} - 2\int \frac{(x+1)dx}{(x^2+2x+2)^2} = \int \frac{d(x+1)}{(x+1)^2+1} - \int \frac{(2x+2)dx}{(x^2+2x+2)^2} =$$

$$(x^2+2x+2)' = \underset{\sim}{2x+2}$$

$$\ominus \operatorname{arctg}(x+1) + \frac{1}{x^2+2x+2} + C$$

5) $\int$

$(x+1)$

$A(x$

$Ax^5 - Ax^4$

$+ Fx^3 +$

$x^5(A+C$

$+(A+B$

$A+$

$-A+$

$A+$

$A+B$

$-A-B$

$A+B$

5) $\int \frac{dx}{(1+x^3)^2} = \int \frac{dx}{(x+1)^2(x^2-x+1)^2}$

$$\frac{1}{(x+1)^2(x^2-x+1)^2} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{Cx+D}{x^2-x+1} + \frac{Fx+E}{(x^2-x+1)^2}$$

$$A(x^2-x+1)^2(x+1) + B(x^2-x+1) + (Cx+D)(x+1)^2(x^2-x+1) + (Fx+E)(x+1)^2 = 1.$$

$$Ax^5 - Ax^4 + Ax^3 + Ax^2 - Ax + A + Bx^2 - Bx + B + Cx^5 + Cx^4 + Cx^2 + Cx + Dx^4 + Dx^3 + Dx + D +$$
$$+ Fx^3 + 2Fx^2 + Fx + Ex^2 + 2Ex + E = 1.$$

$$x^5(A+C) + x^4(-A+C+D) + x^3(A+D+F) + x^2(A+B+C+2F+E) + x(-A-B+C+D+F+2E) +$$
$$+ (A+B+D+E) = 1$$

$$\begin{cases} A+C=0 \\ -A+C+D=0 \\ A+D+F=0 \\ A+B+C+2F+E=0 \\ -A-B+C+D+F+2E=0 \\ A+B+D+E=1 \end{cases}$$

Метод Остроградского.

$$\frac{P(x)}{Q(x)}$$

$Q(x)$ имеет кратные корни

$Q(x) = Q_1(x) \cdot Q_2(x)$, где корни $Q_1(x)$ – это корни многочлена $Q(x)$ с кратностями на 1 меньше, а $Q_2(x)$ содержит все оставшиеся корни $Q(x)$

$$\int \frac{P(x)}{Q(x)}dx = \frac{M(x)}{Q_1(x)} + \int \frac{N(x)}{Q_2(x)}dx \quad (*)$$

$M(x)$ и $N(x)$ – многочлены с неопр. коэф., степени которых на единицу меньше, чем у многочленов в знаменателе.

Коэф-ы вычисляются после дифференцирования рав-ва (*)

(1) $$\int\frac{dx}{x(x^2+1)^2} = \frac{Ax+B}{x^2+1} + \int\frac{Cx^2+Dx+E}{x(x^2+1)}dx$$

производная

$$\frac{1}{x(x^2+1)^2} = \left(\frac{A(x^2+1)-(Ax+B)\cdot 2x}{(x^2+1)^2}\right)^{x} + \frac{Cx^2+Dx+E}{x(x^2+1)}^{x^2+1}$$

$$x(Ax^2+A-2Ax^2-2Bx) + (x^2+1)(Cx^2+Dx+E) = 1$$

$$Cx^4 + x^3(-A+D) + x^2(-2B+E+C) + x(A+D) + E = 1.$$

$$\begin{cases} C=0 \\ -A+D=0 \\ -2B+C+E=0 \\ A+D=0 \\ E=1 \end{cases} \qquad \begin{aligned} &A=C=D=0 \\ &B=\tfrac{1}{2} \\ &E=1 \end{aligned}$$

$$\int\frac{dx}{x(x^2+1)^2} = \frac{1}{2(x^2+1)} + \int\frac{dx}{x(x^2+1)} = \frac{1}{2(x^2+1)} + \int\frac{dx}{x^3(1+\frac{1}{x^2})} =$$

$$= \frac{1}{2(x^2+1)} - \frac{1}{2}\int\frac{d(\frac{1}{x^2})}{1+\frac{1}{x^2}} = \frac{1}{2(x^2+1)} - \frac{1}{2}\ln\left(1+\frac{1}{x^2}\right) + C$$

(2) $$\int\frac{x\,dx}{(x-1)^2(x+1)^3} = \frac{Ax^2+Bx+C}{(x-1)(x+1)^2} + \int\frac{Dx+E}{(x-1)(x+1)}\cdot dx$$

уменьш. на 1 — оставшиеся

дифференцируем.

Интегрирование тригонометрических функций.

① $$\int \frac{dx}{8-4\sin x+7\cos x} = \left\{ \begin{array}{l} tg\frac{x}{2}=t \\ \frac{x}{2}=arctg\, t \end{array} \rightarrow x = 2arctg\, t \quad dx = \frac{2dt}{1+t^2} \quad \begin{array}{l} \sin x = \frac{2t}{1+t^2} \\ \cos x = \frac{1-t^2}{1+t^2} \end{array} \right\} =$$

$$= \int \frac{2dt}{(1+t^2)\left(8-\frac{8t}{1+t^2}+\frac{7(1-t^2)}{1+t^2}\right)} = 2\int \frac{dt}{t^2-8t+15} = 2\int \frac{dt}{(t-3)(t-5)} =$$

$$\frac{1}{(t-3)(t-5)} = \frac{A}{t-3}+\frac{B}{t-5}$$

$$= \int \frac{t-3-(t-5)\,dt}{(t-3)(t-5)} = \int \left(\frac{1}{t-5}-\frac{1}{t-3}\right)dt =$$

$$= \ln|t-5| - \ln|t-3| + C =$$

$$= \ln\left|tg\frac{x}{2}-5\right| - \ln\left|tg\frac{x}{2}-3\right| + C$$

② $$\int \frac{dx}{\cos x+2\sin x+3} = \left\{ \begin{array}{l} tg\frac{x}{2}=t \\ dx=\frac{2dt}{1+t^2} \end{array} \quad \begin{array}{l} \sin x = \frac{2t}{1+t^2} \\ \cos x = \frac{1-t^2}{1+t^2} \end{array} \right\} =$$

$$= \int \frac{2dt}{(1+t^2)\left(\frac{1-t^2}{1+t^2}+\frac{4t}{1+t^2}+3\right)} = \int \frac{2dt}{(1+t^2)\left(\frac{1-t^2+4t+3+3t^2}{1+t^2}\right)} =$$

$$= \int \frac{2dt}{2t^2+4t+4} = \int \frac{dt}{t^2+2t+2} = \int \frac{dt}{(t+1)^2+1} = \int \frac{d(t+1)}{1+(t+1)^2} =$$

$$= arctg(t+1)+C = arctg\left(tg\frac{x}{2}+1\right)+C$$

③ $$\int \frac{dx}{\sin^2 x+tg^2 x} = \left\{ \begin{array}{l} t=tg\,x \rightarrow x = arctg\, t \quad dx = \frac{dt}{1+t^2} \\ \sin^2 x = tg^2 x\cdot\cos^2 x; \\ 1+tg^2 x = \frac{1}{\cos^2 x} \quad \cos^2 x = \frac{1}{1+tg^2 x} \\ \sin^2 x = \frac{tg^2 x}{1+tg^2 x} \end{array} \right\} =$$

$$= \int \frac{dt}{(1+t^2)\left(t^2+t^2\cdot\frac{1}{1+t^2}\right)} = \int \frac{dt}{(1+t^2)t^2\left(1+\frac{1}{1+t^2}\right)} =$$

$$= \int \frac{dt\,(1+t^2)}{(1+t^2)t^2(1+t^2+1)} = \int \frac{dt}{t^2(t^2+2)} = \frac{1}{2}\int \frac{t^2+2-t^2}{t^2(t^2+2)}dt =$$

$$= \frac{1}{2}\int\frac{dt}{t^2} - \frac{1}{2}\int\frac{dt}{t^2+2} = -\frac{1}{2t} - \frac{1}{2\sqrt{2}}arctg\frac{t}{\sqrt{2}} + C =$$

$$= -\frac{1}{2tg\,x} - \frac{1}{2\sqrt{2}}arctg\frac{tg\,x}{\sqrt{2}} + C = -\frac{ctg\,x}{2} - \frac{1}{2\sqrt{2}}arctg\left(\frac{tg\,x}{\sqrt{2}}\right)+C$$

(4) $\int \frac{dx}{\sin^4 x+\cos^4 x}$ $(=)$

$\sin^4 x+\cos^4 x = \left(\frac{1-\cos 2x}{2}\right)^2 + \left(\frac{1+\cos 2x}{2}\right)^2 =$

$= \frac{1}{4}\left(2+2\cos^2 2x\right) = \frac{1+\cos^2 2x}{2}$

$(=) \; 2\int \frac{dx}{1+\cos^2 2x} = \int \frac{d(2x)}{1+\cos^2 2x} = \left\{ \begin{array}{l} \operatorname{tg} 2x = t \\ 1+\operatorname{tg}^2 2x = \frac{1}{\cos^2 x} \\ 1+t^2 = \frac{1}{\cos^2 2x} \to \cos^2 x = \frac{1}{1+t^2} \\ 2x = \operatorname{arctg} t \\ d(2x) = \frac{dt}{1+t^2} \end{array} \right.$

$= \int \frac{dt}{(1+t^2)\left(1+\frac{1}{1+t^2}\right)} = \int \frac{dt}{2+t^2} = \frac{1}{\sqrt{2}} \operatorname{arctg} \frac{t}{\sqrt{2}} + C =$

$= \frac{1}{\sqrt{2}} \operatorname{arctg}\left(\frac{\operatorname{tg} 2x}{\sqrt{2}}\right) + C$

(5) $\int \frac{\sin x}{\sqrt{2}+\sin x+\cos x}\, dx$

$\sin x = A(\sqrt{2}+\sin x+\cos x) + B(\cos x-\sin x) + C$ — линейная комбинация

$\sin x = \sin x(A-B) + \cos x(A+B) + A\cdot\sqrt{2} + C$

$\begin{cases} A-B=1 \\ A+B=0 \\ A\sqrt{2}+C=0 \end{cases} \qquad A=\frac{1}{2}, \quad B=-\frac{1}{2}, \quad C=-\frac{1}{\sqrt{2}}$

$\int \frac{\sin x}{\sqrt{2}+\sin x+\cos x}\, dx = \int \frac{\frac{1}{2}(\sqrt{2}+\sin x+\cos x) - \frac{1}{2}(\cos x-\sin x) - \frac{1}{\sqrt{2}}}{\sqrt{2}+\sin x+\cos x}\, dx$

$= \frac{1}{2}\int dx - \frac{1}{2}\int \frac{d(\sqrt{2}+\sin x+\cos x)}{\sqrt{2}+\sin x+\cos x} - \frac{1}{\sqrt{2}}\int \frac{dx}{\sqrt{2}+\sin x+\cos x} =$

$= \frac{x}{2} - \frac{1}{2}\ln|\sqrt{2}+\sin x+\cos x| - \frac{1}{\sqrt{2}}\int \frac{dx}{\sqrt{2}+\sin x+\cos x}$ $(=)$

$\sqrt{2}+\sin x+\cos x = \sqrt{2}+\sqrt{2}\left(\underbrace{\frac{\sqrt{2}}{2}}_{\sin\frac{\pi}{4}}\sin x + \underbrace{\frac{\sqrt{2}}{2}}_{\cos\frac{\pi}{4}}\cos x\right) = \sqrt{2}+\sqrt{2}\left(\cos\left(x-\frac{\pi}{4}\right)\right)$

$= \sqrt{2}\left(1+\cos\left(x-\frac{\pi}{4}\right)\right) = \sqrt{2}\cdot 2\cos^2\left(\frac{x}{2}-\frac{\pi}{8}\right)$

$a\sin x + b\cos x = \sqrt{a^2+b^2}\left(\underbrace{\frac{a}{\sqrt{a^2+b^2}}}_{\sin\varphi}\sin x + \underbrace{\frac{b}{\sqrt{a^2+b^2}}}_{\cos\varphi}\cos x\right) =$

$= \sqrt{a^2+b^2}\left(\cos x\cos\varphi + \sin x\sin\varphi\right) =$

$= \sqrt{a^2+b^2}\left(\cos(x-\varphi)\right)$

$$= \frac{x}{2} - \frac{1}{2}\ln|\sqrt{2}+\sin x+\cos x| - \frac{1}{\sqrt{2}}\cdot\frac{1}{2\sqrt{2}}\int\frac{dx}{\cos^2\left(\frac{x}{2}-\frac{\pi}{8}\right)} =$$

$$= \frac{x}{2} - \frac{1}{2}\ln|\sqrt{2}+\sin x+\cos x| - \frac{1}{2}\,tg\left(\frac{x}{2}-\frac{\pi}{8}\right)+C$$

Домашняя работа.

(2025) $$\int\frac{dx}{2\sin x-\cos x+5} = \left\{ tg\frac{x}{2}=t;\quad \frac{x}{2}=arctg\,t;\quad dx=\frac{2dt}{1+t^2};\quad \sin x=\frac{2t}{1+t^2};\quad \cos x=\frac{1-t^2}{1+t^2} \right\} =$$

$$= \int\frac{2dt}{(1+t^2)\left(\frac{4t}{1+t^2}-\frac{1-t^2}{1+t^2}+5\right)} = \int\frac{2dt}{(1+t^2)\left(\frac{4t-1+t^2+5+5t^2}{1+t^2}\right)} =$$

$$= \int\frac{2dt}{6t^2+4t+4} = \int\frac{dt}{3t^2+2t+2} = \cancel{\int\frac{dt}{\sqrt{\ }}} = \frac{1}{3}\int\frac{dt}{t^2+2\cdot\frac{1}{3}t+\frac{1}{9}+\frac{5}{9}} =$$

$$= \frac{1}{3}\int\frac{dt}{\left(t+\frac{1}{3}\right)^2+\frac{5}{9}} = \frac{1}{\sqrt{5}}arctg\frac{\left(t+\frac{1}{3}\right)3}{\sqrt{5}} + C =$$

$$= \frac{1}{\sqrt{5}}arctg\left(\frac{3tg\frac{x}{2}+1}{\sqrt{5}}\right)+C$$

(2026) $$\int\frac{dx}{(2+\cos x)\sin x} = \int\frac{\cancel{2}dt\,(1+t^2)}{\cancel{(1+t^2)}\left(2+\frac{1-t^2}{1+t^2}\right)\cdot\cancel{2}t} = \int\frac{dt\,(1+t^2)}{(2+2t^2+1-t^2)t} =$$

$$= \int\frac{(1+t^2)\,dt}{(t^2+3)t} = \frac{2}{3}\int\frac{t\,dt}{t^2+3} + \frac{1}{3}\int\frac{dt}{t} = \frac{1}{3}\int\frac{d(t^2+3)}{(t^2+3)} + \frac{1}{3}\int\frac{dt}{t} =$$

$$\frac{1+t^2}{(t^2+3)t} = \frac{At+B}{t^2+3}+\frac{C}{t}$$

$$= \frac{1}{3}\ln(t^2+3)+\frac{1}{3}\ln t + C =$$

$$= \frac{1}{3}\ln|t(t^2+3)|+C$$

$(At+B)t + C(t^2+3) = 1+t^2$

$At^2+Bt+Ct^2+3C = 1+t^2$

$t^2(A+C)+Bt+3C = t^2+1$

$$\begin{cases} A+C=1 \\ B=0 \\ 3C=1 \end{cases} \Rightarrow C=\frac{1}{3};\quad A=\frac{2}{3}$$

(2029) $$\int\frac{\sin^2 x}{1+\sin^2 x}dx = \int\frac{(1-\cos 2x)\,dx}{2\left(1+\frac{1-\cos 2x}{2}\right)} = \int\frac{(1-\cos 2x)\,dx\cdot\cancel{2}}{\cancel{2}(3-\cos 2x)} =$$

$$\sin^2 x = \frac{1-\cos 2x}{2}$$

$$= \int\frac{1-\cos 2x}{3-\cos 2x}dx = \left\{ tg\,x=t;\quad x=arctg\,t;\quad dx=\frac{dt}{1+t^2};\quad \cos 2x=\frac{1-t^2}{1+t^2} \right\} =$$

$$= \int\frac{\left(1-\frac{1-t^2}{1+t^2}\right)}{\left(3-\frac{1-t^2}{1+t^2}\right)}\cdot\frac{dt}{(1+t^2)} = \int\frac{\left(\frac{1+t^2-1+t^2}{1+t^2}\right)dt}{\frac{(3+3t^2-1+t^2)\cdot\cancel{(1+t^2)}}{\cancel{(1+t^2)}}} =$$

$$= \int \frac{2t^2\,dt}{(4t^2+2)(1+t^2)} = \int \frac{t^2\,dt}{(2t^2+1)(1+t^2)} =$$

$$\frac{t^2}{(2t^2+1)(1+t^2)} = \frac{At+B}{2t^2+1} + \frac{Ct+D}{1+t^2}$$

$$(At+B)(1+t^2) + (Ct+D)(2t^2+1) = t^2$$

$$At+At^3+B+Bt^2+2Ct^3+Ct+2Dt^2+D = t^2$$

$$t^3(A+2C) + t^2(B+2D) + t(A+C) + B + D = t^2$$

$$\begin{cases} A+2C=0 \\ B+2D=1 \\ A+C=0 \\ B+D=0 \end{cases} \rightarrow A=0;\ C=0 \qquad D=1 \Rightarrow B=-1$$

$$= \int\frac{dt}{1+t^2} - \frac{1}{2}\int\frac{dt}{\frac{1}{2}+t^2} = \operatorname{arctg} t - \frac{\sqrt{2}}{2}\operatorname{arctg}\frac{t\sqrt{2}}{1} + C =$$

$$= \cancel{tg}\, x - \frac{1}{\sqrt{2}}\operatorname{arctg}(\sqrt{2}\, tg\, x) + C$$

2030. $$\int\frac{dx}{a^2\sin^2x+b^2\cos^2x} = \{ tg\, x = t \} =$$

$$= \int\frac{\frac{dx}{\cos^2x}}{a^2 tg^2x+b^2} = \frac{1}{a^2}\int\frac{d(tg\,x)}{tg^2x+\frac{a^2}{b^2}} = \frac{b}{a^2\,a}\operatorname{arctg}\left(\frac{b\,tg\,x}{a}\right)+C$$

$$d\,tg\,x = \frac{dx}{\cos^2x}$$

$$1+ctg^2x = \frac{1}{\sin^2x} \Rightarrow \sin^2x = \frac{1}{1+ctg^2x}$$

2034. $$\int\frac{\sin x\,dx}{\sin^3x+\cos^3x} = \left\{ \begin{matrix} t = \cancel{ctg\,x} & \cancel{x = \operatorname{arctg} t \rightarrow dx = \frac{dt}{1+t^2}} \\ dt = -\frac{dx}{\cos^2x} & \cancel{\Rightarrow dx = \cos^2x\,dt} \end{matrix} \right\} =$$

$$= \int\frac{\cancel{\sin x}\,dx}{\cancel{\sin x}(\sin^2x+\cos^2x\cdot ctg\,x)} = \int\frac{dx}{\sin^2x\,(1+ctg^3x)} =$$

$$= -\int\frac{d(ctg\,x)}{1+ctg^3x} = \{ ctg\,x = t \} =$$

$$d(ctg\,x) = -\frac{dx}{\sin^2x}$$

$$= -\int\frac{dt}{1+t^3} = -\int\frac{dt}{(1+t)(1-t+t^2)}$$

$$\frac{1}{(1+t)(1-t+t^2)} = \frac{A}{1+t} + \frac{Bt+C}{1-t+t^2}$$

$$\cancel{1-At+At^2+Bt+Bt^2+C+Ct} = 1 \quad A-At+At^2+Bt+Bt^2+C+Ct=1$$

$$\cancel{t^2(A+B)+t(B+C)} \qquad t^2(A+B)+t(-A+B+C)+A+C=1.$$

$$\begin{cases} A+B=0 & A=-B \\ A+C=1, & C=1+B \\ -A+B+C=0 \end{cases}$$

$$B+B+1+B=0 \Rightarrow B=-\frac{1}{3};\ A=\frac{1}{3};\ C=\frac{2}{3}$$

$$= \left(\frac{1}{3}\int \frac{dt}{1+t} + \int \frac{-\frac{1}{3}t+\frac{2}{3}}{t^2-t+1}\,dt\right) = -\left(\frac{1}{3}\ln|1+t|\right) - \frac{1}{3}\int\frac{(t-2)\,dt}{t^2-t+1} =$$

$$= \cancel{\frac{1}{6}\ln} -\frac{1}{3}\ln|1+t| + \frac{1}{6}\int\frac{(2t-4)\,dt}{t^2-t+1} = -\frac{1}{3}\ln|1+t| + \frac{1}{6}\cdot$$

$$\cdot\int\frac{(2t-1)\,dt}{t^2-t+1} - \frac{3}{6}\int\frac{dt}{t^2-t+1} = -\frac{1}{3}\ln|1+t| + \frac{1}{6}\ln|t^2-t+1| -$$

$$-\frac{1}{2}\int\frac{dt}{t^2-2\cdot\frac{1}{2}t+\frac{1}{4}+\frac{3}{4}} = -\frac{1}{6}\left(2\ln|1+t| - \ln(t^2-t+1)\right) -$$

$$-\frac{1}{2}\int\frac{dt}{\frac{3}{4}+(t-\frac{1}{2})^2} = -\frac{1}{6}\ln\frac{(1+t)^2}{(t^2-t+1)} - \frac{1}{\sqrt{3}}\operatorname{arctg}\frac{2t-\frac{2}{2}}{\sqrt{3}} + C =$$

$$= -\frac{1}{6}\ln\left(\frac{(1+\operatorname{ctg}x)^2}{\operatorname{ctg}^2x-\operatorname{ctg}x+1}\right) - \frac{1}{\sqrt{3}}\operatorname{arctg}\frac{2\operatorname{ctg}x-1}{\sqrt{3}} + C$$

2037. $$\int\frac{\sin^2x-\cos^2x}{\sin^4x+\cos^4x}\,dx = \int\frac{\frac{\operatorname{tg}^2x}{\cos^2x}-\frac{1}{\cos^2x}}{\operatorname{tg}^4x+1}\,dx =$$

$$= \int\frac{\operatorname{tg}^2x-1}{\operatorname{tg}^4x+1}\cdot\frac{dx}{\cos^2x} = \int\frac{\operatorname{tg}^2x-1}{\operatorname{tg}^4x+1}\,d(\operatorname{tg}x) = \{\operatorname{tg}x = t\} =$$

$$d(\sin 2x) = \frac{2\cos 2x\cdot dx}{\cancel{2}}$$

$$= \int\frac{t^2-1}{t^4+1}\,dt$$

$$\int\frac{\sin^2x-\cos^2x}{\sin^4x+\cos^4x}\,dx = -\int\frac{\cos 2x}{1-\frac{1}{2}\sin^2 2x}\,dx = -\frac{1}{2}\int\frac{\cancel{2}\,d(\sin 2x)}{\frac{1}{2}(2-\sin^2 2x)} =$$

$$= -\frac{1}{\cancel{2}\sqrt{2}}\ln\left|\frac{\sqrt{2}+\sin 2x}{\sqrt{2}-\sin 2x}\right| + C$$

2041. $$\int\frac{dx}{a\sin x+b\cos x} = \int\frac{dx}{\sqrt{a^2+b^2}\left(\underbrace{\frac{a}{\sqrt{a^2+b^2}}}_{\cos\varphi}\sin x + \underbrace{\frac{b}{\sqrt{a^2+b^2}}}_{\sin\varphi}\cos x\right)} =$$

$$= \cancel{\int\frac{dx}{\sqrt{a^2+b^2}(\cos x\cos\varphi+\sin x\sin\varphi)} = \frac{1}{\sqrt{a^2+b^2}}\int\frac{dx}{\cos(x-\varphi)}}$$

$$= \frac{1}{\sqrt{a^2+b^2}}\int\frac{dx}{\sin(x+\varphi)} = \cancel{\frac{1}{\sqrt{a^2+b^2}}}\int = \left\{\sin(x+\varphi) = \frac{2\operatorname{tg}\left(\frac{x+\varphi}{2}\right)}{1+\operatorname{tg}^2\left(\frac{x+\varphi}{2}\right)}\right.$$

$$= \frac{1}{\sqrt{a^2+b^2}}\int\frac{\left(1+\operatorname{tg}^2\left(\frac{x+\varphi}{2}\right)\right)dx}{2\operatorname{tg}\frac{x+\varphi}{2}} = \frac{1}{2\sqrt{a^2+b^2}}\int\frac{d\left(\operatorname{tg}\frac{x+\varphi}{2}\right)}{\operatorname{tg}\frac{x+\varphi}{2}} =$$

$$= \frac{1}{2\sqrt{a^2+b^2}}\ln\left|\operatorname{tg}\frac{x+\varphi}{2}\right| + C$$

2039. $\int \frac{dx}{\sin^6 x + \cos^6 x} =$

~~$\sin^6 x = (1+co$~~

$\sin^6 x = \left(\frac{1-\cos 2x}{2}\right)^3$ ш/

$\cos^6 x = \left(\frac{1+\cos 2x}{2}\right)^3$

$\sin^6 x + \cos^6 x = \left(\frac{1-\cos 2x}{2}\right)^3 + \left(\frac{1+\cos 2x}{2}\right)^3 = \frac{1}{8}\left((1-\cos 2x)^3 + (1+\cos 2x)^3\right) = \frac{1}{8}\left(-\cancel{\cos^3 2x} + 3\cos^2 2x - \cancel{3\cos 2x} + 1 + 1 + \cancel{3\cos 2x} + 3\cos^2 2x + \cancel{\cos^3 2x}\right) =$

$= \frac{1}{8}(2 + 6\cos^2 2x) = \frac{1}{4}(1 + 3\cos^2 2x)$

$= \frac{1}{4}\int \frac{dx}{1+3\cos^2 2x} = \left\{ \begin{array}{l} \operatorname{tg} 2x = t \\ 1 + \operatorname{tg}^2 2x = \frac{1}{\cos^2 2x} \\ 2x = \operatorname{arctg} t \end{array} \right. \quad \cos^2 2x = \frac{1}{1+t^2} \quad d(2x) = \frac{dt}{1+t^2}$

$= \cancel{\frac{1}{4}} \int \frac{2\, d(2x)}{1+3\cos^2 2x} = 2\int \frac{dt}{(1+t^2)\left(1+\frac{3}{1+t^2}\right)} = 2\int \frac{dt}{(t^2+4)} =$

$= \operatorname{arctg} \frac{t}{2} + C = \operatorname{arctg} \frac{\operatorname{tg} 2x}{2} + C.$

① $\int \sin^2 x \cdot \cos^4 x \, dx = \frac{1}{4}\int \sin^2 2x \cdot \cos^2 x \, dx = \frac{1}{4}\int \sin^2 2x \cdot \left(\frac{1+\cos 2x}{2}\right) dx =$

$\sin 2x = 2\sin x \cos x$

$\sin x \cos x = \frac{1}{2}\sin 2x$

$= \frac{1}{8}\int (\sin^2 2x \, dx + \sin^2 2x \cdot \cos 2x) \, dx =$

$= \frac{1}{8}\int \frac{1-\cos 4x}{2} dx + \frac{1}{8\cdot 2}\int \sin^2 2x \cdot \cos 2x \, d(2x) =$

$= \frac{1}{16}\int (1-\cos 4x) \, dx + \frac{1}{16}\cdot\frac{\sin^3 2x}{3} = \frac{x}{16} - \frac{\sin 4x}{64} + \frac{\sin^3 2x}{48} + C$

② $\int \sin^4 x \cos^5 x \, dx = \int \sin^4 x \cos^4 x \, d(\sin x) = \int \sin^4 x (1-\sin^2 x)^2 \, d(\sin x) =$

$= \{\sin x = t\} = \int t^4 (1 - 2t^2 + t^4) \, dt = \int (t^4 - 2t^6 + t^8) \, dt =$

$= \frac{t^5}{5} - \frac{2t^7}{7} + \frac{t^9}{9} + C = \frac{\sin^5 x}{5} - \frac{2\sin^7 x}{7} + \frac{\sin^9 x}{9} + C$

③ $\int \frac{\cos^4 x}{\sin^3 x} dx = \left\{ \begin{array}{l} u = \cos^3 x, \quad du = -3\cos^2 x \sin x \, dx \\ dv = \frac{\cos x \, dx}{\sin^3 x} \to v = \int \frac{d(\sin x)}{\sin^3 x} = -\frac{1}{2\sin^2 x} \end{array} \right\} (=)$

$\cancel{\sin^3 x} \quad t^{-3} dt = \frac{t^{-2}}{-2} = -\frac{1}{2t^2}$

$(=) -\frac{\cos^3 x}{2\sin^2 x} - \int \frac{3\cos^2 x \sin x \, dx}{2 \sin^2 x} = -\frac{\cos^3 x}{2\sin^2 x} - \int \frac{3\cos^2 x}{2\sin x} dx =$

$= -\frac{\cos^3 x}{2\sin^2 x} - \frac{3}{2}\int \frac{1-\sin^2 x}{\sin x} dx = -\frac{\cos^3 x}{2\sin^2 x} - \frac{3}{2}\left(\int \frac{dx}{\sin x} - \int \sin x \, dx\right) =$

$= -\frac{\cos^3 x}{2\sin^2 x} - \frac{3}{2}\cos x - \frac{3}{2}\int \frac{dx}{2\sin(\frac{x}{2})\cos(\frac{x}{2})} = -\frac{\cos^3 x}{2\sin^2 x} - \frac{3}{2}\cos x -$

$- \frac{3}{2}\int \frac{dx}{2\cos^2(\frac{x}{2}) \cdot tg(\frac{x}{2})} = -\frac{\cos^3 x}{2\sin^2 x} - \frac{3}{2}\cos x - \frac{3}{2}\int \frac{d(tg\frac{x}{2})}{tg(\frac{x}{2})} =$

$= -\frac{\cos^3 x}{2\sin^2 x} - \frac{3}{2}\cos x - \frac{3}{2}\ln|tg\frac{x}{2}| + C$

④ $\int \frac{1 \, dx}{\cos^3 x} = \int \frac{\sin^2 x + \cos^2 x}{\cos^3 x} dx = \int \frac{\sin^2 x \, dx}{\cos^3 x} + \int \frac{dx}{\cos x} (=)$

$\int \frac{\sin^2 x}{\cos^3 x} dx = \left\{ \begin{array}{l} u = \sin x \\ dv = \frac{\sin x \, dx}{\cos^3 x} \end{array} \quad \begin{array}{l} du = \cos x \, dx \\ v = \int \frac{\sin x \, dx}{\cos^3 x} \end{array} \right\} \to v = -\int \frac{d(\cos x)}{\cos^3 x} = \frac{1}{2\cos^2 x} \Big\} =$

$= \frac{\sin x}{2\cos^2 x} - \frac{1}{2}\int \frac{\cos x \, dx}{\cos^2 x} = \frac{\sin x}{2\cos^2 x} - \frac{1}{2}\int \frac{dx}{\cos x}$

$(=) \frac{\sin x}{2\cos^2 x} - \frac{1}{2}\int \frac{dx}{\cos x} + \int \frac{dx}{\cos x} = \frac{\sin x}{2\cos^2 x} + \frac{1}{2}\int \frac{dx}{\cos x} = \frac{\sin x}{2\cos^2 x} + \frac{1}{2}\int \frac{dx}{\sin(x+\frac{\pi}{2})}$

$= \frac{\sin x}{2\cos^2 x} + \frac{1}{2}\int \frac{dx}{2\sin(\frac{x}{2}+\frac{\pi}{4})\cos(\frac{x}{2}+\frac{\pi}{4})} = \frac{\sin x}{2\cos^2 x} + \frac{1}{2}\int \frac{d(\frac{x}{2}+\frac{\pi}{4})}{\cos^2(\frac{x}{2}+\frac{\pi}{4}) \, tg(\frac{x}{2}+\frac{\pi}{4})}$

$$= \frac{\sin x}{2\cos^2 x} + \frac{1}{2}\ln\left|tg\left(\frac{x}{2}+\frac{\pi}{4}\right)\right| + C$$

5) $$\int \frac{dx}{\sin^3 x \cos^5 x} = \int \frac{d(tg\,x)}{\sin^3 x \cos^3 x} = \int \frac{8\,d(tg\,x)}{\sin^3 2x} = \left\{ \begin{array}{l} \sin 2x = \frac{2tg\,x}{1+tg^2 x} \\ tg\,x = t \end{array} \right\} =$$

$$= 8\int \frac{dt\,(1+t^2)^3}{8t^3} = \int \frac{(1+3t^2+3t^4+t^6)}{t^3}\,dt = -\frac{1}{2t^2} + 3\ln|t| +$$

$$+ \frac{3}{2}t^2 + \frac{t^4}{4} + C = \frac{tg^4 x}{4} + \frac{3}{2}tg^2 x + 3\ln|tg\,x| - \frac{1}{2tg^2 x} + C$$

6) $$\int \frac{\sin x - \cos x}{\sin x + 2\cos x}\,dx =$$

производная

$$\sin x - \cos x = A(\sin x + 2\cos x) + B(\cos x - 2\sin x) =$$

$$\sin x - \cos x = \sin x(A - 2B) + \cos x(2A + B)$$

$$\begin{cases} A - 2B = 1 \;|\cdot 2 \\ 2A + B = -1 \end{cases} - \begin{cases} 2A - 4B = 2 \\ 2A + B = -1 \end{cases} \quad -5B = 3 \rightarrow B = -\frac{3}{5}$$

$$A = 2B + 1 = -\frac{6}{5} + 1 = -\frac{1}{5}$$

$$= \int \frac{-\frac{1}{5}(\sin x + 2\cos x) - \frac{3}{5}(\cos x - 2\sin x)}{\sin x + 2\cos x}\,dx =$$

$$= -\frac{1}{5}\int dx - \frac{3}{5}\int \frac{(\cos x - 2\sin x)\,dx}{\sin x + 2\cos x} = -\frac{x}{5} - \frac{3}{5}\ln|\sin x + 2\cos x|$$

7) $$\int ctg^6 x\,dx = \int ctg^4 x \cdot ctg^2 x\,dx = \int ctg^4 x\left(\frac{1}{\sin^2 x} - 1\right)dx =$$

$$= \int ctg^4 x \cdot \frac{dx}{\sin^2 x} - \int ctg^4 x\,dx = -\int ctg^4 x\,d(ctg\,x) - \int ctg^2 x\left(\frac{1}{\sin^2 x} - 1\right)$$

$$= -\frac{ctg^5 x}{5} + \frac{ctg^3 x}{3} + \int ctg^2 x\,dx =$$

$$= -\frac{ctg^5 x}{5} + \frac{ctg^3 x}{3} + \int\left(\frac{1}{\sin^2 x} - 1\right)dx =$$

$$= -\frac{ctg^5 x}{5} + \frac{ctg^3 x}{3} - ctg\,x - x + C$$

Домашняя работа.

1) $\int \frac{\sin^3 x}{\cos^4 x}dx = \int \frac{\sin^3 x}{(1-\sin^2 x)^2}dx = \int \frac{\sin^3 x \cdot dx}{1+2\sin^2 x+\sin^4 x} =$

$= \left\{ \begin{array}{l} u = \sin x \quad du = \cos x \cdot dx \\ dV = \frac{\sin^2 x \cdot dx}{\cos^4 x} = \frac{dx}{\cos^2 x}\cdot tg^2 x = tg^2 x \cdot d(tg x) \Rightarrow V = \frac{tg^3 x}{3} \end{array} \right\} =$

$= \frac{\sin x \cdot tg^3 x}{3} - \int \frac{tg^3 x}{3}\cdot \cos x \cdot dx = \frac{\sin x \cdot tg^3 x}{3} - \frac{1}{3}\int \frac{\sin^3 x \cdot \cos x}{\cos^3 x}\cdot dx =$

$= \frac{\sin x \cdot tg^3 x}{3} - \frac{1}{3}\int \sin^3 x \cdot \frac{dx}{\cos^2 x}$

$-\int \frac{\sin^2 x \cdot d(\cos x)}{\cos^4 x} =$

$\sin^2 x = \frac{1-\cos 2x}{2}$; $\cos^4 x = \left(\frac{1+\cos 2x}{2}\right)^2 = \frac{(1+\cos 2x)^2}{4}$

$= -\int \frac{4(1-\cos 2x)\cdot d(\cos x)}{2\cdot(1+\cos 2x)^2} = -\int$

$dt = -\sin x \, dx \Rightarrow$

2) $\int \frac{\sin^3 x}{\cos^4 x}dx = \left\{ \begin{array}{l} \cos x = t \rightarrow \cos^4 x = t^4 \\ x = \arccos t \Rightarrow dx = -\frac{dt}{\sqrt{1-t^2}} \end{array} \right.$ $\Rightarrow dx = -\frac{dt}{\sin x}$

$\sin^2 x = 1-t^2$

$= -\int \frac{(1-t^2)\sin x \, dt}{\sin x \cdot t^4} = -\int \frac{(1-t^2)}{t^4}dt = -\int t^{-4}dt + \int t^{-2}dt =$

$= \frac{1}{3t^3} - \frac{1}{t} + C = \frac{1}{3\cos^3 x} - \frac{1}{\cos x} + C.$

3) $\int \frac{dx}{\sin^3 x} =$

$\sin 3x = 3\sin x - 4\sin^3 x \Rightarrow 4\sin^3 x = 3\sin x - \sin 3x$

$\sin^3 x = \frac{3}{4}\sin x - \frac{1}{4}\sin 3x$

$= -\int \frac{d(ctg x)}{\sin x}.$

$\int \frac{dx}{\sin x \cdot \sin^2 x} =$

$1+ctg^2 x = \frac{1}{\sin^2 x} \Rightarrow \sin^2 x = \frac{1}{1+ctg^2 x} = \frac{1}{1+\frac{1}{tg^2 x}} = \frac{tg^2 x}{tg^2 x+1}.$

x ; $+2\cos x| + C$; $x\left(\frac{1}{\sin^2 x}-1\right)dx =$

3) $\int \frac{dx}{\sin^3 x} = \int \frac{(\sin^2 x + \cos^2 x)\,dx}{\sin^3 x} = \int \frac{dx}{\sin x} + \int \frac{\cos^2 x\,dx}{\sin^3 x} \ominus$

$\cancel{\int \frac{dx}{2\sin\frac{x}{2}\cos}}$ $\quad \int \frac{dx}{\sin x} = \int \frac{dx}{2\sin\frac{x}{2}\cdot\cos\frac{x}{2}} = \frac{1}{2}\int \frac{dx}{tg\frac{x}{2}\cos^2\frac{x}{2}} = \frac{2}{2}\ln|tg$

$\boxed{\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2}}$ $\quad \int \frac{\cos^2 x\,dx}{\sin^3 x} = \cancel{\int\frac{(1-\sin^2 x)\,dx}{\sin^3 x}} =$

$= \begin{cases} u = \cos x & du = -\sin x \cdot dx \\ dv = \frac{\cos x\,dx}{\sin^3 x} & v = \int \frac{d(\sin x)}{\sin^3 x} = -\frac{1}{2\sin^2 x} \end{cases}$

$= -\frac{\cos x}{2\sin^2 x} - \int \frac{\sin x\,dx}{2\sin^2 x} = -\frac{\cos x}{2\sin^2 x} - \frac{1}{2}\ln|tg\frac{x}{2}|$

$\ominus -\frac{\cos x}{2\sin^2 x} + \frac{1}{2}\ln|tg\frac{x}{2}|$

4) $\int \frac{dx}{\sin^4 x\cdot\cos^4 x} = \left\{ \begin{array}{l} t = tg\,x \quad x = arctg\,t \Rightarrow dx = \frac{dt}{1+t^2} \\ \sin^4 x = \left(\frac{t^2}{1+t^2}\right)^2 = \frac{t^4}{(1+t^2)^2} \\ 1 + tg^2 x = \frac{1}{\cos^2 x} \Rightarrow \cos^2 x = \frac{1}{1+tg^2 x} \\ \cos^4 x = \frac{1}{(1+t^2)^2} \end{array} \right\} \ominus$

$\ominus \int \frac{dt\cdot(1+t^2)^2\cdot(1+t^2)^2}{\cancel{(1+t^2)}t^4} = \int \frac{(1+t^2)^3\,dt}{t^4} =$

$= \int \frac{(t^6 + 3t^4 + 3t^2 + 1)\,dt}{t^4} = \int t^2 dt + \int 3dt + \int \frac{3\,dt}{t^2} + \int \frac{dt}{t^4} =$

$= \frac{t^3}{3} + 3t - \frac{3}{t} - \frac{1}{5t^5} + C = \frac{tg^3 x}{3} + 3tg\,x - \frac{3}{tg\,x} - \frac{1}{5tg^5 x} + C$

5) $\int \frac{dx}{\sin x\,\cos^4 x} = \int \frac{(\sin^2 x + \cos^2 x)\,dx}{\sin x\cdot\cos^4 x} = \int \frac{\sin x\,dx}{\cos^4 x} + \int \frac{dx}{\sin x\cdot\cos^2 x} =$

$= -\int \frac{d(\cos x)}{\cos^4 x} + \int \frac{(\sin^2 x + \cos^2 x)\,dx}{\sin x\cdot\cos^2 x} = \cancel{+\frac{1}{5\cos^5 x} + \int\frac{\sin x\,dx}{\cos^2 x}} +$

$\cancel{+\int\frac{dx}{\sin x} \quad \frac{1}{5\cos^5 x}} \quad \frac{1}{3\cos^3 x} - \frac{1}{\cos x} + \ln|tg\frac{x}{2}| + C.$

6) $\int \cos x\cdot\cos 2x\cdot\cos 3x\,dx = \int \frac{1}{2}(\cos 3x + \cos(-x))\cdot\cos 3x\cdot dx =$

$= \frac{1}{2}\int(\cos x + \cos 3x)\cdot\cos 3x\cdot dx = \frac{1}{2}\int\cos x\cdot\cos 3x\,dx + \frac{1}{2}\int\cos 3x\cdot\cos$

$= \frac{1}{4}\int(\cos 4x + \cos 2x)\,dx + \frac{1}{2}\int\cos^2 3x\cdot dx$

$d(4x) = 4dx \Rightarrow dx = \frac{d(4x)}{4}$; $d(3x) = 3dx \rightarrow dx = \frac{d(3x)}{3}$

$= \frac{1}{16}\int \cos 4x \, d(4x) + \frac{1}{8}\int \cos 2x \, d(2x) + \frac{1}{2}\int \frac{1+\cos 6x}{2} dx =$

$= \frac{\sin 4x}{16} + \frac{\sin 2x}{8} + \frac{x}{4} + \frac{\sin 6x}{24} + C.$

4) $\int \frac{1\,dx}{\sin(x+a)\cos(x+b)} = \int \frac{dx}{(\sin x \cdot \cos a + \cos x \cdot \sin a)(\cos x \cdot \cos b - \sin x \sin b)}$

Определённый интеграл и его св-ва

$\frac{1}{2}$

1) $\int\limits_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} \frac{dx}{1+x^2} = \operatorname{arctg}\sqrt{3} - \operatorname{arctg}\frac{1}{\sqrt{3}} = \frac{\pi}{3} - \frac{\pi}{6} = \frac{\pi}{6}$

2) $\int\limits_{-\frac{1}{2}}^{\frac{1}{2}} \frac{dx}{\sqrt{1-x^2}} = \arcsin\frac{1}{2} - \arcsin\left(-\frac{1}{2}\right) = \frac{\pi}{6} + \frac{\pi}{6} = \frac{\pi}{3}$

3) $\int\limits_0^2 |1-x|\,dx = \int\limits_0^1 (1-x)\,dx + \int\limits_1^2 (x-1)\,dx =$

$+ \quad 1 \quad -$

$= \left(-\frac{(1-x)^2}{2}\Big|_0^1\right) + \left(\frac{(x-1)^2}{2}\Big|_1^2\right) =$

$= +\frac{1}{2} + \frac{1}{2} = 1.$

5) $\int\limits_0^{\ln 2} x e^{-x}\,dx = \left\{ \cancel{u = e^{-x} \quad du = -e^{-x}dx \quad dV = x\,dx \to V = \frac{x^2}{2}} \quad e^{-x} = t \quad dx = -\frac{dt}{t} \quad -x = \ln t \quad x = -\ln t \right\} =$

$= \cancel{\int\limits_0^{\ln 2} + \ln t \cdot t \cdot \frac{dt}{t} = \int\limits_0^{\ln 2} \ln t \cdot dt} = \int\limits_0 \ln t \cdot dt = \{u =$

6) $\int\limits_0^{\pi} x \sin x\,dx = \left\{ \begin{array}{l} u = x \to du = dx \\ dV = \sin x\,dx \Rightarrow V = -\cos x \end{array} \right\} =$

$= -\cos x \cdot x \Big|_0^{\pi} + \int\limits_0^{\pi} \cos x\,dx = \pi + \sin x \Big|_0^{\pi} = \pi.$

7) $\int\limits_0^{2\pi} x^2 \cos x\,dx = \left\{ \begin{array}{l} x^2 = u \to du = 2x\,dx \\ dV = \cos x\,dx \Rightarrow V = \sin x \end{array} \right\} =$

$= \underbrace{\sin x \cdot x^2 \Big|_0^{2\pi}}_{=0} - 2\int\limits_0^{2\pi} \sin x \cdot x\,dx = \left\{ \begin{array}{l} u = x \to du = dx \\ \sin x\,dx = dV \Rightarrow V = -\cos x \end{array} \right\} =$

$= +2x \cdot \cos x \Big|_0^{2\pi} + \int\limits_0^{2\pi} \cos x\,dx = +4\pi$

8) $\int\limits_{\frac{1}{e}}^{e} |\ln x|\,dx = \int\limits_{\frac{1}{e}}^{1} -\ln x \cdot dx + \int\limits_1^{e} \ln x \cdot dx = \left\{ \begin{array}{ll} u = \ln x & du = \frac{dx}{x} \\ dV = dx & V = x \end{array} \right\} =$

$= -x\ln x \Big|_{\frac{1}{e}}^{1} - \int\limits_{\frac{1}{e}}^{1} dx + \ln x \cdot x \Big|_{\frac{1}{e}}^{1} + \int\limits_{\frac{1}{e}}^{1} dx = -\frac{1}{e} - \left(\cancel{1} - \frac{1}{e}\right) +$

$+\frac{1}{e}+1-\frac{1}{e}$

9) $\int_0^1 \arccos x\, dx = \left\{\begin{matrix} u=\arccos x & du=-\frac{dx}{\sqrt{1-x^2}} \\ dV=dx \to & V=x \end{matrix}\right\} =$

$=\underbrace{x\cdot\arccos x \Big|_0^1}_{=0} + \frac{1}{2}\int \frac{d(x^2)}{(1-x^2)^{\frac{1}{2}}} = \frac{1}{2}\int_0^1 (1-x^2)^{-\frac{1}{2}}\, d(x^2) =$

$=\frac{\cancel{2}}{\cancel{2}}\cdot -\sqrt{1-x^2} \quad = -\sqrt{1-x^2}\Big|_0^1 = +1$

10) $\int_0^{\sqrt{3}} x\cdot \operatorname{arctg} x\, dx = \left\{\begin{matrix} u=\operatorname{arctg} x & du=\frac{dx}{1+x^2} \\ dV=x\,dx \to & V=\frac{x^2}{2} \end{matrix}\right\} =$

$+1-1$

$=\frac{x^2}{2}\operatorname{arctg} x\Big|_0^{\sqrt{3}} - \frac{1}{2}\int\frac{x^2\,dx}{1+x^2} =$

$=\frac{3}{2}\cdot\frac{\pi}{3} - \frac{1}{2}\int_0^{\sqrt{3}} dx - \frac{1}{2}\int_0^{\sqrt{3}}\frac{dx}{1+x^2} =$

$=\frac{\pi}{2} - \frac{1\cdot\sqrt{3}}{2} - \frac{1}{2}\left(\operatorname{arctg} x\Big|_0^{\sqrt{3}}\right) =$

$=\frac{\pi}{2}-\frac{\sqrt{3}}{2} - \frac{1}{2}\cdot\frac{\pi}{3} = \frac{\pi}{2}-\frac{\pi}{6}-\frac{\sqrt{3}}{2} = \frac{2\pi}{6}-\frac{\sqrt{3}}{2} = \frac{\pi}{3}-\frac{\sqrt{3}}{2}$

Замена переменной в определённом интеграле

① $\displaystyle\int_0^4 \frac{dx}{\sqrt{x}+1} = \left\{\begin{array}{l} \sqrt{x}+1=t \\ \sqrt{x}=t-1 \\ x=(t-1)^2 \\ dx=2(t-1)\,dt \\ x=0 \quad t=1 \\ x=4 \quad t=3 \end{array}\right\} = \int_1^3 \frac{2(t-1)\,dt}{t} = \int_1^3 \frac{2t-2}{t}\,dt =$

$\displaystyle= \int_1^3 2\,dt - 2\int_1^3 \frac{dt}{t} = 2\left(t\Big|_1^3 - \ln t\Big|_1^3\right) =$

$= 2(3-1-\ln 3+0) = 2(2-\ln 3)$

② $\displaystyle\int_0^1 x^7\sqrt{1+3x^4}\,dx = \left\{\begin{array}{l} \sqrt{1+3x^4}=t \\ 1+3x^4=t^2 \\ x^4=\frac{t^2-1}{3} \\ 4x^3dx=\frac{1}{3}(2t\,dt \\ x^3dx=\frac{1}{6}t\,dt \\ x=0 \quad t=1 \\ x=1 \quad t=\sqrt{1+3}=2 \end{array}\right\} =$

$\displaystyle= \int_1^2 t\cdot\underbrace{x^4}_{\frac{t^2-1}{3}}\cdot\underbrace{x^3dx}_{\frac{1}{6}t\cdot dt} = \int_1^2 \frac{t\cdot(t^2-1)t}{18}\,dt =$

$\displaystyle= \frac{1}{18}\int_1^2 (t^4-t^2)\,dt = \frac{1}{18}\cdot\left(\frac{t^5}{5}\Big|_1^2\right) - \frac{1}{18}\left(\frac{t^3}{3}\Big|_1^2\right) =$

$\displaystyle= \frac{1}{18}\left(\frac{32}{5}-\frac{1}{5}\right) - \frac{1}{18}\left(\frac{8}{3}-\frac{1}{3}\right) = \frac{31}{18\cdot 5} - \frac{7}{18\cdot 3} =$

$\displaystyle= \frac{31}{6\cdot 3\cdot 5} - \frac{7}{18\cdot 3} = \frac{29}{135}$

③ $\displaystyle\int_{-1}^0 \frac{\sqrt{x+1}}{\sqrt[3]{x+1}+1}\,dx = \left\{\begin{array}{l} \sqrt[6]{x+1}=t \quad x+1=t^6 \quad ; x=t^6-1 \\ \sqrt{x+1}=t^3 \quad\quad\quad\quad ; dx=6t^5dt \\ \sqrt[3]{x+1}=t^2 \\ x=-1 \quad t=0 \\ x=0 \quad t=1 \end{array}\right\} =$

$\displaystyle= \int_0^1 \frac{t^3\cdot 6t^5\cdot dt}{t^2+1} = 6\int_0^1 \frac{t^8\,dt}{t^2+1} = \cancel{6\int_0^1 \frac{(t^6-t^4+t^2-1)\,dt}{1}} + \frac{dt}{t^2+1}$

$t^8 \;|\; t^2+1$
$-\;t^8+t^6 \;|\; t^6-t^4+t^2-1.$
$-t^6$
$-\;-t^6-t^4$
t^4
$-\;t^4+t^2$
$-t^2$
$-\;-t^2-1 \quad +1.$

$\displaystyle= 6\int_0^1\left(t^6-t^4+t^2-1+\frac{1}{1+t^2}\right)dt =$

$\displaystyle= 6\left(\frac{t^7}{7}-\frac{t^5}{5}+\frac{t^3}{3}-t+\operatorname{arctg} t\right)\Big|_0^1$

$\displaystyle= 6\left(\frac{1}{7}-\frac{1}{5}+\frac{1}{3}-\frac{1}{1}+\frac{\pi}{4}\right)$

④ $\displaystyle\int_{\ln 3}^{3\ln 2} \frac{dx}{\sqrt{1+e^x}} = \left\{\begin{array}{l} \sqrt{1+e^x} = t \\ 1+e^x = t^2 \\ e^x = t^2-1 \end{array}\right. \Rightarrow x = \ln(t^2-1) \quad dx = \frac{2t\,dt}{t^2-1}$

$x = \ln 3 \rightarrow t = 2$
$x = 3\ln 2 \rightarrow t = 3$ $\Big\} =$

$$= \int_2^3 \frac{2t\,dt}{(t^2-1)\cdot t} = 2\int_2^3 \frac{dt}{t^2-1} = \frac{2}{2}\int_2^3 \frac{t+1-(t-1)}{(t-1)(t+1)}\,dt =$$

$$= \int_2^3 \frac{dt}{t-1} - \int_2^3 \frac{dt}{t+1} = \left(\ln|t-1| - \ln|t+1|\right)\Big|_2^3 =$$

$$= \left(\ln\left|\frac{t-1}{t+1}\right| \Big|_2^3\right) = \ln\frac{2}{4} - \ln\left(\frac{1}{3}\right) = \ln\frac{3}{2}$$

⑤ $\displaystyle\int_0^{\frac{\pi}{2}} \frac{dx}{3+2\cos x} = \left\{\begin{array}{ll} \operatorname{tg}\frac{x}{2} = t & \\ \frac{x}{2} = \operatorname{arctg} t & dx = \frac{2dt}{1+t^2} \\ x = 2\operatorname{arctg} t & \cos x = \frac{1-t^2}{1+t^2} \end{array}\right\} =$

$x = 0 \quad t = 0$
$x = \frac{\pi}{2} \quad t = 1$

$$= \int_0^1 \frac{2dt}{(1+t^2)\left(3 + \frac{2-2t^2}{1+t^2}\right)} = \int_0^1 \frac{2dt}{t^2+5} = 2\int_0^1 \frac{dt}{t^2+5} =$$

$$= 2\cdot\frac{1}{\sqrt5}\operatorname{arctg}\frac{t}{\sqrt5}\Big|_0^1 = \frac{2}{\sqrt5}\cdot\operatorname{arctg}\frac{1}{\sqrt5}$$

⑥ $\displaystyle\int_0^{\frac{\sqrt3}{2}} \sqrt{1-x^2}\,dx = \left\{\begin{array}{l} -1 \le x \le 1 \\ x = \sin t, \quad t \in [-\frac{\pi}{2}, \frac{\pi}{2}] \\ \sqrt{1-\sin^2 t} = \sqrt{\cos^2 t} = |\cos t| \\ \text{т.к. } t \in [-\frac{\pi}{2}; \frac{\pi}{2}], \text{ то } \cos t \ge 0 \\ dx = \cos t\,dt \\ x = 0 \quad t = 0 \\ x = \frac{\sqrt3}{2} \quad t = \frac{\pi}{3} \end{array}\right\} =$

$$= \int_0^{\frac{\pi}{3}} \cos^2 t\,dt = \int_0^{\frac{\pi}{3}} \frac{1+\cos 2t}{2}\,dt = \frac{1}{2}\left(t + \frac{\sin 2t}{2}\right)\Big|_0^{\frac{\pi}{3}} = \frac{\pi}{6} + \frac{\sqrt3}{8}$$

⑦ $\displaystyle\int_{\sqrt2}^{\sqrt3} \frac{dx}{x^2\sqrt{4-x^2}} = \left\{\begin{array}{l} x = 2\sin t \\ dx = 2\cos t\,dt \\ \sqrt{4-x^2} = \sqrt{4-4\sin^2 t} = 2|\cos t| \\ x = \sqrt2 \rightarrow t = \frac{\pi}{4} \\ x = \sqrt3 \rightarrow t = \frac{\pi}{3} \\ t \in [\frac{\pi}{4}; \frac{\pi}{3}], \cos t > 0 \end{array}\right\} =$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{2\cos t\,dt}{4\sin^2 t\cdot 2\cos t} = \frac{1}{4}\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{dt}{\sin^2 t} = -\frac{1}{4}\operatorname{ctg} t\Big|_{\frac{\pi}{4}}^{\frac{\pi}{3}} =$$

$$= -\frac{1}{4}\left(\frac{\sqrt3}{3} - 1\right) = \frac{3-\sqrt3}{12}$$

⑧ $\int_0^3 \frac{dx}{\sqrt{1+x^2}} = \left\{ \begin{array}{l} x = sh\,t \\ dx = ch\,t\,dt \\ \sqrt{1+x^2} = \sqrt{1+sh^2 t} = \sqrt{ch^2 t} = |ch\,t| = ch\,t \end{array} \right\} =$

$=$ $x=0 \Rightarrow t=0$

$x=3 \quad sh\,t = 3$

$\frac{e^t - e^{-t}}{2} = 3$

$e^t = z, \quad z>0 \quad z^2 - 6z - 1$

$z_{1,2} = 3 \pm \sqrt{9+1} = 3 \pm \sqrt{10}$

$t = \ln(3+\sqrt{10}) \quad ; \quad e^t = 3+\sqrt{10}$

$= \int_0^{\ln(3+\sqrt{10})} \frac{ch\,t\,dt}{ch\,t} = t \Big|_0^{\ln(3+\sqrt{10})} = \ln(3+\sqrt{10})$

⑨ $\int_0^{\frac{1}{2}} \sqrt{\frac{1+x}{1-x}}\,dx = \left\{ \begin{array}{l} \frac{1+x}{1-x} \geq 0 \qquad -1 \leq x < 1 \\ x = \cos t \qquad dx = -\sin t\,dt \\ \sqrt{\frac{1+x}{1-x}} = \sqrt{\frac{1+\cos t}{1-\cos t}} = \sqrt{\frac{2\cos^2\frac{t}{2}}{2\sin^2\frac{t}{2}}} = |ctg\,\frac{t}{2}| = \end{array} \right.$

$x = 0 \Rightarrow t = \frac{\pi}{2} \qquad \frac{t}{2} \in [\frac{\pi}{6}; \frac{\pi}{4}] \Rightarrow ctg\frac{t}{2}$

$x = \frac{1}{2} \Rightarrow t = \frac{\pi}{3}$

$= \cancel{\int_{\frac{\pi}{2}}^{\frac{\pi}{3}} ctg\frac{t}{2}\,dt = \int_{\frac{\pi}{3}}} = -\int_{\frac{\pi}{2}}^{\frac{\pi}{3}} ctg\frac{t}{2} \cdot \sin t\,dt = \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} ctg\frac{t}{2} \cdot 2\sin\frac{t}{2}\cos\frac{t}{2}\,dt =$

$= \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} 2\cos^2\frac{t}{2}\,dt = \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} (1+\cos t)\,dt = (t + \sin t)\Big|_{\frac{\pi}{3}}^{\frac{\pi}{2}} =$

$= \frac{\pi}{2} + 1 - \frac{\pi}{3} - \frac{\sqrt{3}}{2} = \frac{\pi}{6} + 1 - \frac{\sqrt{3}}{2}$

⑩ $\int_0^3 \arccos\sqrt{\frac{x}{1+x}}\,dx = \left\{ \begin{array}{l} \arccos\sqrt{\frac{x}{1+x}} = t \\ \sqrt{\frac{x}{1+x}} = \cos t \end{array} \right.$ $\qquad dx = d(ctg^2 t)$

$\frac{x}{1+x} = \cos^2 t \;\; |\cdot(1+x) \quad \Rightarrow x = ctg^2 t$

Дорешать.!

$x = 0 \quad t = \arccos 0 = \frac{\pi}{2}$

$x = 3 \quad t = \arccos\frac{\sqrt{3}}{2} = \frac{\pi}{6}$

$= \int_{\frac{\pi}{2}}^{\frac{\pi}{6}} t\,d(ctg^2 t) = t\,ctg^2 t\Big|_{\frac{\pi}{2}}^{\frac{\pi}{6}} - \int_{\frac{\pi}{2}}^{\frac{\pi}{6}} ctg^2 t\,dt =$

$= \frac{\pi}{6}\,ctg^2\frac{\pi}{6} - \int_{\frac{\pi}{2}}^{\frac{\pi}{6}} \left(\frac{1}{\sin^2 t} - 1\right)dt = \frac{\pi}{2} + |ctg\,t + t|\Big|_{\frac{\pi}{2}}^{\frac{\pi}{6}} =$

$= \frac{\pi}{2}$

Домашняя работа.

① $\int_{-1}^{1} \frac{x\,dx}{\sqrt{5-4x}} = \left\{ \begin{array}{l} \sqrt{5-4x} = t \\ 5-4x = t^2 \end{array} \right. \to \begin{array}{l} 4x = 5-t^2 \\ x = \frac{5}{4} - \frac{t^2}{4} \end{array} \quad dx = -\frac{1t\,dt}{2} \Big\} =$

$x = 1 \to t = 1$
$x = -1 \to t = 3$

$\int_3^1 \frac{(5-t^2)}{4t} \cdot \left(-\frac{1t\,dt}{2}\right) = \int_1^3 \frac{(5-t^2)}{8} dt = \frac{1}{8}\int_1^3 \left(5 - t^2\right) dt =$

$= \frac{1}{8}\left(5t - \frac{t^3}{3}\right)\Big|_1^3 = \frac{1}{8}\left(15 - \frac{27}{3} - 5 + \frac{1}{3}\right) = \frac{1}{8}\left(\frac{4}{3}\right) = \frac{1}{6}$

② $\int_0^a x^2\sqrt{a^2-x^2}\,dx = \int_0^a \left\{ \begin{array}{l} (a-x)(a+x) \geq 0 \\ -a \leq x \leq a \end{array} \right.$ $\quad -a \quad a$

$\Rightarrow x = a\sin t \quad t \in \left(-\frac{\pi}{2}; \frac{\pi}{2}\right)$

$dx = a\cos t\,dt$

$x^2 = a^2\sin^2 t$

$x = a \Rightarrow \sin t = 1$
$t = \frac{\pi}{2}$; $\quad x = 0 \Rightarrow t = 0$

$\sqrt{a^2 - a^2\sin^2 t}$

$= \int_0^{\frac{\pi}{2}} a^2\sin^2 t\sqrt{a^2 - a^2\sin^2 t} \cdot a\cos t\,dt =$

$= \int_0^{\frac{\pi}{2}} a^4 \sin^2 t \cdot \underline{\cos t \cdot \cos t}\,dt = a^4 \int_0^{\frac{\pi}{2}} \frac{1}{2}\sin 2t \cdot \frac{1}{2}\sin 2t\,dt =$

$= \frac{a^4}{4}\int_0^{\frac{\pi}{2}} \sin^2 2t\,dt = \frac{a^4}{4}\int_0^{\frac{\pi}{2}} \frac{1-\cos 4t}{2} dt = \frac{a^4}{8}\int_0^{\frac{\pi}{2}} (1-\cos 4t)\,dt =$

$= \frac{a^4}{8}\left(t + \frac{\sin 4t}{4}\right)\Big|_0^{\frac{\pi}{2}} = \frac{a^4}{8}\left(\frac{\pi}{2} + 0\right) = \frac{\pi a^4}{16}$

③ $\int_0^{\ln 2} \sqrt{e^x - 1}\,dx = \left\{ \begin{array}{l} \sqrt{e^x-1} = t \\ e^x - 1 = t^2 \\ x = \ln(t^2-1) \end{array} \right. \quad dx = \frac{2t\,dt}{t^2-1} \quad \begin{array}{l} \to x = \ln 2 \Rightarrow t = 1 \\ x = 0 \to t = 0 \end{array} \Big\} =$

$= \int_0^1 \frac{2t^2\,dt}{t^2-1} = 2\int_0^1 \frac{(t^2-1+1)\,dt}{t^2-1} = 2\int_0^1 \left(1 + \frac{1}{t^2-1}\right) dt =$

$\cancel{2(t) - 2\int_0^1 \left(1 - \frac{1}{t^2-1}\right) dt = 2(t - \frac{1}{2}\ln\left|\frac{1+t}{1-t}\right|}\ =$

$= 2\int_0^1 \frac{t^2-1+1}{t^2-1} dt = 2\int_0^1 \left(1 + \frac{1}{(t-1)(t+1)}\right) dt =$

$$= 2\int_0^1 \left(1 + \frac{t+1-(t-1)}{2(t-1)(t+1)}\right) dt =$$

$$= 2\int_0^1 dt + \cancel{2}\int_0^1 \frac{dt}{t-1} - \int_0^1 \frac{dt}{t+1} =$$

$$= 2(t)\Big|_0^1 + \ln|t-1|\Big|_0^1 - \ln|t+1|\Big|_0^1 =$$

$$= 2 + \underline{\ln 0}? - \ln 2 \qquad ?$$

4) $$\int_0^{\frac{1}{4}} \frac{\arcsin\sqrt{x}}{\sqrt{x(1-x)}}\,dx = \int_0^{\frac{1}{4}} \frac{\arcsin \boxed{x^{\frac{1}{2}}}}{\sqrt{x-x^2}}\,dx = \begin{cases} t = \arcsin(x^{\frac{1}{2}}) \\ dt = \dfrac{1\, dx}{2\sqrt{1-x}\,\sqrt{x}} \end{cases}$$

$$dx = \frac{2\sqrt{x-x^2}\,dt}{1}$$

$$\left.\begin{matrix} x = \frac{1}{4} \to t = \frac{\pi}{6} \\ x = 0 \to t = 0 \end{matrix}\right\}$$

$$= \int_0^{\frac{\pi}{6}} \frac{t \cdot 2\cancel{\sqrt{x-x^2}}\,dt}{\cancel{\sqrt{x-x^2}}} = 2\int_0^{\frac{\pi}{6}} t\,dt = \frac{2\cdot t^2}{2}\Big|_0^{\frac{\pi}{6}} = \frac{\pi^2}{36}$$

⑤ $$\int_1^9 x\sqrt[3]{1-x}\,dx = \begin{cases} \cancel{u = \sqrt[3]{1-x} \to du = \frac{dx}{3(1-x)^{\frac{2}{3}}}} \\ t = \sqrt[3]{1-x} \\ t^3 = 1-x \quad \Rightarrow x = 1-t^3 \mid dx = -3t^2\,dt \\ x = 9 \to t = -2 \\ x = 1 \to t = 0 \end{cases} =$$

$$= \int_0^{-2} (1-t^3)\cdot t \cdot (-3t^2)\,dt = 3\int_{-2}^{0} (t^3 - t^5)\,dt =$$

$$= 3\left(\frac{t^4}{4} - \frac{t^6}{6}\right)\Big|_{-2}^{0} = 3\left(-\frac{16}{4} + \frac{64}{6}\right) = 20$$

⑥ $\int_0^1 x^{15}\sqrt{1+3x^8}\,dx = \begin{cases} t=\sqrt{1+3x^8} \\ t^2 = 1+3x^8 \end{cases} \Rightarrow 3x^8 = t^2-1;$

$dt = \dfrac{24x^7\cdot dx}{2\sqrt{1+3x^8}} \Rightarrow \Big\} =$

$= \int_0^1 x^8\sqrt{1+3x^8}\cdot x^7\cdot dx = \frac{1}{8}\int_0^1 x^8\sqrt{1+3x^8}\cdot d(x^8) = \begin{Bmatrix} x^8 = t \end{Bmatrix} =$

$d(x^8) = 8\cdot x^7 dx$

$x=1 \Rightarrow t=1$

$x=0 \Rightarrow t=0$

$= \frac{1}{8}\int_0^1 t\sqrt{1+3t}\cdot dt =$

$= \frac{1}{8}\int_0^1 t\sqrt{1+3t}\cdot dt = \begin{cases} \sqrt{1+3t} = u \\ 1+3t = u^2 \rightarrow t = \frac{u^2}{3} - \frac{1}{3} \end{cases}$

$dt = \frac{2u\,du}{3}$

$t=1 \rightarrow u=2$

$t=0 \rightarrow u=1$ $\Big\} =$

$= \frac{1}{8}\cdot\frac{2}{3}\int_1^2 \frac{(u^2-1)\cdot u\cdot du}{3} = \frac{1}{36}\int_1^2 (u^3-u)\,du =$

$= \frac{1}{36}\left(\frac{u^4}{4} - \frac{u^2}{2}\right)\Big|_1^2 = \frac{1}{36}\left(\frac{16}{4} - \frac{4}{2} - \frac{1}{4} + \frac{1}{2}\right) = \frac{1}{16}$?

⑦ $\int_0^{\frac{\sqrt3}{2}} \sqrt{\frac{1-x}{1+x}}\,dx = \Big\{$

$-\ \ -1\ \ +\ \ 1\ \ -$

$\Rightarrow -1 < x \leq 1$

$x = \cos t \qquad dx = -\sin t\,dt.$

$\sqrt{\frac{1-x}{1+x}} = \sqrt{\frac{1-\cos t}{1+\cos t}} = \sqrt{\frac{2\sin^2\frac{t}{2}}{2\cos^2\frac{t}{2}}} = \left|tg\frac{t}{2}\right|$

$x=0 \Rightarrow t = \frac{\pi}{2}$

$x = \frac{\sqrt3}{2} \Rightarrow t = \frac{\pi}{6} \quad \rightarrow tg\frac{t}{2} > 0$ $\Big\} =$

$= -\int tg\frac{t}{2}\cdot\sin t\,dt = -\int \frac{\sin\frac{t}{2}\cdot 2\sin\frac{t}{2}\cdot\cancel{\cos\frac{t}{2}}}{\cancel{\cos\frac{t}{2}}}\,dt =$

$= -2\int \sin^2\frac{t}{2}\,dt = -2\int\frac{1-\cos t}{2}\,dt = -\int(1-\cos t)\,dt =$

$= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}}(1-\cos t)\,dt = (t-\sin t)\Big|_{\frac{\pi}{6}}^{\frac{\pi}{2}} = \frac{\pi}{2} - 1 - \frac{\pi}{6} + \frac{1}{2} = \frac{\pi}{3} - \frac{1}{2}$

⑧ $\int_0^3 \arcsin\sqrt{\frac{x}{1+x}}\,dx = \Big\{$

$+\ \ -1\ \ -\ \ 0\ \ +$

$x > 0$

$x < -1$

$\sqrt{\frac{x}{1+x}} = t \qquad \frac{x}{1+x} = t^2$

$dx = \frac{2t}{(1-t^2)^2}$

$1 - \frac{1}{x+1} = t^2$

$\frac{x+1-1}{x+1} = t^2$

$\frac{1}{x+1} = 1 - t^2 \Rightarrow x+1 = \frac{1}{1-t^2}$

$x = \frac{1}{1-t^2} - 1$

$$\arcsin\sqrt{\frac{x}{1+x}} = t \Rightarrow \sin t = \sqrt{\frac{x}{1+x}}$$

$$\sin^2 t = \frac{x+1-1}{1+x}$$

$$\sin^2 t = 1 - \frac{1}{1+x}$$

$$\frac{1}{1+x} = \cos^2 t$$

$$1+x = \frac{1}{\cos^2 t} \Rightarrow x = \frac{1}{\cos^2 t} - 1$$

$$x = tg^2 t$$

$$dx = \frac{2tgt}{\cos^2 t} \cdot dt$$

$x = 0 \Rightarrow t = 0$

$x = 3 \to \sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2} \Rightarrow t = \frac{\pi}{3}$

$$= \int_0^{\frac{\pi}{3}} \frac{t \cdot 2tg t \cdot dt}{\cos^2 t} = 2\int_0^{\frac{\pi}{3}} \frac{\sin t \cdot t \cdot dt}{\cos t \cdot \cos^2 t} = 2\int_0^{\frac{\pi}{3}} \frac{\sin t}{\cos^3 t} \cdot t\,dt =$$

$$= \left\{ \begin{array}{l} u = t \\ dV = \frac{\sin t}{\cos(t)^3} dt \end{array} \Rightarrow \begin{array}{l} du = dt \\ V = \frac{1}{2\cos^2 t} \end{array} \right\} =$$

$$= 2 \cdot \left(\frac{t}{2\cos^2 t} \Big|_0^{\frac{\pi}{3}} - \frac{1}{2}\int_0^{\frac{\pi}{3}} \frac{dt}{\cos^2 t} \right) =$$

$$= 2 \cdot \left(\frac{t}{2\cos^2 t} \Big|_0^{\frac{\pi}{3}} - \frac{1}{2}\left(tg t \Big|_0^{\frac{\pi}{3}}\right) \right) =$$

$$= 2 \cdot \left(\frac{\pi \cdot 4}{6 \cdot 1} - 0 - \frac{1}{2}\left(tg\frac{\pi}{3} - 0\right) \right) = 2\left(\frac{2\pi}{3} - \frac{\sqrt{3}}{2}\right) =$$

$$= \frac{4}{3}\pi - \sqrt{3}$$

25.12.2020.

Несобственные интегралы с бесконечными пределами.

① $\int\limits_{-\infty}^{+\infty} \frac{dx}{x^2+2x+2} = \lim\limits_{\substack{A\to-\infty \\ B\to+\infty}} \int\limits_A^B \frac{dx}{x^2+2x+2} = \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) = \pi$

$\int\limits_A^B \frac{d(x+1)}{(x+1)^2+1} = \operatorname{arctg}(x+1)\Big|_A^B = \operatorname{arctg}(B+1) - \operatorname{arctg}(A+1)$

② $\int\limits_1^{+\infty} \frac{dx}{x^2(x+1)} = \int\limits_1^{+\infty} \frac{dx}{x^3\left(1+\frac{1}{x}\right)} = -\int\limits_1^{+\infty} \frac{d\left(\frac{1}{x}\right)}{x\left(1+\frac{1}{x}\right)} = \left\{ \begin{array}{l} \frac{1}{x} = t \\ x=1,\ t=1 \\ x=+\infty;\ t=0 \end{array} \right\} =$

$= -\int\limits_1^0 \frac{t\,dt}{1+t} = \int\limits_0^1 \frac{t+1-1}{t+1}\,dt = \int\limits_0^1 \left(1 - \frac{1}{t+1}\right) dt = t - \ln|t+1|\Big|_0^1 =$

$= 1 - \ln 2$

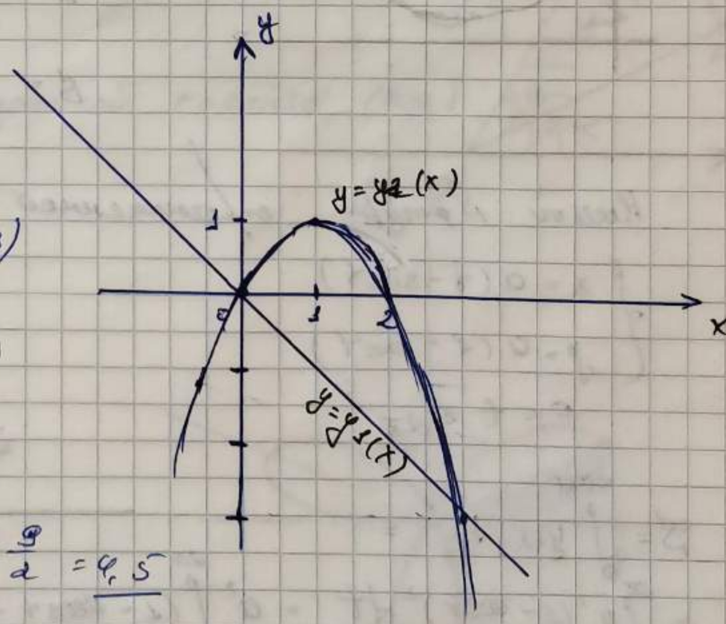
Приложения определённого интеграла.

I. Вычисление площадей плоских фигур

① S − ?

$y = 2x - x^2$
$x + y = 0$

$2x - x^2 = -x$
$x^2 - 3x = 0$
$x(x-3) = 0 \to$ т. пер. $(0;0)$ и $(3;-3)$



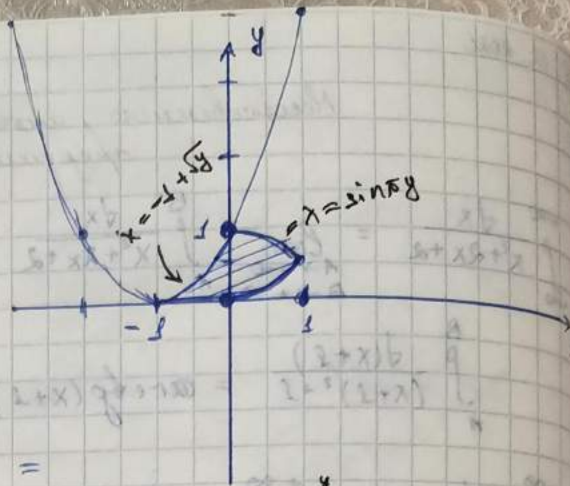
$S = \int\limits_a^b (y_2(x) - y_1(x))\,dx =$

$= \int\limits_0^3 (2x - x^2 - (-x))\,dx =$

$= \left(\frac{3x^2}{2} - \frac{x^3}{3}\right)\Big|_0^3 = \frac{27}{2} - 9 = \frac{9}{2} = 4{,}5$

② $y = (x+1)^2$, $x = \sin \pi y$, $y = 0$, $0 \le y \le 1$

$$S = \int_c^d (x_2(y) - x_1(y))\,dy \;\ominus$$



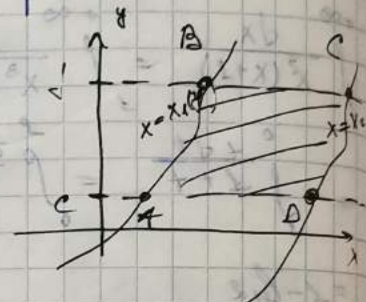
$x + 1 = \pm\sqrt{y}$

$x = -1 \pm \sqrt{y}.$

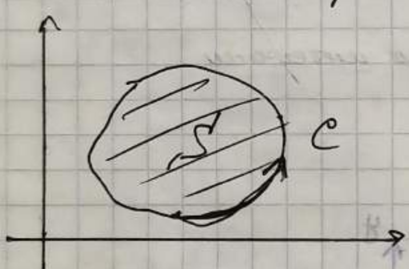
$$\ominus \int_0^1 (\sin\pi y + 1 - \sqrt{y})\,dy =$$

$$= \left(-\frac{1}{\pi}\cos\pi y + y - \frac{2}{3}y^{\frac{3}{2}}\right)\Big|_0^1 =$$

$$= \left(\frac{1}{\pi} + 1 - \frac{2}{3}\right) - \left(-\frac{1}{\pi}\right) = \frac{2}{\pi} + \frac{1}{3}$$



③ $\begin{cases} x = x(t) \\ y = y(t) \\ 0 \le t \le T \end{cases}$, что фигура ограниченная этой кривой остаётся слева.



$$S = \frac{1}{2}\int_0^T (x(t)\cdot y'(t) - y(t)\cdot x'(t))\,dt$$

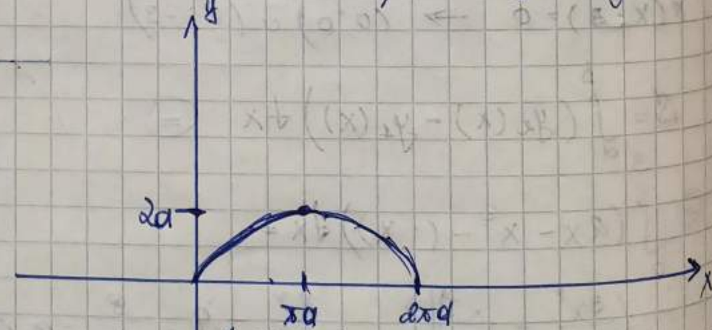
$$S = \int_0^T x(t)\,y'(t)\,dt$$

$$S = -\int_0^T y(t)\,x'(t)\,dt$$ – Если меняе ориентацию знак меня

Найти S фигуры ограниченной первой аркой циклоиды.

$\begin{cases} x = a(t - \sin t) \\ y = a(1 - \cos t) \end{cases}$

$0 \le t \le 2\pi$



$$S = \int_0^{2\pi a} y(x)\,dx =$$

$$= \int_0^{2\pi} a^2(1-\cos t)^2\,dt = a^2\int_0^{2\pi}\left(1 - 2\cos t + \frac{1+\cos 2t}{2}\right)dt =$$

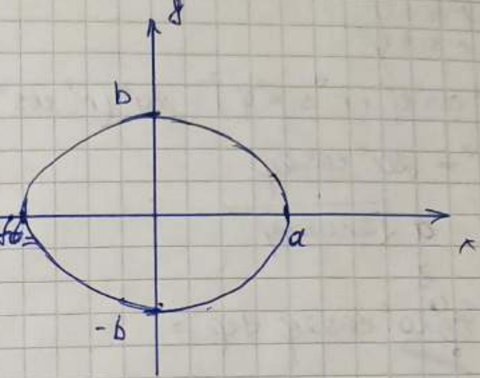
$dx = a(1 - \cos t)\,dt$

$y(x(t)) = a(1 - \cos t)$

$$= a^2\left(\frac{3}{2}t - 2\sin t + \frac{\sin 2t}{4}\right)\Big|_0^{2\pi} = 3\pi a^2$$

б) $S - ?$ $\quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$

$$\begin{cases} x = a\cos t \\ y = b\sin t \\ t \in [0; 2\pi] \end{cases}$$



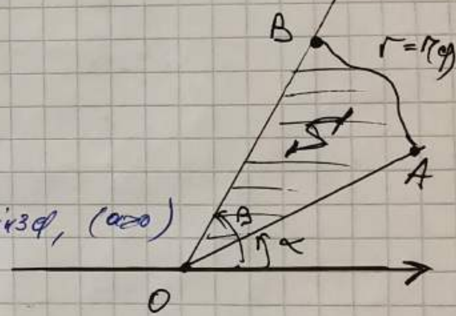
$$S = \frac{1}{2}\int_0^{2\pi} (a\cos t \cdot b\cos t + b\sin t \, a\sin t)\,dt =$$

$$= \frac{ab}{2}\int_0^{2\pi} (\cos^2 t + \sin^2 t)\,dt =$$

$$= \frac{ab}{2}\cdot t\Big|_0^{2\pi} = \pi ab$$

Площадь сектора OAB, ограниченного кривой $r = r(\varphi)$, заданной в полярной системе координат и двумя лучами $\varphi = \alpha$, $\varphi = \beta$ $(\alpha < \beta)$

$$S = \frac{1}{2}\int_\alpha^\beta r^2(\varphi)\cdot d\varphi$$



в) S_φ, ограниченной кривой, заданной $r = a\sin 3\varphi$, $(a>0)$

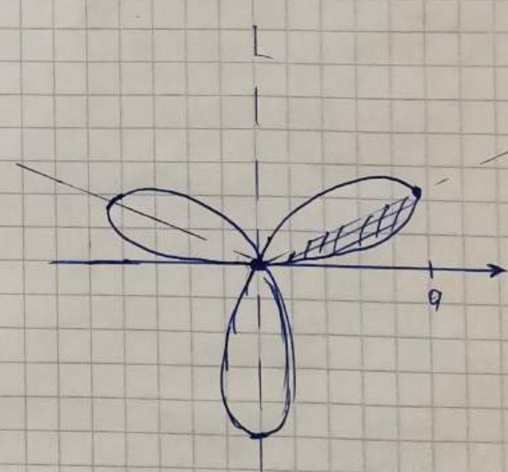
$$S = 6\cdot\frac{1}{2}\int_0^{\frac{\pi}{6}} (a\sin 3\varphi)^2\,d\varphi =$$

$$= 3a^2\int_0^{\frac{\pi}{6}} \sin^2 3\varphi\,d\varphi =$$

$$= \frac{3a^2}{2}\int_0^{\frac{\pi}{6}} (1 - \cos 6\varphi)\,d\varphi =$$

$$= \frac{3a^2}{2}\left(\varphi - \frac{1}{6}\sin 6\varphi\right)\Big|_0^{\frac{\pi}{6}} =$$

$$= \frac{3a^2}{2}\cdot\frac{\pi}{6} = \frac{\pi a^2}{4}$$



г) $(x^2+y^2) = 2a^2(x^2-y^2)$

$x = r\cos\varphi$
$y = r\sin\varphi$

$(r^2\cos^2\varphi + r^2\sin^2\varphi)^2 = 2a^2(r^2\cos^2\varphi - r^2\sin^2\varphi)$

$r^2 = 2a^2\cos 2\varphi$

$r = a\sqrt{2\cos 2\varphi}$

$S = 4\cdot\frac{1}{2}\int_0^{\frac{\pi}{4}} 2a^2\cos 2\varphi\, d\varphi =$

$= \frac{8a^2}{2}\cdot\frac{\sin 2\varphi}{2}\Big|_0^{\frac{\pi}{4}} = 2a^2$